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Simpler analysis of LP extreme points for traveling salesman and survivable network design problems

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ABSTRACT

We consider the Survivable Network Design Problem (SNDP) and the Symmetric Traveling Salesman Problem (STSP). We give simpler proofs of the existence of a $\frac{1}{2}$-edge and 1-edge in any extreme point of the natural LP relaxations for the SNDP and STSP, respectively. We formulate a common generalization of both problems and show our results by a new counting argument. We also obtain a simpler proof of the existence of a $\frac{1}{2}$-edge in any extreme point of the set-pair LP relaxation for the *element connectivity* Survivable Network Design Problem (SNDP_{elt}).

1. Introduction

We consider two well-studied combinatorial optimization problems, the Survivable Network Design Problem (SNDP) and the Symmetric Traveling Salesman Problem (STSP). Given an undirected graph $G = (V, E)$ and connectivity requirements r_{uv} for all undirected pairs $u, v \in V$ of vertices, a *Steiner network* is a subgraph of G in which there are at least r_{uv} edge-disjoint paths between u and v for all pairs $u, v \in V$. The Survivable Network Design Problem is a general network design problem where we are given an edge-weighted graph $G = (V, E)$ and connectivity requirements $\left\{r_{uv} \mid (u, v) \in \binom{V}{2}\right\}$, and the task is to find a minimum-cost Steiner network.

A *Hamiltonian cycle* in graph $G = (V, E)$ is a connected subgraph of G that has degree 2 at every vertex of V. In the Symmetric Traveling Salesman problem (STSP), we are given an edge-weighted undirected graph $G = (V, E)$, and the goal is to compute a minimum-cost Hamiltonian cycle.

Linear programming methods have been successfully used in solving both these problems in practice [1,8]. Strong theoretical results have also been obtained by analyzing linear programming (LP) relaxations for these problems [7,8]. We present a common generalization of these problems and its natural LP relaxation. Using this LP and a new counting argument, we prove the following results in Section 2.

Theorem 1.1. *Given any extreme point x of the LP relaxation* ($\mathsf{LP}_{\mathrm{sndp}}$) *for* Survivable Network Design, *there exists an edge e such that $x_e \geq \frac{1}{2}$.*

Theorem 1.2. *Given any extreme point x of the LP relaxation* ($\mathsf{LP}_{\mathrm{stsp}}$) *for the* Symmetric Traveling Salesman, *there exists an edge e such that $x_e = 1$.*

Theorem 1.1 was originally proved by Jain [9], and Theorem 1.2 by Boyd and Pulleyblank [3]. In fact [3] showed that any extreme point of the LP relaxation to the STSP has at least *three* 1-edges.

We also consider the *element connectivity* Survivable Network Design Problem (SNDP_{elt}) in Section 3. This is a well-known generalization of the usual (edge-connectivity) SNDP, where the input is an edge-weighted undirected graph $G = (V, E)$, a set $U \subseteq V$ of terminals, and connectivity requirements r_{uv} for all undirected pairs $u, v \in U$ of terminals. The vertices $V \setminus U$ and edges E of the graph are called *elements*. The goal in SNDP_{elt} is to find the minimum-cost subgraph that contains at least r_{uv} *element-disjoint* paths between u and v for every $u, v \in U$. Using the new counting argument, we provide a shorter proof of the following theorem for its natural LP relaxation considered in Fleischer et al. [5].

Theorem 1.3. *Given any extreme point x of the LP relaxation* ($\mathsf{LP}_{\mathrm{elt}}$) *for element connectivity* Survivable Network Design, *there exists an edge e such that $x_e \geq \frac{1}{2}$.*

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This result is originally due to Fleischer et al. [5], where they used it to obtain a 2-approximation algorithm for SNDP_{elt}. Recently, Chuzhoy and Khanna [4] gave a very elegant reduction from the (even more general) *vertex-connectivity* SNDP to the element-connectivity SNDP; using this 2-approximation for SNDP_{elt}, they obtained an $O(k^3 \log n)$-approximation algorithm for the vertex-connectivity SNDP (here k is the maximum requirement and n is the number of vertices).

