Explicit near-fully X-Ramanujan graphs

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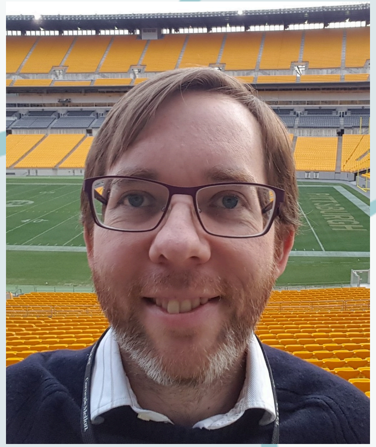
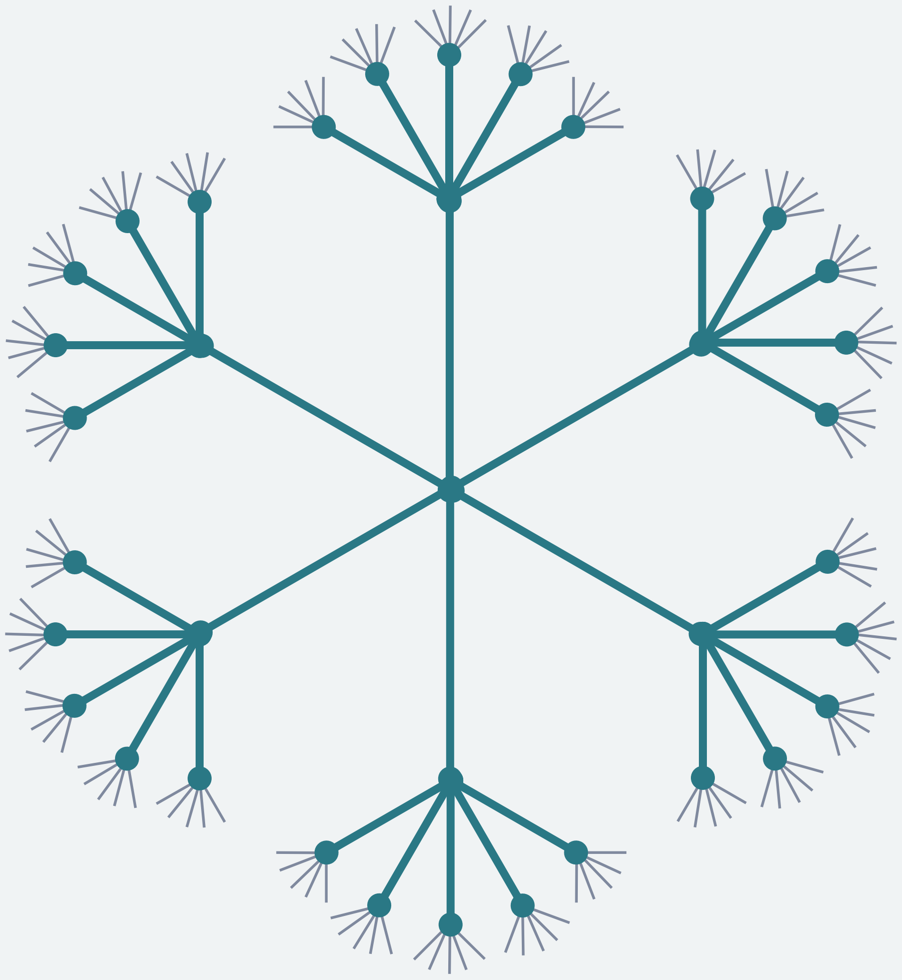
