21-122 Midterm II review solutions/hints

In order to get as many questions as possible in the review, I haven’t double-checked my solutions. Don’t hesitate to email me if the solutions don’t look right. Don’t hesitate to email me with general questions, for that matter.

1. Divergent \( p \)-series.

2. Converges. Root test or ratio test, recalling that

\[
\lim_{n \to \infty} \frac{p(n+1)}{p(n)}, \quad \lim_{n \to \infty} p(n)^{1/n}
\]

for any polynomial \( p(n) \).

3. Divergent. Compare to \( 1/n \).

4. Divergent by divergence test.

5. Convergent by alternating series test.

6. \((-1, 3)\). Divergence test for the endpoints.

7. Use the table of MacLaurin series. This is \( \sin(\pi) = 0 \).

8. \[
\sum_{n=0}^{\infty} \frac{1}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} \cdot 1^n = e^1 = e.
\]

9. Two solutions. Either just plug in the series for \( \ln(1 + x) \) directly, or note that

\[
\frac{d}{dx} [(1 + x) \ln(1 + x) - x] = \ln(1 + x)
\]

so the series is given by

\[
(1 + x) \ln(1 + x) - x = \int \ln(1 + x) dx = \int \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n dx = C + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n(n+1)} x^{n+1}
\]

and plugging in \( x = 0 \) gives \( C = 0 \).

10. We have \( \frac{e^x - 1}{x} = \frac{1}{x} \sum_{n=1}^{\infty} \frac{1}{n!} x^n = \sum_{n=1}^{\infty} \frac{1}{n!} x^{n-1} \). Integrate this to get

\[
C + \sum_{n=1}^{\infty} \frac{1}{n \cdot (n!)} x^n
\]

11. We have

\[
\frac{1}{(1-x)^2} = \frac{d}{dx} \frac{1}{1-x} = \frac{d}{dx} \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} (n+1)x^n
\]

so

\[
\frac{d}{dx} \frac{1}{(1-x)^2} = \frac{d}{dx} \sum_{n=0}^{\infty} (n+1)x^n = \sum_{n=0}^{\infty} (n+1)(n+2)x^n
\]
12. Separate variables, integrate etc. You get

\[ |y| = e^{x^2/2+C}, \quad \text{or} \quad y = \pm e^{x^2/2+C} \]

and with the initial condition \( y(0) = 1 \),

\[ y = e^{x^2/2}. \]

13. Separate variables, integrate, get

\[ \int \frac{1}{\tan y} dy = \int \frac{1}{x} dx \]

which gives

\[ \ln |\sin y| = \ln |x| + C \]

or

\[ |\sin y| = |x|e^C. \]

You can stop here.

14. Euler’s method is

\[ y_{i+1} = y_i + hF(x_i, y_i) \]

where \( F(x, y) \) is the right-hand side of the DE, i.e. \( F(x, y) = x - y \). You’ll get \( y_0 = -1, y_1 = -1 + .2(0 - (-1)) = -0.8, y_2 = -0.8 + .2(.2 - (-0.8)) = -0.6 \). We stop at \( y_2 \) since \( x_2 = x_0 + 2h = 0.4 \).

15. Geometric series with \( a = 9/2, r = 1/2 \), so

\[ \sum_{n=1}^{\infty} 9 \left( \frac{1}{2} \right)^n = \sum_{n=1}^{\infty} \frac{9}{2} \left( \frac{1}{2} \right)^{n-1} = \frac{9/2}{1-1/2} = 9. \]

16. Geometric series. Write out the first few terms:

\[ \left( \frac{1}{3} \right)^{-2} + \left( \frac{1}{3} \right)^{-1} + \left( \frac{1}{3} \right)^{0} + \ldots \]

The important information here is that the first term is 9 and the common ratio is 1/3, so

\[ \sum_{n=3}^{\infty} \left( \frac{1}{3} \right)^{n-5} = \sum_{n=1}^{\infty} 9 \left( \frac{1}{3} \right)^{n-1} = \frac{9}{1-1/3} = \frac{27}{2} \]

17. Telescoping series. Note that

\[ \frac{4}{n(n+1)} = \frac{4}{n} - \frac{4}{n+1} \]

and \( 4/n \to 0 \) as \( n \to \infty \) so

\[ \sum_{n=1}^{\infty} \frac{4}{n(n+1)} = 4 \cdot \frac{1}{1} = 4. \]
18. We have
\[
\frac{2 \cos x - 2 + x^2}{x^3} = 2 \left( 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \cdots \right) - 2 + x^2 = \frac{1}{x^3} \sum_{n=2}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = \sum_{n=2}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n-3}.
\]

Plugging in \( x = 0 \), each term is zero so the limit is zero.

19. We have
\[
\frac{\ln(1 + x) - x}{e^x - 1 - x} = \frac{\left( x - \frac{x^2}{2} + \frac{x^3}{3!} - \cdots \right) - x}{\left( 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots \right) - 1 - x} = \frac{-\frac{x^2}{2} + \frac{x^3}{3!} - \cdots}{\frac{x^2}{2} + \frac{x^3}{3!} + \cdots} = -\frac{1}{2} + \frac{x}{3} + \cdots
\]

and as \( x \to 0 \), the terms that survive are \( -\frac{1}{2} = -1 \).