Our proofs are based on a new counting argument that involves distributing *fractional tokens*. This idea was used earlier in Bansal et al. [2] for degree-bounded network design problems in directed graphs, and also appears implicit in the proofs of Gabow et al. [6] for the k-edge connected subgraph problem.

Notation. For any subset $F \subseteq E$ of edges, the *characteristic vector* $\chi(F) \in \{0, 1\}^E$ (also denoted χ_F) contains a 1 corresponding to each edge $e \in F$, and a 0 otherwise. For any assignment $x : E \to \mathbb{R}_+$ of non-negative real values to the edges and any subset $F \subseteq E$, $x(F)$ denotes the sum $\sum_{e\in F} x_e$.

2. The STSP and the edge-connectivity SNDP

Given a subset $S \subseteq V$, let $\delta(S) = \{(u, v) \in E \mid u \in S,\ v \notin S\}$ denote the set of edges with exactly one end-point in S. We also denote $\delta(\{v\})$ by $\delta(v)$. Then, the classical LP relaxation ($\mathsf{LP}_{\mathsf{stsp}}$) for the STSP has the following constraints:

$$x(\delta(S)) \ge 2 \quad \forall\, \emptyset \subsetneq S \subsetneq V \text{ (cut constraints)}$$
$$x(\delta(v)) = 2 \quad \forall\, v \in V \text{ (degree constraints)}$$
$$0 \le x_e \le 1 \quad \forall\, e \in E.$$

We now consider the LP relaxation ($\mathrm{LP}_{\mathrm{sndp}}$) for the SNDP. A function f from subsets of V to the integers is called *weakly supermodular* if $f(V) = f(\emptyset) = 0$ and, for all $S, T \subseteq V$, one of the following holds.

$$f(S) + f(T) \le f(S \cup T) + f(S \cap T), \quad \text{or}$$
$$f(S) + f(T) \le f(S \setminus T) + f(T \setminus S).$$

It is easy to see that the function f defined by $f(S) = \max_{u\in S, v\notin S} r_{uv}$ for each subset $S \subseteq V$ is weakly supermodular. It can be verified that the above function encodes the connectivity requirements $\{r_{u,v}\}$. We state the LP relaxation [9] for any network design problem with weakly supermodular connectivity requirement (which contains the SNDP as a special case).

$$x(\delta(S)) \ge f(S) \quad \forall\, S \subseteq V \text{ (cut constraints)}$$
$$0 \le x_e \le 1 \quad \forall\, e \in E.$$

Now we present the LP relaxation of a generalization of both the SNDP and the STSP. The input consists of an undirected graph $G = (V, E)$ with edge-costs $c : E \to \mathbb{R}_+$, a weakly supermodular function $f : 2^V \to \mathbb{Z}$, and a designated subset $W \subseteq V$ of vertices. The LP corresponding to this is as follows.

$$\begin{array}{lll} \text{(LP)} & \text{minimize} \displaystyle\sum_{e\in E} c_e x_e & \\ \text{subject to} & x(\delta(S)) \ge f(S) & \forall\, S \subseteq V \\ & x(\delta(v)) = f(v) & \forall\, v \in W \\ & 0 \le x_e \le 1 & \forall\, e \in E. \end{array}$$

Note that the first set of constraints above enforces the connectivity requirements f, the second set of constraints enforces the degree constraints on W, and the last set of constraints ensures that only a subgraph is chosen.

Given graph G, edge-costs c and connectivity requirements $\left\{r_{u,v} \mid (u, v) \in \binom{V}{2}\right\}$, the LP relaxation ($\mathrm{LP}_{\mathrm{sndp}}$) of this SNDP instance is obtained by setting, in (LP), $f(S) = \max_{u\in S, v\notin S} r_{uv}$ for each subset $S \subseteq V$ and $W = \emptyset$. For an instance of the STSP given by graph G and edge-costs c, the corresponding LP relaxation ($\mathsf{LP}_{\mathsf{stsp}}$) is obtained by setting $f(S) = 2$ for each $\emptyset \subsetneq S \subsetneq V$, $f(\emptyset) = f(V) = 0$, and $W = V$.

We prove the following theorem, which implies Theorems 1.1 and 1.2.

