A Closed-form Formula of a Particular Gram-Schmidt Process Related to Shifted Legendre Polynomials

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1 Introduction

For some $N \in \mathbb{Z}^+$, let V be an $\mathbb{R}$-vector space of real polynomials with degree at most N, with inner product defined as

$$\langle p, q \rangle = \int_0^1 p(x)q(x)dx$$

for all $p, q \in V$. This is a well-known inner product space.

Consider the list $\mathcal{L} = (1, x, x^2, \ldots, x^m)$ for some $m \in \{0, 1, \ldots, N\}$.

The Gram-Schmidt Orthogonalization Algorithm [2] produces an orthonormal list of vectors $\mathcal{O} = (\vec{w}_0, \vec{w}_1, \ldots, \vec{w}_m)$ such that $\text{span}(\mathcal{O}) = \text{span}(\mathcal{L})$.

Matthew Milunic conjectured [1] a closed-form formula for $\vec{w}_n$ for all $n \in \{0, 1, \ldots, m\}$:

Conjecture 1 (Milunic's Conjecture)**.** Let $n \in \{0, 1, \ldots, m\}$. We have

$$\vec{w}_n(x) = \sqrt{2n+1} \cdot \sum_{k=0}^{n} (-1)^{n+k} \binom{n}{k} \binom{n+k}{k} x^k.$$

We will prove that this conjecture holds.

2 Exploring an Interesting Polynomial

Let $n \in \mathbb{Z}^+$. Consider the polynomial $f_n(x) = x^n(1-x)^n$.

Lemma 2.

$$f_n(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} x^{n+k}.$$

Proof. By the binomial theorem, we have

$$\begin{aligned}
f_n(x) &= x^n(1-x)^n \\
&= (x-x^2)^n \\
&= \sum_{j=0}^{n} \binom{n}{j} x^j \cdot (-x^2)^{n-j} \\
&= \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{n-j} x^{n+(n-j)} \\
&= \sum_{k=0}^{n} (-1)^k \binom{n}{k} x^{n+k}.
\end{aligned}$$

□

This polynomial is interesting to us, because we can repeatedly take its derivative and obtain an expression similar to the conjectured summation.

For any function g, denote $g^{(n)}(x) = \frac{d^n}{dx^n} g(x)$.

Lemma 3.

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n+k}{k} x^k = \frac{f_n^{(n)}(x)}{n!}.$$

Proof. By Lemma 2, we have

$$\begin{aligned}
f_n^{(n)}(x) &= \frac{d^n}{dx^n} f_n(x) \\
&= \frac{d^n}{dx^n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} x^{n+k} \\
&= \sum_{k=0}^{n} (-1)^k \binom{n}{k} \cdot \frac{(n+k)!}{k!} \cdot x^k \\
&= \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n+k}{k} \cdot n! \cdot x^k \\
&= n! \cdot \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n+k}{k} \cdot x^k.
\end{aligned}$$

Rearranging, we have $\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n+k}{k} x^k = \frac{f_n^{(n)}(x)}{n!}$, as desired. □

3 Orthogonality

Now, for some $n \in \mathbb{Z}^+$, consider the polynomial p_n defined by

$$p_n(x) = \frac{1}{\binom{2n}{n}} \sum_{k=0}^{n} (-1)^{n+k} \binom{n}{k} \binom{n+k}{k} x^k.$$

We will show that p_n is a monic polynomial that is orthogonal to the subspace of all real polynomials of degree at most $n-1$.

Lemma 4. p_n is a monic polynomial.

Proof. The coefficient of x^n is equal to

$$\begin{aligned} &\frac{1}{\binom{2n}{n}} \cdot (-1)^{n+n} \binom{n}{n} \binom{n+n}{n} \\ &= \frac{\binom{2n}{n}}{\binom{2n}{n}} \cdot (-1)^{2n} \cdot 1 \\ &= 1, \end{aligned}$$

so p_n is indeed a monic polynomial. $\square$

Lemma 5. Consider the inner product space V as defined above. If W is the subspace of all polynomials of degree at most $n-1$, then $p_n \perp q$ for all $q \in W$.

Proof. Let $q \in W$.

We have $\langle p_n, q \rangle = \displaystyle\int_0^1 p_n(x) q(x) dx$.

Observe that some algebra gives

$$p_n(x) = \frac{(-1)^n}{\binom{2n}{n}} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n+k}{k} x^k.$$

By Lemma 3, we have $p_n(x) = \dfrac{(-1)^n}{\binom{2n}{n} \cdot n!} f_n^{(n)}(x)$.

