15-453
FORMAL LANGUAGES, AUTOMATA AND COMPUTABILITY
THE ARITHMETIC HIERARCHY

THURSDAY, MAR 6
ORACLE TMs

Is (M,w) in $A_{TM}$?

Oracle for $A_{TM}$

INPUT

INFINITE TAPE
ORACLE MACHINES

An ORACLE is a set $B$ to which the TM may pose membership questions "Is $w$ in $B$?" (formally: TM enters state $q_?$) and the TM always receives a correct answer in one step (formally: if the string on the "oracle tape" is in $B$, state $q_?$ is changed to $q_{YES}$, otherwise $q_{NO}$)

This makes sense even if $B$ is not decidable! (We do not assume that the oracle $B$ is a computable set!)
We say **A is semi-decidable in B** if there is an oracle TM **M** with oracle **B** that semi-decides **A**

We say **A is decidable in B** if there is an oracle TM **M** with oracle **B** that decides **A**
Language A “Turing Reduces” to Language B

if A is decidable in B, ie if there is an oracle TM M with oracle B that decides A

A ≤ₜ B
Theorem: If $A \leq_m B$ then $A \leq_T B$

Proof:

If $A \leq_m B$ then there is a computable function $f : \Sigma^* \rightarrow \Sigma^*$, where for every $w$,

$$w \in A \iff f(w) \in B$$

We can thus use an oracle for $B$ to decide $A$.

Theorem: $\neg \text{AT}_{TM} \leq_T \text{AT}_{TM}$

Theorem: $\neg \text{AT}_{TM} \leq_m \text{AT}_{TM}$
THE ARITHMETIC HIERARCHY

$\Delta^0_1 = \{ \text{decidable sets} \}$  (sets = languages)

$\Sigma^0_1 = \{ \text{semi-decidable sets} \}$

$\Sigma^0_{n+1} = \{ \text{sets semi-decidable in some } B \in \Sigma^0_n \}$

$\Delta^0_{n+1} = \{ \text{sets decidable in some } B \in \Sigma^0_n \}$

$\Pi^0_n = \{ \text{complements of sets in } \Sigma^0_n \}$
\[\Delta_1^{0} \subseteq \Sigma_1^{0} \subseteq \Delta_2^{0} \subseteq \Sigma_2^{0} \subseteq \Delta_3^{0}\]
Decidable Languages

Semi-decidable Languages

Co-semi-decidable Languages

\[ \Delta^0_1 = \Sigma^0_1 \cap \Pi^0_1 \]

Decidable Languages
Semi-decidable Languages

\[ \sum^0_3 \]

\[ \sum^0_2 \]

\[ \sum^0_1 \]

\[ \Delta^0_3 \]

\[ \Delta^0_2 \]

\[ \Delta^0_1 \]

Decidable Languages

\[ = \sum^0_1 \cap \Pi^0_1 \]

Co-semi-decidable Languages

\[ \Pi^0_3 \]

\[ \Pi^0_2 \]

\[ \Pi^0_1 \]
Theorem

\[ \Sigma_{1}^{0} = \{ \text{semi-decidable sets} \} \]
\[ = \text{languages of the form } \{ x \mid \exists y \ R(x,y) \} \]

\[ \Pi_{1}^{0} = \{ \text{complements of semi-decidable sets} \} \]
\[ = \text{languages of the form } \{ x \mid \forall y \ R(x,y) \} \]

\[ \Delta_{1}^{0} = \{ \text{decidable sets} \} \]
\[ = \Sigma_{1}^{0} \cap \Pi_{1}^{0} \]

Where R is a decidable predicate
Theorem

\[ \Sigma^0_2 = \{ \text{sets semi-decidable in some semi-dec. } B \} \]
\[ = \text{languages of the form } \{ x \mid \exists y_1 \forall y_2 \ R(x,y_1,y_2) \} \]

\[ \Pi^0_2 = \{ \text{complements of } \Sigma^0_2 \text{ sets} \} \]
\[ = \text{languages of the form } \{ x \mid \forall y_1 \exists y_2 \ R(x,y_1,y_2) \} \]

\[ \Delta^0_2 = \Sigma^0_2 \cap \Pi^0_2 \]

Where \( R \) is a decidable predicate
Theorem

\[
\sum^0_n = \text{languages} \left\{ x \mid \exists y_1 \forall y_2 \exists y_3 \ldots Qy_n R(x,y_1,\ldots,y_n) \right\}
\]

\[
\Pi^0_n = \text{languages} \left\{ x \mid \forall y_1 \exists y_2 \forall y_3 \ldots Qy_n R(x,y_1,\ldots,y_n) \right\}
\]

\[
\Delta^0_n = \sum^0_n \cap \Pi^0_n
\]

Where \(R\) is a decidable predicate
\[ \sum_1^0 = \text{languages of the form } \{ x \mid \exists y \ R(x,y) \} \]

We know that \( A_{TM} \) is in \( \sum_1^0 \) Why?