Theorem 2.1. *Let x be a basic feasible solution to (LP) where f is weakly supermodular.*

A. *There exists an edge $e \in E$ such that $x_e \ge \frac{1}{2}$.*
B. *Moreover, if $f(S)$ is even for each subset $S \subseteq V$, then there exists an edge $e \in E$ such that $x_e = 1$.*

The first part of Theorem 2.1 was at the heart of the iterative 2-approximation algorithm for the SNDP [9].

Before the proof of Theorem 2.1, we state some properties of tight constraints of extreme points. Two sets X, Y are *intersecting* if $X \cap Y$, $X - Y$ and $Y - X$ are nonempty. A family of sets is *laminar* if no two sets are intersecting. The proof of the following lemma is immediate from the uncrossing lemma in Jain [9].

Lemma 2.2 ([9]). *Let x be a basic feasible solution to (LP) with f being weakly supermodular, such that $0 < x_e < 1$ for all edges $e \in E$. Then, there exists a laminar family $\mathcal{L}$ of subsets such that*

1. *x is the unique solution to $\{x(\delta(S)) = f(S), \forall\, S \in \mathcal{L}\}$;*
2. *the vectors $\chi_{\delta(S)}$ for $S \in \mathcal{L}$ are linearly independent; and*
3. *$|E| = |\mathcal{L}|$.*

Proof. Lemma 4.3 in [9] proves this lemma when $W = \emptyset$; that proof is based on standard *uncrossing* arguments. In the general case, there are additional *equalities* for singleton vertex-sets corresponding to W. Let (LP′) denote the polytope given by just the first and third sets of constraints in (LP), i.e., without equality constraints on W. Note that the polytope (LP) is a face of polytope (LP′). Hence any extreme point in (LP) is also an extreme point in (LP′), for which the lemma from [9] applies. □

We now prove Theorem 2.1. Let x be any basic feasible solution to (LP).

Proof of Theorem 2.1 (*A*). We first prove that $x_e \ge \frac{1}{2}$ for some edge $e \in E$. Suppose for the sake of contradiction that $x_e < \frac{1}{2}$ for each $e \in E$. If $x_e = 0$ for some $e \in E$, we can remove edge e from the graph G and variable x_e from (LP). The residual solution x remains a basic feasible solution to the modified (LP). Thus we assume without loss of generality that $x_e > 0$ for all $e \in E$, and so Lemma 2.2 applies.

We will show a contradiction to Lemma 2.2 by means of a new counting argument. The counting argument proceeds as follows. We assign one token to each edge in E, and then reassign the tokens such that we can collect strictly more than one token per set in the laminar family $\mathcal{L}$: this would imply $|E| > |\mathcal{L}|$, which is the desired contradiction.

For any sets $S, R \in \mathcal{L}$, we say that S is the parent of R (or equivalently, that R is a child of S) if S is the smallest set in $\mathcal{L}$ containing R. Each edge $e = (u, v) \in E$ is given a unit token, which it reassigns as follows.

1. (*Rule* 1) Let S be the smallest set in $\mathcal{L}$ containing u, and R be the smallest set in $\mathcal{L}$ containing v. Then e assigns x_e tokens to each of S and R.
2. (*Rule* 2) Let T be the smallest set in $\mathcal{L}$ containing both u and v. Then e assigns $1 - 2x_e$ tokens to T.

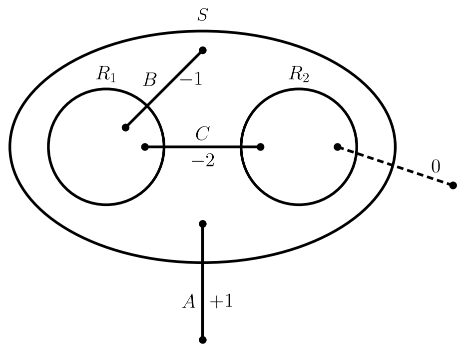

Fig. 1. Example for the expression $x(\delta(S)) - \sum_{i=1}^{k} x(\delta(R_i))$ with $k = 2$ children. The dashed edges cancel out in the expression. Edge-sets A, B, C are shown with their respective coefficients.