Therefore,

$$\langle p_n, q \rangle = \frac{(-1)^n}{\binom{2n}{n} \cdot n!} \int_0^1 f_n^{(n)}(x) q(x) dx$$

Let us analyze the integral $\displaystyle\int_0^1 f_n^{(n)}(x) q(x) dx$. Using integration by parts repeatedly, we can see it is equal to

$$\sum_{j=0}^{n-1} (-1)^j f_n^{(n-j-1)}(x) q^{(j)}(x) \Big|_0^1 + (-1)^n \int_0^1 f_n(x) q^{(n)}(x) dx.$$

Let us analyze $f_n^{(r)}$ for $r = n - j - 1$, where $j \in \{0, 1, \ldots, n-1\}$. By the general Leibniz rule, we have

$$f_n^{(r)} = (x^n(1-x)^n)^{(r)} = \sum_{k=0}^{r} \binom{r}{k} (x^n)^{(r-k)} ((1-x)^n)^{(k)}.$$

Note that since $r < n$, for all $k \in \{0, 1, \ldots, r\}$, we have $x(1-x) \mid (x^n)^{(r-k)} ((1-x)^n)^{(k)}$. Therefore, it must be that $f_n^{(r)}(0) = f_n^{(r)}(1) = 0$.

Substituting back $r = n - j - 1$, we have $f_n^{(n-j-1)}(0) = f_n^{(n-j-1)}(1) = 0$, so

$$\sum_{j=0}^{n-1} (-1)^j f_n^{(n-j-1)}(x) q^{(j)}(x) \Big|_0^1 = 0.$$

Furthermore, because q has degree at most $n-1$, $q^{(n)}$ is the zero polynomial, so

$$(-1)^n \int_0^1 f_n(x) q^{(n)}(x) dx = 0.$$

Therefore, we have $\langle p_n, q \rangle = \dfrac{(-1)^n}{\binom{2n}{n} \cdot n!}(0+0) = 0$, which means $p_n \perp q$, as desired. □

With this lemma, we are ready to prove the main theorem.

4 The Main Theorem

We will now prove the original conjecture about the Gram-Schmidt Orthogonalization Algorithm by strong induction. Let us formally restate the theorem:

Theorem 6 (the main theorem)**.** For some fixed $N \in \mathbb{Z}^+$, let V be the $\mathbb{R}$-inner product space of real polynomials with degree at most N, with inner product defined as

$$\langle p, q \rangle = \int_0^1 p(x) q(x) dx$$

for all $p, q \in V$.

Let $m \in \{0, 1, \ldots, N\}$.

Consider the list $\mathcal{L} = (1, x, x^2, \ldots, x^m)$. Suppose that $\mathcal{O} = (\vec{w}_0, \vec{w}_1, \ldots, \vec{w}_m)$ is the orthonormal list of vectors produced by the Gram-Schmidt Orthogonalization Algorithm.

Then, we have

$$\vec{w}_n(x) = \sqrt{2n+1} \cdot \sum_{k=0}^{n} (-1)^{n+k} \binom{n}{k} \binom{n+k}{k} x^k$$

for all $n \in \{0, 1, \ldots, m\}$.

Proof. We will prove this theorem by induction on m.

Base case. Consider the case where $m = 0$. We have $\vec{w}_0 = x^0 = 1$ by definition. Note that

$$\sqrt{2(0)+1} \cdot \sum_{k=0}^{0} (-1)^{0+k} \binom{0}{k} \binom{0+k}{k} x^0 = 1$$

also, so the equation holds for $n = 0$, and thus the conjecture holds for $m = 0$.

Inductive step. Let $m \in \{1, 2, \ldots, N\}$. For our inductive hypothesis, assume that

$$\vec{w}_n(x) = \sqrt{2n+1} \cdot \sum_{k=0}^{n} (-1)^{n+k} \binom{n}{k} \binom{n+k}{k} x^k$$

holds for all $n \in \{0, 1, \ldots, m-1\}$. It remains to show that it also holds for $n = m$.

Let $n = m$.

Let W be the subspace of all polynomials of degree at most $n - 1$.

The Gram-Schmidt Orthogonalization Algorithm on an orthonormal list gives us a (not necessarily normalized) vector

$$\vec{v}_n = x^n - \sum_{k=0}^{n-1} \langle \vec{w}_k, x^n \rangle \, \vec{w}_k$$

such that $\vec{w}_n = \dfrac{\vec{v}_n}{||\vec{v}_n||}$.

Note that $\vec{v_n}$ is a monic polynomial of degree n, because by our inductive hypothesis, $\vec{w}_k$ has degree at most k for all $k \in \{0, 1, \ldots, n-1\}$, so the only instance of x^n comes from the initial x^n, which has coefficient 1.

Furthermore, a known property of the algorithm is that $\vec{v}_n$ is orthogonal to W.

Consider the polynomial p_n defined by

$$p_n(x) = \frac{1}{\binom{2n}{n}} \sum_{k=0}^{n} (-1)^{n+k} \binom{n}{k} \binom{n+k}{k} x^k.$$

p_n is a polynomial of degree n. By Lemma 4, p_n is also a monic polynomial. By Lemma 5, p_n is also orthogonal to W.