Show it can be described in this form:

\[ A_{TM} = \{ \langle M, w \rangle \mid \exists t \ [M \text{ accepts } w \text{ in } t \text{ steps}] \} \]

Example

\[ A_{TM} = \{ \langle M, w \rangle \mid \exists v \ (v \text{ is an accepting computation history of } M \text{ on } w) \} \]
Definition: A decidable predicate \( R(x,y) \) is some proposition about \( x \) and \( y \), where there is a TM \( M \) such that

for all \( x, y \), \( R(x,y) \) is TRUE \( \Rightarrow \) \( M(x,y) \) accepts
\( R(x,y) \) is FALSE \( \Rightarrow \) \( M(x,y) \) rejects

We say \( M \) “decides” the predicate \( R \).

EXAMPLES:
\( R(x,y) = \) “\( x + y \) is less than 100”
\( R(<N>,y) = \) “\( N \) halts on \( y \) in at most 100 steps”
Kleene’s T predicate, \( T(<M>, x, y) \): \( M \) accepts \( x \) in \( y \) steps

1. \( x, y \) are positive integers or elements of \( \sum^* \)
Definition: A decidable predicate $R(x,y)$ is some proposition about $x$ and $y^1$, where there is a TM $M$ such that for all $x, y$, $R(x,y)$ is TRUE $\Rightarrow$ $M(x,y)$ accepts $R(x,y)$ is FALSE $\Rightarrow$ $M(x,y)$ rejects

We say $M$ “decides” the predicate $R$.

EXAMPLES:
$R(x,y) = \text{“}x + y \text{ is less than 100}\text{”}$
$R(<N>,y) = \text{“}N \text{ halts on } y \text{ in at most 100 steps}\text{”}$
Kleene’s T predicate, $T(<M>, x, y)$: $M$ accepts $x$ in $y$ steps

Note: $A$ is decidable $\iff A = \{x \mid R(x,\epsilon)\}$, for some decidable predicate $R$. 
Theorem

\[ \sum_{n=0}^{\infty} = \text{languages } \{ x \mid \exists y_1 \forall y_2 \exists y_3 \ldots Q y_n R(x,y_1,\ldots,y_n) \} \]

\[ \Pi_{n=0}^{\infty} = \text{languages } \{ x \mid \forall y_1 \exists y_2 \forall y_3 \ldots Q y_n R(x,y_1,\ldots,y_n) \} \]

\[ \Delta_{n=0}^{\infty} = \sum_{n=0}^{\infty} \cap \Pi_{n=0}^{\infty} \]

Where R is a decidable predicate
Theorem: A language $A$ is semi-decidable if and only if there is a decidable predicate $R(x, y)$ such that: $A = \{ x | \exists y \ R(x, y) \}$

Proof:

(1) If $A = \{ x | \exists y \ R(x, y) \}$ then $A$ is semi-decidable
Because we can enumerate over all $y$’s

(2) If $A$ is semi-decidable, then $A = \{ x | \exists y \ R(x, y) \}$

Let $M$ semi-decide $A$
Then, $A = \{ x | \exists y \ T(<M>, x, y) \}$ (Here $M$ is fixed.)
where

Kleene’s $T$ predicate, $T(<M>, x, y)$: $M$ accepts $x$ in $y$ steps.
THE PAIRING FUNCTION

Theorem. There is a 1-1 and onto computable function \(<,> : \Sigma^* \times \Sigma^* \rightarrow \Sigma^*\) and computable functions \(\pi_1\) and \(\pi_2 : \Sigma^* \rightarrow \Sigma^*\) such that

\[ z = <w, t> \implies \pi_1(z) = w \text{ and } \pi_2(z) = t \]

Proof: Let \(w = w_1\ldots w_n \in \Sigma^*, t \in \Sigma^*\).
Let \(a, b \in \Sigma, a \neq b\).