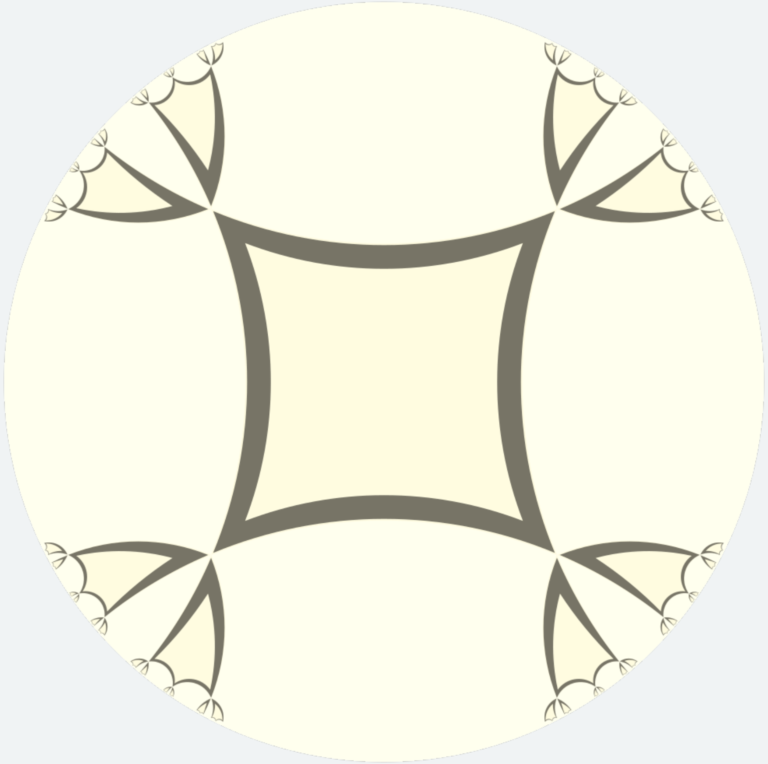
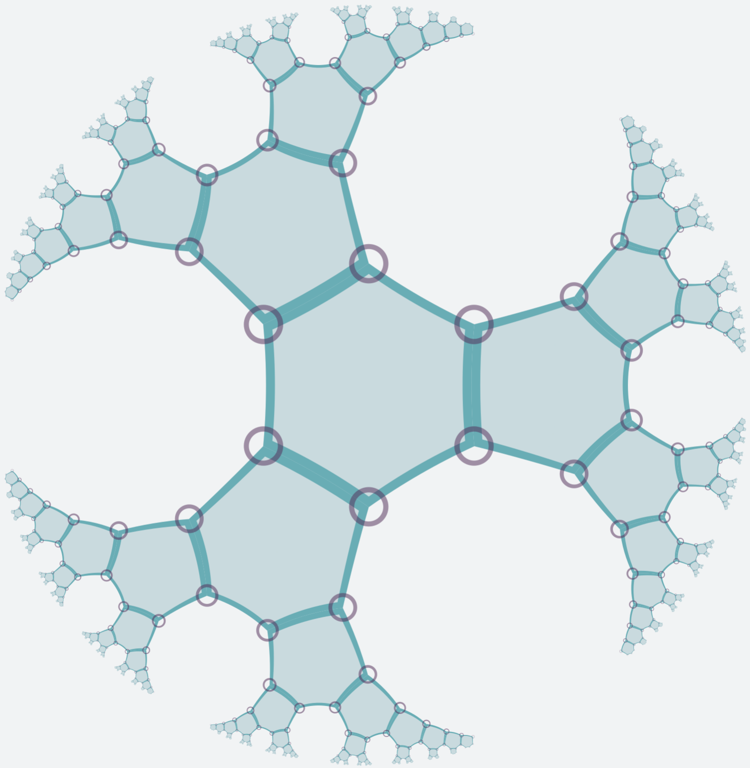
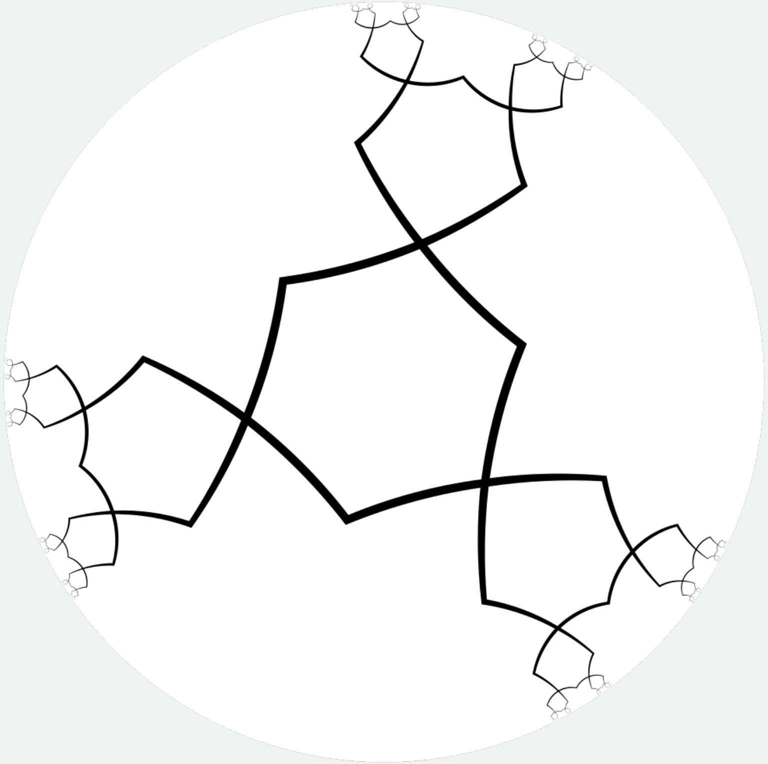
Joint work with Ryan O’Donnell