20. Find the pattern for \( f^{(n)}(\pi/2) \). It is 1, 0, -1, 0, 1, 0, -1, 0, 1, \ldots, so
\[
\sin x = \sum_{n=0}^{\infty} \frac{f^{(n)}(\pi/2)}{n!} \left( x - \frac{\pi}{2} \right)^n = 1 - \frac{1}{2!} \left( x - \frac{\pi}{2} \right)^2 + \frac{1}{4!} \left( x - \frac{\pi}{2} \right)^4 - \frac{1}{6!} \left( x - \frac{\pi}{2} \right)^6 + \cdots
\]

and we see that
\[
\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left( x - \frac{\pi}{2} \right)^{2n}
\]

We have
\[
R = \lim_{n \to \infty} \left| \frac{c_n}{c_{n+1}} \right|, \text{ where } c_n = \frac{(-1)^n}{(2n)!}.
\]

We get \( R = +\infty \), so the IOC is \( \mathbb{R} \).

21. Calculate
\[
f^{(0)}(x) = \frac{1}{x^2}, \quad f^{(1)}(x) = -\frac{2}{x^3}, \quad f^{(2)}(x) = \frac{6}{x^4}, \quad \ldots, \quad f^{(n)}(x) = (-1)^n \frac{(n+1)!}{x^{n+2}}
\]

so
\[
f^{(n)}(2) = (-1)^n \frac{(n+1)!}{2^{n+2}}
\]

and
\[
\frac{1}{x^2} = \sum_{n=0}^{\infty} \frac{1}{n!} (-1)^n \frac{(n+1)!}{2^{n+2}} (x - 2)^n = \sum_{n=0}^{\infty} (-1)^n \frac{n+1}{2^{n+2}} (x - 2)^n
\]

The radius of convergence is 2, and the interval of convergence is \((0, 4)\). Use the divergence test at the endpoints.

22. Use the comparison test. This is roughly \( 1/n^2 \) for large \( n \), so we expect convergence. Is it true that
\[
\frac{n+1}{n^3 + 8n} \leq \frac{1}{n^2}?
\]

Not quite, but compare to \( 100/n^2 \) to establish convergence.

23. You can use the root test or the integral test. It is convergent.
24. We have
\[ \frac{n^{2/3} + 1}{\sqrt{n}(1 + \sqrt{n})} = \frac{1 + n^{-2/3}}{n^{1/2 - 2/3} + n^{1/3}}. \]

The numerator tends to 1, the denominator to +\( \infty \), so the limit is zero.

25. Write
\[ f(x) = \frac{x + 1}{x^2 + 1}. \]

We have
\[ f'(x) = \frac{x^2 + 1 - 2x(x + 1)}{(x^2 + 1)^2} = \frac{1 - 2x - x^2}{(x^2 + 1)^2} < 0 \]
for \( x > 1 \), so \( f(x) \) is decreasing which implies that \( a_n \) is decreasing.

26. Not decreasing. For example, \( a_3 = -2/3 \) while \( a_4 = 1/4 > a_3 \).

27. Fast solution: note that \( y = e^x - 1 \) and plug it into \( y' - y = 1 \).

Slow solution: note that \( y' = \sum_{n=1}^{\infty} \frac{n x^{n-1}}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} x^n \) and plug the series into the equation.

28. Note that \( a = 0 \) is the midpoint of the interval. For \(-1/2 \leq x \leq 1/2\) we have
\[ \left| \frac{1}{1-x} - 1 - x - x^2 - x^3 \right| \leq M |x - 0|^5 \leq \frac{16 \cdot 24}{5!} \left( \frac{1}{2} \right)^5 = A. \]
You do not need to simplify this.

29. Here \( f(x) = e^x \), and all derivatives are also \( e^x \). So
\[ |f^{(3)}(x)| = e^x \leq e = M \]
for all \(-1 \leq x \leq 1\). So
\[ \left| e^x - 1 - x - \frac{x^2}{2} \right| \leq \frac{e}{3!} |x - 0|^3 \leq \frac{e}{6} = A. \]

30. First few terms of the two series:
\[ \sum_{n=1}^{\infty} \frac{\sin \left( \frac{\pi}{2} n \right)}{n} = 1 + 0 - \frac{1}{3} + 0 + \frac{1}{5} + 0 - \frac{1}{7} + 0 + \frac{1}{9} + 0 - \ldots \]
\[ \sum_{n=0}^{\infty} \frac{(-1)^n}{2n + 1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots \]

31. Yes. Use the previous problem to see that it’s an alternating series, and use the alternating series test.

32.

33.

34.
35. It is true by the comparison test: \( \frac{a_n}{n} \leq a_n \) for all \( n \).

36. It is false. A counterexample is given by \( a_n = \frac{1}{n^2} \). Then

\[
\sum_{n=1}^{\infty} na_n = \sum_{n=1}^{\infty} \frac{1}{n}
\]

which is divergent, while \( \sum \frac{1}{n^2} \) is convergent.

