1. Problem:
   (a) Perform long division on the rational function
   \[
   \frac{x^4 - 2x^2 + 2}{(x - 1)(x^2 + 1)}.
   \]
   (b) Find the partial fraction decomposition of your result from (a).

Solution:
(a) Write \((x - 1)(x^2 + 1) = x^3 - x^2 + x - 1\). Long division: (please excuse the bad formatting)

\[
\begin{array}{c|cc|cc|cc|cc|}
\hline
x & +1 & & & & & & \\
\hline
x^4 & -2x^2 & +2 & & & & & \\
\hline
-x^3 & +x^2 & -x & & & & & \\
\hline
\end{array}
\]

so we have
\[
\frac{x^4 - 2x^2 + 2}{(x - 1)(x^2 + 1)} = x + 1 + \frac{-2x^2 + 3}{(x - 1)(x^2 + 1)}.
\]

(b) We aim to decompose
\[
\frac{-2x^2 + 3}{(x - 1)(x^2 + 1)}.
\]

The denominator cannot be factored further: \(x - 1\) and \(x^2 + 1\) are irreducible polynomials.

Set
\[
\frac{-2x^2 + 3}{(x - 1)(x^2 + 1)} = \frac{A}{x - 1} + \frac{Bx + C}{x^2 + 1}.
\]

Multiply this equation by \((x - 1)(x^2 + 1)\) to get
\[-2x^2 + 3 = A(x^2 + 1) + (Bx + C)(x - 1)
\]
or
\[-2x^2 + 3 = (A + B)x^2 + (-B + C)x + (A - C).
\]

By identifying coefficients we get
\[
\begin{align*}
x^2 & : & A + B &= -2 \\
x & : & -B + C &= 0 \\
1 & : & A &- C &= 3
\end{align*}
\]

which has the unique solution \(A = \frac{1}{2}, B = C = -\frac{5}{2}\). So
\[
\frac{-2x^2 + 3}{(x - 1)(x^2 + 1)} = \frac{1}{2(x - 1)} + \frac{-5x - 5}{2(x^2 + 1)}.
\]
2. **Problem:** Evaluate the indefinite integral

\[ \int \frac{x^3}{x^3 - 3x^2 - 6x + 8} \, dx. \]

**Solution:** The degree of the numerator is not smaller than the degree of the denominator, so we need to perform long division. Instead of going through long division we notice that the numerator is simple enough for a shortcut:

\[
\frac{x^3}{x^3 - 3x^2 - 6x + 8} = \frac{x^3 - 3x^2 - 6x + 8 + 3x^2 + 6x - 8}{x^3 - 3x^2 - 6x + 8} = 1 + \frac{3x^2 + 6x - 8}{x^3 - 3x^2 - 6x + 8}
\]

(This method of adding and subtracting is fast and simple in some cases, but long division is more fool-proof.)

The next step is to find the partial fraction decomposition for

\[
\frac{3x^2 + 6x - 8}{x^3 - 3x^2 - 6x + 8}.
\]

In order to do this we need to factor \(x^3 - 3x^2 - 6x + 8\). By plugging in some simple values (e.g. \(x = 0, 1, -1, 2, -2, 1/2, -1/2, 3, \ldots\)) we quickly find that \(x = 1\) is a root of this polynomial. Perform long division to get

\[
\frac{x^3 - 3x^2 - 6x + 8}{x - 1} = x^2 - 2x - 8.
\]

The two roots of \(x^2 - 2x - 8\) are \(x = -2\) and \(x = 4\), so we have

\[
x^3 - 3x^2 - 6x + 8 = (x - 1)(x^2 - 2x - 8) = (x - 1)(x + 2)(x - 4).
\]

So we introduce constants \(A, B, C\) and set

\[
\frac{3x^2 + 6x - 8}{x^3 - 3x^2 - 6x + 8} = \frac{3x^2 + 6x - 8}{(x - 1)(x + 2)(x - 4)} = \frac{A}{x - 1} + \frac{B}{x + 2} + \frac{C}{x - 4}.
\]