We now show that each set in $\mathcal{L}$ receives at least one token. Let $S \in \mathcal{L}$ have k children $R_1, \ldots, R_k$ in $\mathcal{L}$ (if S does not have any children then $k = 0$). We have the following tight inequalities for the extreme point x.

$$x(\delta(S)) = f(S) \quad \text{and} \quad x(\delta(R_i)) = f(R_i) \quad \forall\, 1 \le i \le k.$$

Subtracting, we obtain

$$x(\delta(S)) - \sum_{i=1}^{k} x(\delta(R_i)) = f(S) - \sum_{i=1}^{k} f(R_i) \Rightarrow$$

$$x(A) - x(B) - 2x(C) = f(S) - \sum_{i=1}^{k} f(R_i)$$

where

$$A = \{e : |e \cap (\cup_i R_i)| = 0,\ |e \cap S| = 1\}$$
$$B = \{e : |e \cap (\cup_i R_i)| = 1,\ |e \cap S| = 2\}$$
$$C = \{e : |e \cap (\cup_i R_i)| = 2,\ |e \cap S| = 2\}.$$

Observe that $A \cup B \cup C \neq \emptyset$: otherwise, we have the dependence $\chi_{\delta(S)} = \sum_{i=1}^{k} \chi_{\delta(R_i)}$. Also, S receives x_e tokens for each edge $e \in A$ (by *Rule* 1), $1 - x_e$ tokens for each edge $e \in B$ (by *Rules* 1 & 2), and $1 - 2x_e$ tokens for each edge $e \in C$ (by *Rule* 2). Hence, the total number of tokens received by S is exactly

$$\sum_{e \in A} x_e + \sum_{e \in B} (1 - x_e) + \sum_{e \in C} (1 - 2x_e)$$
$$= x(A) + |B| - x(B) + |C| - 2x(C)$$
$$= |B| + |C| + f(S) - \sum_{i=1}^{k} f(R_i). \qquad (1)$$

Observe that, for every edge $e \in E$, $x_e,\ 1 - x_e,\ 1 - 2x_e > 0$ since $0 < x_e < \frac{1}{2}$; combined with the fact that $A \cup B \cup C \neq \emptyset$, the number of tokens assigned to S is strictly positive (using the first expression in Eq. (1)). On the other hand, the last expression in (1) implies that the number of tokens assigned to S is integral. Thus every $S \in \mathcal{L}$ gets at least one token in this assignment (Fig. 1).

Now we show that there are some unassigned tokens, thereby showing the strict inequality $|\mathcal{L}| < |E|$. Let R be a maximum-cardinality set in $\mathcal{L}$; note that none of the sets in $\mathcal{L} \setminus \{R\}$ contains R and $R \neq V$ since $f(V) = 0$. Consider any edge $e \in \delta(R) \neq \emptyset$: the token by *Rule* 2 for edge e is unassigned as there is no set such that $|T \cap e| = 2$. This gives us the desired contradiction, and proves the first part of Theorem 2.1. □

Proof of Theorem 2.1 (B). We now consider the case when $f(S)$ is even for each $S \subseteq V$, and show that, for any basic feasible solution x to (LP), there is always an edge $e \in E$ with $x_e = 1$. The proof follows the same approach as above but with a scaled token assignment. For the sake of contradiction, we assume that $x_e < 1$ for each $e \in E$. As before, we can assume without loss of generality that $x_e > 0$. Again, we will show a contradiction to Lemma 2.2 by showing that $|\mathcal{L}| < |E|$. The counting argument proceeds as follows. We assign one token to each edge $e = (u, v) \in E$, which it redistributes as follows.

1. (*Rule* 1′) Let S be the smallest set in $\mathcal{L}$ containing u, and R be the smallest set in $\mathcal{L}$ containing v. Then e assigns $\frac{x_e}{2}$ tokens to each of S and R.
2. (*Rule* 2′) Let T be the smallest set in $\mathcal{L}$ containing both u and v. Then e assigns $1 - x_e$ tokens to T.