6-regular infinite tree $\mathbb{T}_6$

Goal: approximate spectrum and structure of infinite graphs with finite graphs

Approximating infinite graphs with finite graphs

d-regular Ramanujan graphs approximate the structure and spectrum of infinite d-regular tree

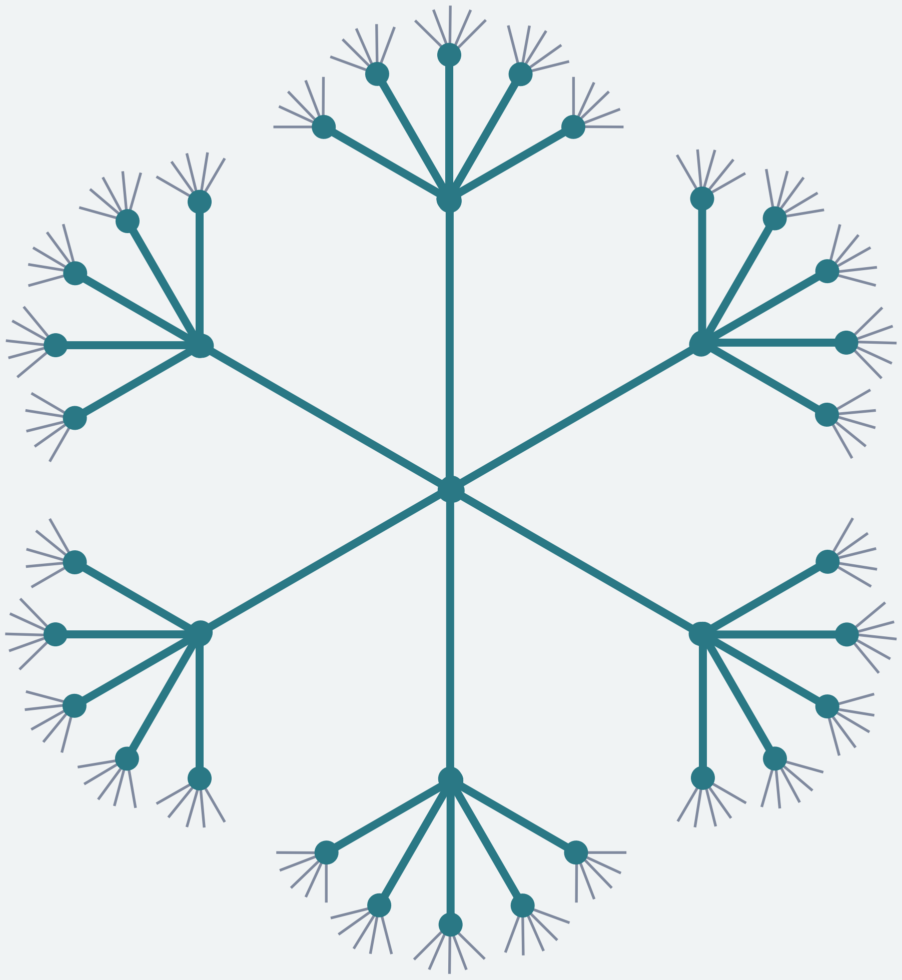
- Ramanujan graphs: best possible expanders
- Expanders have many applications in TCS