Multiply the equation by \((x - 1)(x + 2)(x - 4)\) to get

\[
3x^2 + 6x - 8 = A(x + 2)(x - 4) + B(x - 1)(x - 4) + C(x - 1)(x + 2)
\]

or

\[
3x^2 + 6x - 8 = (A + B + C)x^2 + (-2A - 5B + C)x + (-8A + 4B - 2C)
\]

so we have

\[
x^2:\quad A + B + C = 3
\]

\[
x:\quad -2A - 5B + C = 6
\]

\[
1:\quad -8A + 4B - 2C = -8
\]

which has solution

\[
A = -\frac{1}{9}, \quad B = -\frac{4}{9}, \quad C = \frac{32}{9}
\]

so

\[
\frac{x^3}{x^3 - 3x^2 - 6x + 8} = 1 + \frac{3x^2 + 6x - 8}{x^3 - 3x^2 - 6x + 8} = 1 - \frac{1}{9(x - 1)} - \frac{4}{9(x + 2)} + \frac{32}{9(x - 4)}
\]

and

\[
\int \frac{x^3}{x^3 - 3x^2 - 6x + 8} \, dx = x - \frac{1}{9} \ln |x - 1| - \frac{4}{9} \ln |x + 2| + \frac{32}{9} \ln |x - 4| + D
\]

where \(D\) is an arbitrary constant.
3. **Problem:** Evaluate the indefinite integral

\[ \int \frac{x - 2}{(x^2 - 1)(x - 1)} \, dx. \]

**Solution:** The degree of the numerator is smaller than the degree of the denominator, so no long division is required. We can factor the denominator by

\[(x^2 - 1)(x - 1) = (x + 1)(x - 1)(x - 1) = (x + 1)(x - 1)^2\]

and we find the partial fraction decomposition of the integrand:

\[ \frac{x - 2}{(x + 1)(x - 1)^2} = \frac{A}{x + 1} + \frac{B}{x - 1} + \frac{C}{(x - 1)^2}. \]

Multiply by \((x + 1)(x - 1)^2\):

\[ x - 2 = A(x - 1)^2 + B(x + 1)(x - 1) + C(x + 1) \]
\[ = (A + B)x^2 + (-2A + C)x + (A - B + C) \]

We have

\[
\begin{align*}
x^2 : & \quad A + B = 0 \\
x : & \quad -2A + C = 1 \\
1 : & \quad A - B + C = -2
\end{align*}
\]

with solution

\[ A = -\frac{3}{4}, \quad B = \frac{3}{4}, \quad C = -\frac{1}{2}. \]

So

\[
\int \frac{x - 2}{(x^2 - 1)(x - 1)} \, dx = \int \frac{-\frac{3}{4}(x - 1)^2 + \frac{3}{4}(x + 1)}{(x + 1)(x - 1)^2} \, dx \]
\[ = -\frac{3}{4}\ln|x + 1| + \frac{3}{4}\ln|x - 1| + \frac{1}{2(x - 1)^2} + C \]
\[ = \frac{3}{4}\ln\left|\frac{x - 1}{x + 1}\right| + \frac{1}{2(x - 1)} + C. \]

4. **Problem:** Evaluate the indefinite integral

\[ \int \frac{2x^2}{(x + 2)(x^2 - 2x + 2)} \, dx. \]

**Solution:** No long division is required. The factor \(x^2 - 2x + 2 = (x - 1)^2 + 1\) has no real zeros, so it is irreducible. Set

\[ \frac{2x^2}{(x + 2)(x^2 - 2x + 2)} = \frac{A}{x + 2} + \frac{Bx + C}{x^2 - 2x + 2}. \]

Multiply by \((x + 2)(x^2 - 2x + 2)\) to get

\[ 2x^2 = A(x^2 - 2x + 2) + (Bx + C)(x + 2) = (A + B)x^2 + (-2A + 2B + C)x + (2A + 2C) \]
and solve

\[
x^2: \quad A + B = 2
\]
\[
x: \quad -2A + 2B + C = 0
\]
\[
1: \quad 2A + 2C = 0
\]
to get

\[
A = \frac{4}{5}, \quad B = \frac{6}{5}, \quad C = -\frac{4}{5}.
\]

So

\[
\int \frac{2x^2}{(x + 2)(x^2 - 2x + 2)} \, dx = \int \frac{4}{5(x + 2)} + \frac{6x - 4}{5(x^2 - 2x + 2)} \, dx
\]
\[
= \int \frac{4}{5(x + 2)} \, dx + \int \frac{6x - 4}{5(x^2 - 2x + 2)} \, dx
\]

We have

\[
\int \frac{4}{5(x + 2)} \, dx = \frac{4}{5} \ln |x + 2| + C.
\]