We now show that each set in $\mathcal{L}$ receives at least one token. As before, let $S \in \mathcal{L}$ have children $R_1, \ldots, R_k$ ($k \ge 0$). We have the following tight inequalities.

$$x(\delta(S)) = f(S) \quad \text{and} \quad x(\delta(R_i)) = f(R_i) \quad \forall\, 1 \le i \le k.$$

Dividing by two and subtracting, we obtain

$$\frac{1}{2}\left[x(\delta(S)) - \sum_i x(\delta(R_i))\right] = \frac{1}{2}\left[f(S) - \sum_i f(R_i)\right] \Rightarrow$$

$$\frac{x(A) - x(B)}{2} - x(C) = \frac{1}{2}\left[f(S) - \sum_i f(R_i)\right]$$

where the edge sets A, B, C are exactly as in the earlier case. Observe that $A \cup B \cup C \neq \emptyset$: else there is a dependence in the constraints for S and its children. Also, S receives $\frac{x_e}{2}$ tokens for each edge $e \in A$ (*Rule* 1′), $1 - \frac{x_e}{2}$ tokens for each edge $e \in B$ (*Rules* 1′ & 2′), and $1 - x_e$ tokens for each edge $e \in C$ (*Rule* 2′). Hence, the total number of tokens received by S is

$$\sum_{e \in A} \frac{x_e}{2} + \sum_{e \in B} \left(1 - \frac{x_e}{2}\right) + \sum_{e \in C} (1 - x_e)$$
$$= \frac{x(A)}{2} + |B| - \frac{x(B)}{2} + |C| - x(C)$$
$$= |B| + |C| + \frac{f(S) - \sum_i f(R_i)}{2}.$$

Following the same reasoning as before, this quantity is a positive integer (here f is an even-valued function, so the number of tokens is still integral). Thus every set $S \in \mathcal{L}$ receives at least one token in this assignment. Finally, note that some tokens corresponding to the maximal sets in $\mathcal{L}$ are unassigned. This shows the strict inequality $|\mathcal{L}| < |E|$, and gives us the desired contradiction. This proves the second part of Theorem 2.1. □

3. The element-connectivity SNDP

In this section, we consider the element-connectivity survivable network design problem (SNDP_{elt}). In this problem, we are given an undirected graph $G = (V, E)$ with edge-costs $c : E \to \mathbb{R}_+$, a set $U \subseteq V$ of terminals, and connectivity requirements r_{uv} for all undirected pairs $u, v \in U$ of terminals. Vertices in $V \setminus U$ are called non-terminals. The edges and non-terminals of the graph are called *elements*. The goal in SNDP_{elt} is to find the minimum-cost subgraph that contains at least r_{uv} *element-disjoint* paths between u and v for every $u, v \in U$. Fleischer et al. [5] used iterative rounding to obtain a 2-approximation algorithm for this problem. They [5] showed that SNDP_{elt} can be formulated as a suitable integer program (defined formally below), such that any extreme point solution to its LP-relaxation contains an edge with solution value at least half. We give a short proof of this result using a new counting argument generalizing the results in the previous section.

A *set-pair* is an ordered tuple (S, S') where $S, S' \subseteq V$. Let $\mathcal{F}$ denote some family of set-pairs. A two-set function $f : \mathcal{F} \to \mathbb{Z}_+$ is called *weakly two-supermodular* if, for any (S, S') and $(T, T') \in \mathcal{F}$, at least one of the following holds.

1. $(S \cap T, S' \cup T')$ and $(S \cup T, S' \cap T') \in \mathcal{F}$, and we have
$$f(S \cap T, S' \cup T') + f(S \cup T, S' \cap T') \geq f(S, S') + f(T, T').$$
2. $(S \cap T', S' \cup T)$ and $(S \cup T', S' \cap T) \in \mathcal{F}$, and we have
$$f(S \cap T', S' \cup T) + f(S \cup T', S' \cap T) \geq f(S, S') + f(T, T').$$

For any set-pair (S, S'), let $E(S, S') = \{e = (u, v) \in E \mid u \in S,\ v \in S'\}$ denote the edges with one end-point in S and the other in S'. For any assignment $x : E \to \mathbb{R}_+$ and set-pair (S, S'), we abbreviate $x(E(S, S'))$ by just $x(S, S')$. The LP-relaxation for SNDP_{elt} considered in [5] is the following.

$$(\mathsf{LP}_{\text{elt}}) \qquad \text{minimize} \quad \sum_{e \in E} c_e x_e$$
$$\text{subject to} \quad x(S, S') \geq f(S, S') \qquad \forall\, (S, S') \in \mathcal{F}$$
$$0 \leq x_e \leq 1 \qquad \forall\, e \in E,$$

where $\mathcal{F} = \left\{(S, S') \mid S \bigcap S' = \emptyset,\ U \subseteq S \bigcup S'\right\}$, and

$$f(S, S') = \max\{r_{uv} \mid u \in S \cap U,\ v \in S' \cap U\} - |V - S - S'| \quad \text{for any } (S, S') \in \mathcal{F}.$$

Note that f is a weakly two-supermodular function on set-pairs $\mathcal{F}$. We will prove the following that immediately implies Theorem 1.3.