\(<w, t> := a w_1\ldots a w_n b t\)

\(\pi_1(z) := \text{“if } z \text{ has the form } a w_1\ldots a w_n b t, \text{ then output } w_1\ldots w_n, \text{ else output } \varepsilon”\)

\(\pi_2(z) := \text{“if } z \text{ has the form } a w_1\ldots a w_n b t, \text{ then output } t, \text{ else output } \varepsilon”\)
Theorem

\[ \Sigma^0_1 = \{ \text{semi-decidable sets} \} \]
\[ = \text{languages of the form } \{ x \mid \exists y \ R(x, y) \} \]
\[ \Pi^0_1 = \{ \text{complements of semi-decidable sets} \} \]
\[ = \text{languages of the form } \{ x \mid \forall y \ R(x, y) \} \]
\[ \Delta^0_1 = \{ \text{decidable sets} \} \]
\[ = \Sigma^0_1 \cap \Pi^0_1 \]

Where \( R \) is a decidable predicate
Theorem

\[ \sum_2^0 = \{ \text{sets semi-decidable in some semi-dec. B} \} = \text{languages of the form } \{ x | \exists y_1 \forall y_2 R(x,y_1,y_2) \} \]

\[ \Pi_2^0 = \{ \text{complements of } \sum_2^0 \text{ sets} \} = \text{languages of the form } \{ x | \forall y_1 \exists y_2 R(x,y_1,y_2) \} \]

\[ \Delta_2^0 = \sum_2^0 \cap \Pi_2^0 \]

Where R is a decidable predicate
$\sum^0_1$ = languages of the form \{ $x$ | $\exists y \ R(x,y)$ \}

We know that $A_{TM}$ is in $\sum^0_1$ Why?

Show it can be described in this form:

$A_{TM} = \{ <(M,w)> | \exists t \ [M \text{ accepts } w \text{ in } t \text{ steps}] \}$

decidable predicate

$A_{TM} = \{ <(M,w)> | \exists t \ T (<M>, w, t) \}$

$A_{TM} = \{ <(M,w)> | \exists v \ (v \text{ is an accepting computation history of } M \text{ on } w) \}$
Decidable languages

Semi-decidable languages

\[ \sum_0^1 \]

Co-semi-decidable languages

\[ \Pi_0^1 \]

\[ \Delta_0^3 \]

\[ \Delta_0^2 \]

\[ \Delta_0^1 \]

\[ \sum_2^0 \cap \Pi_2^0 \]

\[ A_{\text{TM}} \]
\( \Pi^0_1 \) = languages of the form \( \{ x \mid \forall y \ R(x,y) \} \)

Show that EMPTY (i.e., \( E_{TM} \)) = \( \{ M \mid L(M) = \emptyset \} \) is in \( \Pi^0_1 \)

EMPTY = \( \{ M \mid \forall w \forall t \ [M \ doesn't \ accept \ w \ in \ t \ steps] \} \)

two quantifiers??
decidable predicate
\[ \Pi^0_1 = \text{languages of the form } \{ x \mid \forall y \, R(x,y) \} \]

Show that \( \text{EMPTY} \) (ie, \( E_{TM} \)) = \{ M \mid L(M) = \emptyset \} \text{ is in } \Pi^0_1 \]

\[ \text{EMPTY} = \{ M \mid \forall w \forall t \left[ \neg T(<M>, w, t) \right] \} \]

\text{two quantifiers??} \quad \text{decidable predicate}
Theorem. There is a 1-1 and onto computable function \( <, > : \Sigma^* \times \Sigma^* \rightarrow \Sigma^* \) and computable functions \( \pi_1 \) and \( \pi_2 : \Sigma^* \rightarrow \Sigma^* \) such that

\[
z = <w, t> \implies \pi_1(z) = w \text{ and } \pi_2(z) = t
\]

\[
\text{EMPTY} = \{ M \mid \forall w \forall t [M \text{ doesn't accept } w \text{ in } t \text{ steps}] \}
\]

\[
\text{EMPTY} = \{ M \mid \forall z [M \text{ doesn't accept } \pi_1(z) \text{ in } \pi_2(z) \text{ steps}] \}
\]

\[
\text{EMPTY} = \{ M \mid \forall z [ \neg T(<M>, \pi_1(z), \pi_2(z)) ] \}
\]
Theorem. There is a 1-1 and onto computable function $\langle , \rangle : \Sigma^* \times \Sigma^* \rightarrow \Sigma^*$ and computable functions $\pi_1$ and $\pi_2 : \Sigma^* \rightarrow \Sigma^*$ such that

$z = \langle w, t \rangle \Rightarrow \pi_1 (z) = w$ and $\pi_2(z) = t$

Proof: Let $w = w_1 \ldots w_n \in \Sigma^*$, $t \in \Sigma^*$. Let $a, b \in \Sigma$, $a \neq b$.