37. (a) For \( x > 0 \) we have

\[
e^x - 1 - x = \sum_{n=2}^{\infty} \frac{1}{n!} x^n > 0.
\]

(b) From (a) we have

\[
e^{1/n} - 1 > \frac{1}{n} \quad \text{for all } n
\]

and we use the comparison test. Since \( \sum \frac{1}{n} \) diverges, so does \( \sum (e^{1/n} - 1) \).

38. (a) See Figure 1.

(b) At \( F(1, 1) \), the direction field is pointing “northeast”, meaning \( F(1, 1) > 0 \).

(c) Same as (b), \( F(3, 1) > 0 \).

(d) Choose any point where the line is horizontal. \((x, y) = (-1, 0.4)\) is a good choice.

(e) See Figure 1. Start at the point \((0, -2)\) and draw a straight line in the direction of the direction field until you hit \( x = 0.5 \). Mark the end of this line segment. From this endpoint, draw a straight line in the direction of the direction field until you hit \( x = 1 \). Keep doing this until you hit \( x = 2 \).

39. Separate the variables and integrate:

\[
\int \frac{1}{y} \, dy = \int \frac{1}{x^2 + 1} \, dx
\]

\[
\ln |y| = \arctan x + C
\]

\[
|y| = e^{\arctan x + C}
\]

\[
y = \pm e^{\arctan x + C}
\]

Since \( e^{\arctan x + C} \neq 0 \) for all \( x \), the sign in \( \pm \) will be the same for all \( x \). Plug in the initial condition \( y(0) = 5 \):

\[
5 = \pm e^{\arctan 0 + C} = \pm e^C.
\]

We get \( \pm = + \) and \( C = \ln 5 \). So

\[
y = e^{\arctan x + \ln 5} = 5e^{\arctan x}.
\]
40. Separate the variables and integrate:

$$\int \frac{1}{y(1-y)} \, dy = \int dx.$$

For the left-hand side, use partial integration:

$$\frac{1}{y(1-y)} = \frac{A}{y} + \frac{B}{1-y}$$

We get

$$1 = A(1 - y) + By$$

so $A = B = 1$. So

$$\int \frac{1}{y(1-y)} \, dy = \int \frac{1}{y} + \frac{1}{1-y} \, dy = \ln |y| - \ln |1-y| + C = \ln \left| \frac{y}{1-y} \right| + C$$

so the equation $\int \frac{1}{y(1-y)} \, dy = \int dx$ becomes

$$\ln \left| \frac{y}{1-y} \right| = x + C$$

or

$$\left| \frac{y}{1-y} \right| = e^{x+C}.$$
This can’t be simplified further without knowing any initial conditions.

41. We can use the definition, but it’s much easier to just apply the MacLaurin series for \( \cos t \) with \( t = x + 1 \):

\[
\cos(x + 1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (x + 1)^{2n}.
\]

- Why does this work? All we’re after is a power series representation of \( \cos(x + 1) \) centered at \( a = -1 \). The above is such a power series.

Another way to see this is by substituting \( u = x + 1 \) and finding a MacLaurin series in \( u \).

42. We have

\[
xe^{x} - e^{2x} = x \sum_{n=0}^{\infty} \frac{x^n}{n!} - \sum_{n=0}^{\infty} \frac{(2x)^n}{n!}
\]

\[
= \sum_{n=0}^{\infty} \frac{1}{n!} x^{n+1} - \sum_{n=0}^{\infty} \frac{2^n}{n!} x^n
\]

\[
= \sum_{n=1}^{\infty} \frac{1}{(n-1)!} x^n - \sum_{n=0}^{\infty} \frac{2^n}{n!} x^n
\]

\[
= \sum_{n=1}^{\infty} \frac{1}{(n-1)!} x^n - \left( 1 + \sum_{n=1}^{\infty} \frac{2^n}{n!} x^n \right)
\]

\[
= -1 + \sum_{n=1}^{\infty} \left[ \frac{1}{(n-1)!} - \frac{2^n}{n!} \right] x^n
\]

\[
= -1 + \sum_{n=1}^{\infty} \frac{n - 2^n}{n!} x^n.
\]

(On the midterm, you would not be expected to simplify it as much as I just did.)

43. Divergent by the divergence test.

44. Convergent: use comparison test.

45. Convergent by the root test: \( L = 4/5 \).

46. We have

\[
\ln(1 + 1) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} 1^n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}
\]

so the answer is \( \ln 2 \).

47. Geometric series with \( a = 1/6 \) and \( r = 1/6 \), so the sum is

\[
\frac{1/6}{1 - 1/6} = \frac{1/6}{5/6} = \frac{1}{5}.
\]

48. This is a telescoping series. Write \( b_n = \frac{n}{n^2 + 1} \). Then

\[
\sum_{n=1}^{\infty} \frac{n + 1}{n^2 + 2n + 2} - \frac{n}{n^2 + 1} = \sum_{n=1}^{\infty} b_{n+1} - b_n
\]
and since $b_n \to 0$, this equals $-b_1 = -1/2$.

49. The radius is $+\infty$. 