Theorem 3.1. *Let x be a basic feasible solution to $(\mathsf{LP}_{\text{elt}})$, where $f : \mathcal{F} \to \mathbb{Z}_+$ is weakly two-supermodular; then there exists an $e \in E$ such that $x_e \geq \frac{1}{2}$.*

As mentioned earlier, this theorem was proved earlier in Fleischer et al. [5], and is the main ingredient in the 2-approximation algorithm for the element-connectivity SNDP.

We first introduce some definitions from [5] that are required for the proof. Define a *partial order* on set-pairs where $(S, S') \leq (T, T')$ iff $S \subseteq T$ and $T' \subseteq S'$; in this case we say that (S, S') is *smaller* than (T, T'). We also say that (S, S') and (T, T') are comparable if either $(S, S') \leq (T, T')$ or $(T, T') \leq (S, S')$; otherwise they are incomparable.

Set-pairs (S, S') and (T, T') are said to *pair-cross* iff *none* of the following holds.

C1. $S \subseteq T$ and $T' \subseteq S'$; i.e., $(S, S') \leq (T, T')$.
C2. $S' \subseteq T'$ and $T \subseteq S$; i.e., $(S, S') \geq (T, T')$.
C3. $S \subseteq T'$ and $T \subseteq S'$.

A collection of set-pairs is *pair-laminar* if no two of them pair-cross. The following result appears as Corollary 4.6 and Lemma 4.7 in Fleischer et al. [5].

Lemma 3.2 ([5]). *Let x be a basic feasible solution to $(\mathsf{LP}_{\text{elt}})$, such that $0 < x_e < 1$ for all edges $e \in E$. Then, there exists a pair-laminar family $\mathcal{L}$ of set-pairs such that*

1. *x is the unique solution to $\{x(S, S') = f(S, S'),\ \forall (S, S') \in \mathcal{L}\}$;*
2. *the vectors $\chi_{E(S,S')}$ for $(S, S') \in \mathcal{L}$ are linearly independent;*
3. *the poset induced by $\leq$ on $\mathcal{L}$ is a forest; i.e., for any (X, X'), (Y, Y'), $(Z, Z') \in \mathcal{L}$ with $(X, X') \leq (Y, Y')$ and $(X, X') \leq (Z, Z')$, the set-pairs (Y, Y') and (Z, Z') are comparable; and*
4. *$|E| = |\mathcal{L}|$.*

We also refer to set-pairs in $\mathcal{L}$ as nodes. For any node $(S, S') \in \mathcal{L}$, its parent is the smallest node $(T, T') \in \mathcal{L} \setminus \{(S, S')\}$ that is larger than (S, S') (i.e., satisfying $(T, T') \geq (S, S')$); in this case (S, S') is called a child of (T, T'). If there is no node in $\mathcal{L} \setminus \{(S, S')\}$ that is larger than (S, S'), then node (S, S') is called a maximal node. Node $(R, R') \in \mathcal{L}$ is a descendent of $(S, S') \in \mathcal{L}$ iff $(R, R') \leq (S, S')$.

Proof of Theorem 3.1. Suppose for a contradiction that the claim does not hold, and let x be an extreme point solution with $x_e < \frac{1}{2}$ for all $e \in E$. If $x_e = 0$ for some $e \in E$, we can remove edge e from the graph G and variable x_e from $(\mathsf{LP}_{\text{elt}})$. The residual solution x remains a basic feasible solution to the modified $(\mathsf{LP}_{\text{elt}})$. Thus we assume without loss of generality that $x_e > 0$ for all $e \in E$, and so Lemma 3.2 applies. We will derive a contradiction using a counting argument similar to the one in the previous section. Each edge $e = (i, j) \in E$ is assigned one unit of token, which it distributes to nodes in $\mathcal{L}$ as follows.