$$<w, t> := a w_1 \ldots a w_n b t$$

$$\pi_1 (z) := \text{"if } z \text{ has the form } a w_1 \ldots a w_n b t, \text{ then output } w_1 \ldots w_n, \text{ else output } \varepsilon"$$

$$\pi_2(z) := \text{"if } z \text{ has the form } a w_1 \ldots a w_n b t, \text{ then output } t, \text{ else output } \varepsilon"$$
Decidable languages

Semi-decidable languages

Co-semi-decidable languages

\[ \sum_0^3 \]

\[ \sum_0^2 \]

\[ \sum_0^1 \]

\[ \Delta_0^0 \cap \Pi_0^0 = \sum_0^2 \cap \Pi_0^2 \]

\[ \Delta_0^1 \]

\[ \Delta_0^2 \]

\[ \Delta_0^3 \]
Show that \( \text{TOTAL} = \{ M \mid M \text{ halts on all inputs} \} \) is in \( \Pi^0_2 \) decidable predicate.
\( \Pi^0_2 = \) languages of the form \( \{ x \mid \forall y \exists z \ R(x,y,z) \} \)

Show that \( \text{TOTAL} = \{ M \mid M \text{ halts on all inputs} \} \) is in \( \Pi^0_2 \)

\( \text{TOTAL} = \{ M \mid \forall w \ \exists t \ [ T(<M>, w, t) ] \} \)

decidable predicate
Decidable languages

Semi-decidable languages

\[ \sum^0_0 \cap \Pi^0_1 = \emptyset \]

\[ \sum^0_1 \cap \Delta^0_2 = \sum^0_2 \cap \Pi^0_2 \]

\[ \Delta^0_1 \cap \Delta^0_3 = \text{TOTAL} \]

Co-semi-decidable languages
\[ \sum_2^0 = \text{languages of the form } \{ x \mid \exists y \forall z \ R(x,y,z) \} \]

Show that \( \text{FIN} = \{ M \mid \text{L}(M) \text{ is finite} \} \) is in \( \sum_2^0 \)

\[
\text{FIN} = \{ M \mid \exists n \forall w \forall t \ [\text{Either } |w| < n, \text{ or } M \text{ doesn’t accept } w \text{ in } t \text{ steps}] \}
\]

\[
\text{FIN} = \{ M \mid \exists n \forall w \forall t ( |w| < n \lor \neg T(<M>,w,t) ) \}
\]

decidable predicate
Decidable languages

Semi-decidable languages

\[ \Sigma^0_1 \]

\[ \Sigma^0_2 \]

\[ \Sigma^0_3 \]

\[ \Delta^0_1 \]

\[ \Delta^0_2 \]

\[ \Delta^0_3 \]

\[ \Pi^0_1 \]

\[ \Pi^0_2 \]

\[ \Pi^0_3 \]

FIN

TOTAL

\[ \sum^0_2 \cap \prod^0_2 \]

\[ A_{TM} \]

Co-semi-decidable languages

Decidable languages

EMPTY
\[ \sum_3^0 \] = languages of the form \( \{ x \mid \exists y \forall z \exists u \ R(x,y,z,u) \} \)

Show that \( \text{COF} = \{ M \mid L(M) \text{ is cofinite} \} \) is in \( \sum_2^0 \)

\( \text{COF} = \{ M \mid \exists n \forall w \exists t \ [ |w| > n \implies M \text{ accept } w \text{ in } t \text{ steps}] \} \)

\( \text{COF} = \{ M \mid \exists n \forall w \exists t \ ( |w| \leq n \lor T(<M>,w,t)) \} \)

decidable predicate
Decidable languages

Semi-decidable languages

Co-semi-decidable languages

\[ \sum_0^1 \]

\[ \sum_0^2 \]

\[ \sum_0^3 \]

\[ \Delta^0_1 \]

\[ \Delta^0_2 \]

\[ \Delta^0_3 \]

\[ \Pi^0_1 \]

\[ \Pi^0_2 \]

\[ \Pi^0_3 \]

\[ \text{COF} \]

\[ \text{FIN} \]

\[ \text{TOTAL} \]

\[ \text{EMPTY} \]

\[ \text{ATM} \]

\[ = \sum_0^2 \cap \Pi_0^2 \]
Decidable languages

Semi-decidable languages

Co-semi-decidable languages

\[ \sum_0^3 \]
\[ \sum_0^2 \]
\[ \sum_0^1 \]

\[ \Delta^0_3 \]
\[ \Delta^0_2 \]
\[ \Delta^0_1 \]