For the second integral note that \( x^2 - 2x + 2 = (x - 1)^2 + 1 \), so we make the substitution \( u = x - 1 \) to get

\[
\int \frac{6x - 4}{5(x^2 - 2x + 2)} \, dx = \int \frac{6(u + 1) - 4}{5(u^2 + 1)} \, du = \int \frac{6u + 2}{5(u^2 + 1)} \, du
\]
and since

\[
\int \frac{u}{u^2 + 1} \, du = \frac{1}{2} \ln(u^2 + 1) + C, \quad \text{and} \quad \int \frac{1}{u^2 + 1} \, du = \arctan u + C
\]
we have

\[
\int \frac{6u + 2}{5(u^2 + 1)} \, du = \frac{3}{5} \ln(u^2 + 1) + \frac{2}{5} \arctan u + C
\]
\[
= \frac{3}{5} \ln(x^2 - 2x + 2) + \frac{2}{5} \arctan(x - 1) + C.
\]

Putting it all together;

\[
\int \frac{2x^2}{(x + 2)(x^2 - 2x + 2)} = \frac{4}{5} \ln |x + 2| + \frac{3}{5} \ln(x^2 - 2x + 2) + \frac{2}{5} \arctan(x - 1) + C.
\]

5. **Problem:** Let

\[
I = \int_0^4 e^{-x^2} \, dx.
\]

(a) Approximate \( I \) using the midpoint rule with \( n = 5 \).

(b) Approximate \( I \) using the trapezoidal rule with \( n = 8 \).

(c) Bound \( |E_T| \) from (b).

**Solution:**
(a) Let \( a = 0, b = 4 \) and \( f(x) = e^{-x^2} \). Divide the interval \([0, 4]\) into 5 intervals of length 
\[ \Delta x = \frac{4 - 0}{4} = 0.8 \] by setting \( x_0 = 0, x_1 = 0.8, x_2 = 1.6, x_3 = 2.4, x_4 = 3.2, x_5 = 4 \). The midpoints of the five intervals are \( \bar{x}_1 = 0.4, \bar{x}_2 = 1.2, \bar{x}_3 = 2, \bar{x}_4 = 2.8, \bar{x}_5 = 3.6 \). So
\[
M_5 = \Delta x(f(\bar{x}_1) + f(\bar{x}_2) + f(\bar{x}_3) + f(\bar{x}_4) + f(\bar{x}_5))
= 0.8(f(0.4) + f(1.2) + f(2) + f(2.8) + f(3.6))
\approx 0.8(0.852144 + 0.236927 + 0.018315 + 0.000393 + 0.000002)
= 0.88622.
\]

(b) Again let \( a = 0, b = 4 \) and \( f(x) = e^{-x^2} \). We have \( n = 8 \) so \( \Delta x = 4/8 = 0.5 \) and
\[ x_0 = 0, x_1 = 0.5, x_2 = 1, x_3 = 1.5, x_4 = 2, x_5 = 2.5, x_6 = 3, x_7 = 3.5, x_8 = 4. \) So
\[
T_8 = \frac{0.5}{2} (f(0) + 2f(0.5) + 2f(1) + 2f(1.5) + \cdots + 2f(3.5) + f(4))
\approx 0.88623
\]

6. **Problem:** Suppose you want to use Simpson’s rule to approximate 
\[
\int_1^3 e^{3x} + 3x^2 - 4x + 7 \, dx.
\]
How large does \( n \) need to be to guarantee that your answer is correct to ten decimal places?

**Solution:** Let \( f(x) = e^{3x} + 3x^2 - 4x + 7 \) and \( a = 1, b = 3 \). The error \( E_S \) for Simpson’s rule is bounded by
\[
|E_S| \leq \frac{K(b-a)^5}{180n^4}
\]
where \( K > 0 \) is such that \( |f^{(4)}(x)| \leq K \) for all \( 1 \leq x \leq 3 \). We have
\[
f'(x) = 3e^{3x} + 6x - 4, \quad f''(x) = 9e^{3x} + 6, \quad f^{(3)}(x) = 27e^{3x}, \quad f^{(4)}(x) = 81e^{3x}
\]
and we have \( 81e^{3x} \leq 81e^9 \) for \( 1 \leq x \leq 3 \) so set \( K = 81e^9 \). We require that
\[
\frac{81e^9(3-1)^5}{180n^4} \leq 0.001
\]
or
\[
n^4 \geq \frac{1000 \cdot 81 \cdot e^9 \cdot 2^5}{180}
\]
so we require
\[
n \geq \left( \frac{1000 \cdot 81 \cdot e^9 \cdot 2^5}{180} \right)^{1/4} \approx 103.9.
\]