Let $q \in W$. Then we have $\langle \vec{v}_n, q \rangle = \langle p_n, q \rangle = 0$.

Thus we have $\langle \vec{v}_n - p_n, q \rangle = \langle \vec{v}_n, q \rangle - \langle p_n, q \rangle = 0 - 0 = 0$, so $\vec{v}_n - p_n$ is orthogonal to W. But because $\vec{v}_n$ and p_n are both monic polynomials of degree n, $\vec{v}_n - p_n$ is a polynomial of degree at most $n - 1$, so $\vec{v}_n - p_n \in W$. The only vector in W that is orthogonal to W is the zero vector, so $\vec{v}_n - p_n = \vec{0}$, which gives $\vec{v}_n = p_n$.

Thus, we have shown that

$$\vec{v}_n = x^n - \sum_{k=0}^{n-1} \langle \vec{w}_k, x^n \rangle \vec{w}_k = \frac{1}{\binom{2n}{n}} \sum_{k=0}^{n} (-1)^{n+k} \binom{n}{k} \binom{n+k}{k} x^k.$$

Now we can find $||\vec{v}_n||$.

By Lemma 3, we have

$$\begin{aligned}
&||\vec{v}_n||^2 \\
&= \int_0^1 \left(\frac{1}{\binom{2n}{n}} \sum_{k=0}^{n} (-1)^{n+k} \binom{n}{k} \binom{n+k}{k} x^k \right)^2 dx \\
&= \frac{1}{\binom{2n}{n}^2} \int_0^1 \left(\sum_{k=0}^{n} (-1)^{k} \binom{n}{k} \binom{n+k}{k} x^k \right)^2 dx \\
&= \frac{1}{\binom{2n}{n}^2 (n!)^2} \int_0^1 \left(f_n^{(n)}(x) \right)^2 dx.
\end{aligned}$$

Using integration by parts repeatedly, and by similar logic as before, the boundary terms are all zero, and we have

$$\begin{aligned}
&\int_0^1 \left(f_n^{(n)}(x) \right)^2 dx \\
&= (-1)^n \int_0^1 f_n(x) \cdot f_n^{(2n)}(x) dx \\
&= (-1)^n \int_0^1 f_n(x) \cdot (2n)!(-1)^n dx \\
&= (2n)! \int_0^1 x^n (1-x)^n dx \\
&= (2n)! \cdot B(n+1, n+1) \\
&= (2n)! \cdot \frac{n!n!}{(2n+1)!} \\
&= \frac{(n!)^2}{2n+1}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&||\vec{v}_n||^2 \\
&= \frac{1}{\binom{2n}{n}^2 (n!)^2} \cdot \frac{(n!)^2}{2n+1} \\
&= \frac{1}{\binom{2n}{n}^2 (2n+1)}.
\end{aligned}$$

This means that $||\vec{v}_n|| = \dfrac{1}{\binom{2n}{n}\sqrt{2n+1}}$, so

$$\begin{aligned}\vec{w}_n &= \frac{\vec{v}_n}{||\vec{v}_n||} \\ &= \frac{1}{\binom{2n}{n}} \sum_{k=0}^{n} (-1)^{n+k} \binom{n}{k}\binom{n+k}{k} x^k \cdot \binom{2n}{n}\sqrt{2n+1} \\ &= \sqrt{2n+1} \cdot \sum_{k=0}^{n} (-1)^{n+k} \binom{n}{k}\binom{n+k}{k} x^k,\end{aligned}$$

as desired.

Since both the base case and the inductive step hold, by the principle of mathematical induction, the theorem holds for all $m \in \{0, 1, \dots, N\}$.

That completes the proof. □

5 Conclusion and Future Works

We have constructed, and proved, a closed-form formula for computing the outputs of the Gram-Schmidt Orthogonalization Algorithm for our particular inner product space and the choice of elementary basis vectors in the vector space:

$$\vec{w}_n = \sqrt{2n+1} \cdot \sum_{k=0}^{n} (-1)^{n+k} \binom{n}{k}\binom{n+k}{k} x^k$$

for all $n \in \{0, 1, \dots, m\}$.

It should be noted that this expression is very closely related to the shifted Legendre Polynomials $\tilde{P}_n(x) = P_n(2x+1)$. In the future, we will try to prove this theorem using properties of Legendre Polynomials and look for more interesting relationships between Gram-Schmidt processes and Legendre Polynomials.

References

[1] Matthew Milunic. “A Fun Closed Formula”. Published Oct 20, 2025.

[2] Sergei Treil. *Linear Algebra Done Wrong*. Version 2024-08-20, CC BY-NC-ND 3.0. Department of Mathematics, Brown University, 2024. URL: https://www.math.brown.edu/~treil/papers/LADW/LADW_2024_08-20.pdf.