\[ \Pi^0_3 \]
\[ \Pi^0_2 \]

\[ \text{REG} \]
\[ \text{FIN} \]
\[ \text{EMPTY} \]

\[ \sum_0^1 \cap \Pi^0_2 = \Delta^0_2 \]

\[ A_{TM} \]
Decidable languages

Semi-decidable languages

Co-semi-decidable languages

\[ \sum_0^3 \]

\[ \sum_0^2 \]

\[ \sum_0^1 \]

\[ \Delta^0_3 \]

\[ \Delta^0_2 \]

\[ \Delta^0_1 \]

\[ \Pi^0_3 \]

\[ \Pi^0_2 \]

\[ \Pi^0_1 \]

\[ \Sigma^0_2 \cap \Pi^0_2 = A_{\text{TM}} \]
Decidable languages

Semi-decidable languages

Co-semi-decidable languages

\[ \Delta^0_3 \]

\[ \Delta^0_2 \]

\[ \Delta^0_1 \]

\[ \Sigma^0_3 \]

\[ \Sigma^0_2 \]

\[ \Sigma^0_1 \]

\[ \Pi^0_3 \]

\[ \Pi^0_2 \]

\[ \Pi^0_1 \]

\[ \sum^0_2 \cap \prod^0_2 = \sum^0_2 \]

\[ \sum^0_1 \cap \prod^0_1 = \Delta^0_1 \]

\[ \sum^0_0 \cap \prod^0_0 = \Delta^0_0 \]

Decidable languages

Semi-decidable languages

Co-semi-decidable languages

CFL

TOTAL

FIN

\[ A_{TM} \]
Each is m-complete for its level in hierarchy and cannot go lower (by next Theorem, which shows the hierarchy does not collapse).

$L$ is m-complete for class $C$ if
i) $L \in C$ and
ii) $L$ is m-hard for $C$,

ie, for all $L' \in C$, $L' \leq_m L$
ATM is m-complete for class $C = \sum_0^1$

i) $ATM \in C$

ii) $ATM$ is m-hard for $C$,

Suppose $L \in C$. Show: $L \leq_m ATM$

Let $M$ semi-decide $L$. Then Map $\sum^* \rightarrow \sum^*$

where $w \rightarrow (M, w)$.

Then, $w \in L \iff (M, w) \in ATM$ QED
FIN is m-complete for class $C = \sum_2^0$

i) $FIN \in C$

ii) $FIN$ is m-hard for $C$,

Suppose $L \in C$. Show: $L \leq_m FIN$

Suppose $L = \{ w | \exists y \forall z \ R(w,y,z) \}$
where $R$ is decided by some TM $D$

Map $\sum^* \to \sum^*$
where $w \to N_{D,w}$
Suppose $L \subseteq \Sigma_2^0$ i.e. $L = \{ w \mid \exists y \forall z \ R(w, y, z) \}$ where $R$ is decided by some TM $D$.

Show: $L \leq_m \text{FIN}$

Map $\Sigma^* \rightarrow \Sigma^*$
where $w \rightarrow N_{D,w}$

Define $N_{D,w}$ on input $s$:

1. Write down all strings $y$ of length $|s|$
2. For each $y$, try to find a $z$ such that $\neg R(w, y, z)$ and accept if all are successful (here use $D$ and $w$)

So, $w \in L \iff N_{D,w} \in \text{FIN}$
ORACLES not all powerful

The following problem cannot be decided, even by a TM with an oracle for the Halting Problem:

SUPERHALT = \{ (M,x) | M, with an oracle for the Halting Problem, halts on x \}

Can use diagonalization here!
Suppose H decides SUPERHALT (with oracle)
Define $D(X) =$ “if $H(X,X)$ accepts (with oracle) then LOOP, else ACCEPT.”
$D(D)$ halts $\iff$ $H(D,D)$ accepts $\iff$ $D(D)$ loops…
ORACLES not all powerful

Theorem: The arithmetic hierarchy is strict. That is, the nth level contains a language that isn’t in any of the levels below n.

Proof IDEA: Same idea as the previous slide.

SUPERHALT^0 = HALT = \{ (M,x) \mid M \text{ halts on } x \}.

SUPERHALT^1 = \{ (M,x) \mid M, \text{ with an oracle for the Halting Problem, halts on } x \}.

SUPERHALT^n = \{ (M,x) \mid M, \text{ with an oracle for SUPERHALT}^{n-1}, \text{ halts on } x \}.
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Read Chapter 6.4 for next time