1. **Problem:** Evaluate the indefinite integral

\[ \int \sin^3 x \cos^2 x \, dx \]

**Solution:** Use the identity \( \sin^2 x + \cos^2 x = 1 \) to rewrite the integral as

\[ \int \sin^3 x \cos^2 x \, dx = \int \sin x(1 - \cos^2 x) \cos^2 x \, dx. \]

We substitute \( u = \cos x \). Then \( du = -\sin x \, dx \), so

\[ \int \sin x(1 - \cos^2 x) \cos^2 x \, dx = -\int (1 - u^2)u^2 \, du = \int u^4 - u^2 \, du \]

\[ = \frac{1}{5} u^5 - \frac{1}{3} u^3 + \frac{1}{5} \cos^5 x - \frac{1}{3} \cos^3 x + C. \]

2. **Problem:** Evaluate the indefinite integral

\[ \int \cos^4 t \, dt. \]

**Solution:** Use the identity \( \cos^2 t = \frac{1 + \cos 2t}{2} \) to get

\[ \int \cos^4 t \, dt = \int \left( \frac{1 + \cos 2t}{2} \right)^2 dt = \int \frac{1}{4} + \frac{1}{2} \cos 2t + \frac{1}{4} \cos^2(2t) dt. \]

We need to apply the same identity again: \( \cos^2(2t) = \frac{1 + \cos 4t}{2} \), to get

\[ \int \frac{1}{4} + \frac{1}{2} \cos 2t + \frac{1}{8} + \frac{1}{8} \cos 4t \, dt \]

\[ = \int \frac{3}{8} + \frac{1}{2} \cos 2t + \frac{1}{8} \cos 4t \, dt \]

\[ = \frac{3}{8} t + \frac{1}{4} \sin 2t + \frac{1}{32} \sin 4t + C. \]

3. **Problem:** Evaluate the definite integral

\[ \int_0^{\pi/3} (2 - \tan x) \sec^4 x \, dx \]

**Solution:** Use the identity \( \sec^2 x = 1 + \tan^2 x \) to get

\[ \int_0^{\pi/3} (2 - \tan x) \sec^4 x \, dx = \int_0^{\pi/3} (2 - \tan x)(1 + \tan^2 x) \sec^2 x \, dx. \]

1
Now set \( u = \tan x \). We get \( du = \sec^2 x \, dx \). When \( x = 0 \) we have \( u = \tan 0 = 0 \) and when \( x = \pi/3 \) we have \( u = \tan(\pi/3) = \sqrt{3} \), so

\[
\int_0^{\pi/3} (2 - \tan x)(1 + \tan^2 x) \sec^2 x \, dx = \int_0^{\sqrt{3}} (2 - u)(1 + u^2) \, du
\]
\[
= \int_0^{\sqrt{3}} 2 - u + 2u^2 - u^3 \, du
\]
\[
= [2u - \frac{1}{2}u^2 + \frac{2}{3}u^3 - \frac{1}{4}u^4]_{u=0}^{\sqrt{3}}
\]
\[
= 2\sqrt{3} - \frac{3}{2} + 2\sqrt{3} - \frac{9}{4} = 4\sqrt{3} - \frac{15}{4}
\]

4. **Problem:** Evaluate the indefinite integral

\[
\int \sin(x) \cos(x) \cos(7x) \, dx
\]

**Solution:** Note that \( \sin x \cos x = \frac{1}{2} \sin 2x \), so

\[
\int \sin(x) \cos(x) \cos(7x) \, dx = \frac{1}{2} \int \sin(2x) \cos(7x) \, dx.
\]

Now we use the formula (page 476)

\[
\sin A \cos B = \frac{1}{2} [\sin(A - B) + \sin(A + B)]
\]

with \( A = 2x \) and \( B = 7x \) to get

\[
\sin 2x \cos 7x = \frac{1}{2} [\sin(-5x) + \sin(9x)] = \frac{1}{2} [\sin 9x - \sin 5x]
\]

so

\[
\int \sin(x) \cos(x) \cos(7x) \, dx = \frac{1}{2} \int \sin(2x) \cos(7x) \, dx
\]
\[
= \frac{1}{4} \int \sin 9x - \sin 5x \, dx
\]
\[
= \frac{1}{4} \left( -\frac{1}{9} \cos 9x + \frac{1}{5} \cos 5x \right) + C.
\]