1. *Rule* I: assign x_e tokens to the smallest node $(S, S') \in \mathcal{L}$ such that either $i \in S$ or $\{i, j\} \cap S' = \emptyset$.
2. *Rule* II: assign x_e tokens to the smallest node $(T, T') \in \mathcal{L}$ such that either $j \in T$ or $\{i, j\} \cap T' = \emptyset$.
3. *Rule* III: assign $1 - 2\, x_e$ tokens to the smallest node $(R, R') \in \mathcal{L}$ such that $\{i, j\} \cap R' = \emptyset$.

Note that both x_e and $1 - 2x_e$ are strictly positive for any edge e. Additionally, by Lemma 4.8 in Fleischer et al. [5], it follows that each of *Rules* I, II and III assigns tokens to at most one node. Hence each edge in E distributes a total of at most one token.

We now show that each node of $\mathcal{L}$ receives a total of at least one token. Consider any node $(S, S') \in \mathcal{L}$ with children $\{(R_i, R_i')\}_{i=1}^k$; if (S, S') is a leaf then $k = 0$. For each $i \in [k]$, we have (A) $R_i' \supseteq S'$, since $(S, S') \geq (R_i, R_i')$; and (B) $R_i' \supseteq R_j$ for all $j \in [k] \setminus \{i\}$, since (R_i, R_i') and (R_j, R_j') are incomparable, and they satisfy condition (C3). Additionally, the $\{R_i\}_{i=1}^k$ are disjoint subsets of S. Define the following edge-sets:

$$H = \bigcup_{i=1}^{k} E(R_i, R_i' \setminus S')$$
$$C = \{e \in H : |e \cap (\cup_i R_i)| = 2\}$$
$$B = \{e \in H : |e \cap (\cup_i R_i)| = 1\}$$
$$D = \bigcup_{i=1}^{k} E(R_i, S')$$
$$A = E\left(S \setminus (\cup_i R_i), S'\right).$$

Thus we can write $\sum_{i=1}^k x(E(R_i, R_i')) = 2 \cdot x(C) + x(B) + x(D)$, and $x(E(S, S')) = x(D) + x(A)$. Recall that the tight LP constraints imply that

$$x(E(S, S')) = f(S, S') \quad \text{and} \quad x(E(R_i, R_i')) = f(R_i, R_i') \quad \forall\, 1 \leq i \leq k.$$

Subtracting, we obtain (since the f-values are all integral)

$$x(E(S, S')) - \sum_{i=1}^{k} x(E(R_i, R_i')) = f(S, S') - \sum_{i=1}^{k} f(R_i, R_i') \in \mathbb{Z}$$
$$\Rightarrow x(A) - x(B) - 2x(C) \in \mathbb{Z}.$$

Adding $|B| + |C|$ (an integer) to the above expression, we obtain

$$\sum_{e \in A} x_e + \sum_{e \in B} (1 - x_e) + \sum_{e \in C} (1 - 2\, x_e) \in \mathbb{Z}.$$

Note that $A \cup B \cup C \neq \emptyset$; otherwise, $\chi(E(S, S')) = \sum_{i=1}^k \chi(E(R_i, R_i'))$, contradicting the linear independence in Lemma 3.2. Since $0 < x_e < \frac{1}{2}$ for all $e \in E$, the left-hand side above is strictly positive, and

$$\sum_{e \in A} x_e + \sum_{e \in B} (1 - x_e) + \sum_{e \in C} (1 - 2\, x_e) \geq 1. \tag{2}$$

We now show that the tokens assigned to (S, S') total to at least the left-hand side in Inequality (2).

