21-122 Homework # 1— Solutions

1. **Problem:** Evaluate the indefinite integral

\[ \int (2 + 4x)^{3/2} dx. \]

**Solution:** Use the substitution \( u = 2 + 4x \). We have \( \frac{du}{dx} = 4 \) so \( dx = \frac{1}{4} du \). So

\[ \int (2 + 4x)^{3/2} dx = \int u^{3/2} \frac{du}{4} = \frac{1}{4} \int u^{3/2} du = \frac{1}{4} \cdot \frac{2}{5} u^{5/2} + C = \frac{1}{10} (2 + 4x)^{5/2} + C, \]

where \( C \) is an arbitrary constant.

2. **Problem:** Evaluate the definite integral

\[ \int_{0}^{\pi/4} \frac{\sin x}{\cos^2 x} dx. \]

**Solution:** Use the substitution \( u = \cos x \). Then \( \frac{du}{dx} = -\sin x \) so \( \sin(x)dx = -du \). Now we apply the substitution rule for definite integrals (page 411 in Stewart). The lower limit \( x = 0 \) corresponds to \( u = \cos(0) = 1 \) and the upper limit \( x = \pi/4 \) corresponds to \( u = \cos(\pi/4) = 1/\sqrt{2} \), so

\[ \int_{0}^{\pi/4} \frac{\sin x}{\cos^2 x} dx = \int_{1}^{1/\sqrt{2}} \frac{1}{u^2} (-du) = - \int_{1}^{1/\sqrt{2}} \frac{1}{u^2} du = \int_{1/\sqrt{2}}^{1} \frac{1}{u^2} du \]

\[ = \left[ -\frac{1}{u} \right]_{u=1/\sqrt{2}}^{u=1} = -\frac{1}{1} + \sqrt{2} = \sqrt{2} - 1. \]

Note that we do not have to change back to \( x \) when using the substitution rule. It is still possible to evaluate this integral without using the substitution rule, but that is not always the case.

3. **Problem:** Evaluate the definite integral

\[ \int_{-2}^{2} x^6 \frac{\sin^5 x}{\cos^4 x} dx. \]

**Remark on solution:** As mentioned in class on Friday 5/20, this integral is divergent\(^1\). The function has asymptotes at \( x = \pm \frac{\pi}{2} \). I didn’t realize this when I typed up the homework. The “spirit” of the problem is to notice that the function is odd and quickly deduce that the integral is zero. The grading does not require you to check for asymptotes, only to apply symmetry.

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\(^1\)The reason that it diverges is slightly too difficult to go through at this point in the course. I might come back to it later.
Solution using symmetry, ignoring asymptotes: See page 412 in Stewart. Let \( f(x) = \frac{x^6 \sin^5 x}{\cos^4 x} \). Then
\[
f(-x) = (-x)^6 \frac{(\sin(-x))^5}{(\cos(-x))^4} = (-x)^6 \frac{(-\sin x)^5}{(-\cos x)^4} = -x^6 \frac{\sin^5 x}{\cos^4 x} = -f(x),
\]
so \( f \) is odd and we have
\[
\int_{-2}^{2} x^6 \frac{\sin^5 x}{\cos^4 x} \, dx = 0.
\]

4. Problem: Evaluate the definite integral
\[
\int_{0}^{2} x e^{2x} \, dx.
\]

Solution: Use integration by parts. The integral becomes easier if we differentiate \( x \) and integrate \( e^{2x} \), so we let
\[
u = x, \quad dv = e^{2x} \, dx,
\]
\[
 du = dx, \quad v = \frac{1}{2} e^{2x}.
\]
Then
\[
\int (x)(e^{2x} \, dx) = x \frac{1}{2} e^{2x} - \int \frac{1}{2} e^{2x} \, dx
\]
\[
= x e^{2x} - \frac{1}{4} e^{2x}.
\]
and so
\[
\int_{0}^{2} x e^{2x} \, dx = \left[ x e^{2x} - \frac{1}{4} e^{2x} \right]_{x=0}^{2} = e^4 - \frac{1}{4} e^4 - 0 + \frac{1}{4} = \frac{3}{4} e^4 + \frac{1}{4}.
\]

5. Problem: Evaluate the indefinite integral
\[
\int e^{2x} \sin x \, dx.
\]

Solution: Here we use a standard trick – see Example 4 on page 466 of Stewart. Let
\[
u = e^{2x}, \quad dv = \sin x \, dx,
\]
\[
 du = 2 e^{2x} \, dx, \quad v = -\cos x.
\]
Let \( I(x) \) denote the integral we are evaluating. Then
\[
I(x) = \int e^{2x} \sin x \, dx = -e^{2x} \cos x + 2 \int e^{2x} \cos x \, dx.
\]
Now set
\[
u = e^{2x}, \quad dv = \cos x \, dx,
\]
\[
 du = 2 e^{2x} \, dx, \quad v = \sin x
\]
so

\[ I(x) = -e^{2x} \cos x + 2 \int e^{2x} \cos x \, dx + C \]

\[ = -e^{2x} \cos x + 2 \left( e^{2x} \sin x - 2 \int e^{2x} \sin x \, dx \right) + C \]

\[ = -e^{2x} \cos x + 2e^{2x} \sin x - 4I(x) + C \]

and we can solve for \( I(x) \):

\[ I(x) = \frac{1}{5} e^{2x} (2 \sin x - \cos x) + C. \]

(Note that \( C \) is always an arbitrary constant – technically we should write \( C/5 \) in the last step but it is understood that \( C \) and \( C/5 \) are essentially the same thing when \( C \) is arbitrary.)