5. **Problem:** Evaluate the definite integral

\[
\int_0^{5/\sqrt{2}} \frac{x^2}{\sqrt{1 - \frac{x^2}{25}}} \, dx
\]
Solution: Set \( x = 5 \sin \theta \). Then \( dx = 5 \cos \theta \ d\theta \). When \( x = 0 \) we have \( \theta = 0 \) and when \( x = 5/\sqrt{2} \) then \( \theta = \pi/4 \). So

\[
\int_{0}^{5/\sqrt{2}} \frac{x^2}{\sqrt{1 - x^2}} \, dx = \int_{0}^{\pi/4} \frac{(5 \sin \theta)^2}{\sqrt{1 - (5 \sin \theta)^2}} \cdot 5 \cos \theta \ d\theta
\]

\[
= 125 \int_{0}^{\pi/4} \frac{\sin^2 \theta \cos \theta}{\sqrt{1 - \sin^2 \theta}} \, d\theta
\]

\[
= 125 \int_{0}^{\pi/4} \sin^2 \theta \ d\theta
\]

\[
= 125 \frac{\pi}{8} - \frac{125}{4}
\]

6. Problem: Evaluate the indefinite integral

\[
\int \frac{1}{(4 + 9x^2)^{3/2}} \, dx.
\]

Solution: First write \( 4 + 9x^2 = 9 \left( \frac{4}{9} + x^2 \right) \) in order to get an expression of the form \( a^2 + x^2 \):

\[
\int \frac{1}{(4 + 9x^2)^{3/2}} \, dx = \int \frac{1}{9^{3/2} \left( \frac{4}{9} + x^2 \right)^{3/2}} \, dx = \frac{1}{27} \int \frac{1}{\left( \frac{4}{9} + x^2 \right)^{3/2}} \, dx.
\]

Here \( \frac{4}{9} + x^2 \) has the form \( a^2 + x^2 \) with \( a = 2/3 \). Set \( x = \frac{2}{3} \tan \theta \). Then \( dx = \frac{2}{3} \sec^2 \theta \ d\theta \) and we have

\[
\frac{1}{27} \int \frac{1}{\left( \frac{4}{9} + x^2 \right)^{3/2}} \, dx = \frac{1}{27} \int \frac{1}{\left( \frac{4}{9} + \frac{4}{9} \tan^2 \theta \right)^{3/2}} \cdot \frac{2}{3} \sec^2 \theta \ d\theta
\]

\[
= \frac{2}{81} \cdot \frac{1}{\left( \frac{4}{9} \right)^{3/2}} \int \sec^2 \theta \left( 1 + \tan^2 \theta \right)^{3/2} \, d\theta
\]

\[
= \frac{1}{12} \int \sec^2 \theta \sec^3 \theta \, d\theta.
\]

Here we used the identity \( 1 + \tan^2 \theta = \sec^2 \theta \) to conclude that the denominator is \( (\sec^2 \theta)^{3/2} = \sec^3 \theta \). Continuing, we have

\[
\frac{1}{12} \int \frac{1}{\sec \theta} \, d\theta = \frac{1}{12} \int \cos \theta \ d\theta = \frac{1}{12} \sin \theta + C.
\]
Since the integral is indefinite, we need to switch back to the original variable $x$. We have

$$x = \frac{2}{3} \tan \theta = \frac{2}{3} \frac{\sin \theta}{\cos \theta} = \frac{2}{3} \frac{\sin \theta}{\sqrt{1 - \sin^2 \theta}}$$

and we solve this equation for $\sin \theta$.

$$3x \sqrt{1 - \sin^2 \theta} = 2 \sin \theta$$
$$9x^2(1 - \sin^2 \theta) = 4 \sin^2 \theta$$
$$9x^2 = (4 + 9x^2) \sin^2 \theta$$
$$\sin \theta = \frac{3x}{\sqrt{4 + 9x^2}}$$

and we conclude that

$$\int \frac{1}{(4 + 9x^2)^{3/2}} dx = \frac{1}{12} \sin \theta + C = \frac{x}{4\sqrt{4 + 9x^2}} + C.$$