- *Edge* $e = (u, v) \in A$. Let $u \in S \setminus (\cup_i R_i)$ and $v \in S'$. We claim that the token assigned by *Rule* I goes to (S, S'). Clearly, (S, S') is the smallest set-pair with $u \in S$. For any descendant (T, T') of (S, S'), we must have $T' \supseteq S' \ni v$; thus we cannot have $u, v \notin T'$. Hence (S, S') receives x_e tokens from e.
- *Edge* $e = (u, v) \in C$. Let $u \in R_i$ and $v \in R_j$ for $i, j \in [k], i \neq j$. We claim that the token assigned by *Rule* III goes to (S, S'). Clearly $u, v \notin S'$. Furthermore, for any child (R_ℓ, R'_ℓ) of (S, S') we have $R'_\ell \supseteq R_i \ni u$ or $R'_\ell \supseteq R_j \ni v$. Hence (S, S') receives $1 - 2x_e$ tokens from e.
- *Edge* $e = (u, v) \in B$. Let $u \in R_i$ and $v \in R'_i \setminus S'$ for some $i \in [k]$. We first claim that the token assigned by *Rule* III goes to (S, S'). Clearly $u, v \notin S'$. We show that $\{u, v\} \cap R'_\ell \neq \emptyset$ for every child (R_ℓ, R'_ℓ) of (S, S').
  1. Suppose $\ell = i$; then $v \in R'_i$.
  2. Suppose $\ell \in [k] \setminus \{i\}$; then $u \in R_i \subseteq R'_\ell$.

  That is, (S, S') receives the token by *Rule* III. We next claim that the token assigned by *Rule* II also goes to (S, S'). Note that $v \notin \cup_i R_i$, so no descendant (T, T') of (S, S') can have $v \in T$. As seen above, (S, S') is the smallest node with $u, v \notin S'$; i.e., (S, S') receives the token by *Rule* II. Hence (S, S') receives in total $1 - x_e$ tokens from e.

Thus each node of $\mathcal{L}$ receives at least a unit token.

We now show that there is some positive amount of unused tokens. Let $(P, P') \in \mathcal{L}$ be any maximal node in $\mathcal{L}$. Note that there is at least one maximal node $(P, P') \in \mathcal{L}$ and $E(P, P') \neq \emptyset$. We claim that the token of any edge $(u, v) \in E(P, P')$ given by *Rule* III is unused. Let $u \in P$ and $v \in P'$. For any descendent (T, T') of (P, P'), we have $T' \supseteq P' \ni v$; so $T' \bigcap \{u, v\} \neq \emptyset$. Any node $(Q, Q') \in \mathcal{L}$ that is not a descendent of (P, P') is incomparable to (P, P'), and we have $Q' \supseteq P \ni u$. Thus $\{u, v\} \cap S' \neq \emptyset$ for all $(S, S') \in \mathcal{L}$, i.e., the *Rule* III token of edge (u, v) is unassigned. Thus there is a positive amount of unused tokens. However, this implies that $|E| > |\mathcal{L}|$, which contradicts Lemma 3.2.

This completes the proof of Theorem 3.1. □

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References

[1] David L. Applegate, Robert E. Bixby, Vasek Chvátal, William J. Cook, The Traveling Salesman Problem: A Computational Study, Princeton University Press, 2006.
[2] N. Bansal, R. Khandekar, V. Nagarajan, Additive guarantees for degree-bounded directed network design, SIAM J. Comput. 39 (4) (2009) 1413–1431.
[3] S.C. Boyd, W.R. Pulleyblank, Optimizing over the subtour polytope of the travelling salesman problem, Math. Program. 49 (2) (1990) 163–187.
[4] J. Chuzhoy, S. Khanna, An $O(k^3 \log n)$-approximation algorithm for vertex-connectivity survivable network design, in: Proceedings of FOCS, 2009.
[5] L. Fleischer, K. Jain, D.P. Williamson, Iterative rounding 2-approximation algorithms for minimum-cost vertex connectivity problems, J. Comput. System Sci. 72 (5) (2006) 838–867.
[6] H.N. Gabow, M.X. Goemans, E. Tardos, D.P. Williamson, Approximating the smallest k-edge connected spanning subgraph by LP-rounding, Networks 53 (4) (2009) 345–357.
[7] M. Goemans, Worst-case comparison of valid inequalities for the TSP, Math. Program. 69 (1–3) (2005) 335–349.
[8] M. Grötschel, C.L. Monma, M. Stoer, Design of Survivable Networks, in: M.O. Ball, et al. (Eds.), Handbooks in OR & MS, vol. 7, 1995, pp. 617–671.
[9] K. Jain, A factor 2-approximation algorithm for the generalized Steiner network problem, Combinatorica 21 (1) (2001) 39–60.