Beyond d-regular graphs

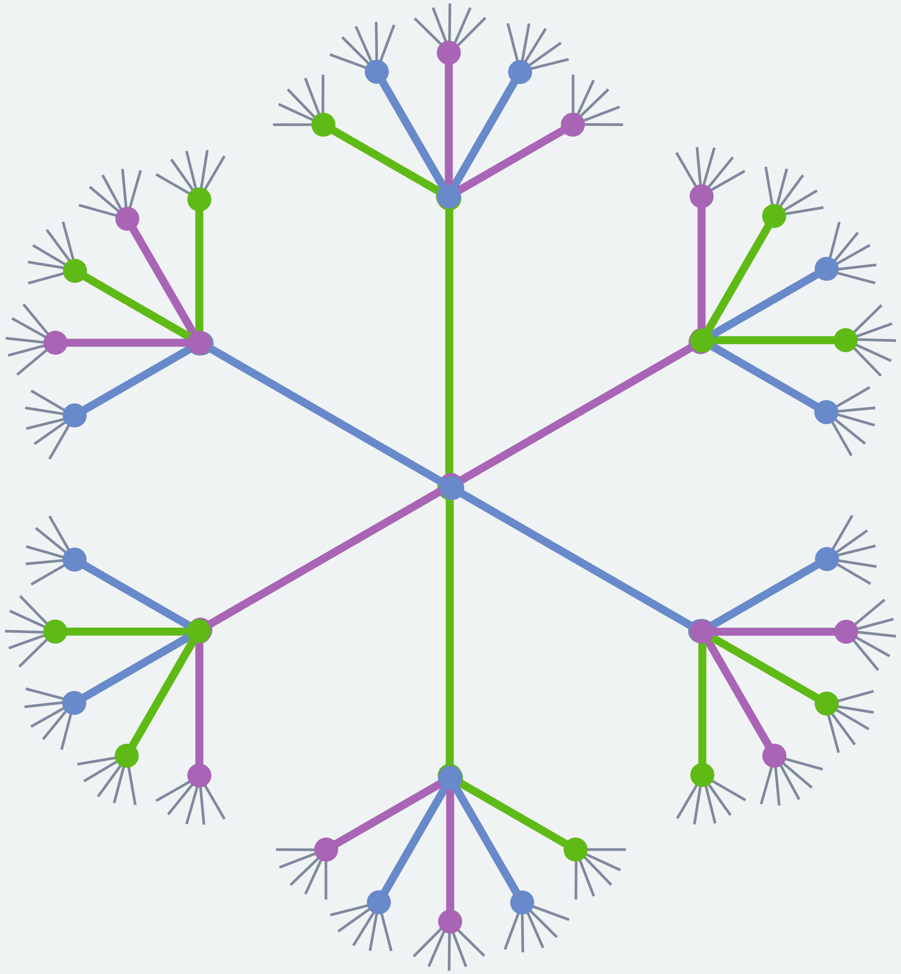
Approximations of more complicated infinite graphs = expanders with local constraints

Example: typical instances of random constraint satisfaction problems

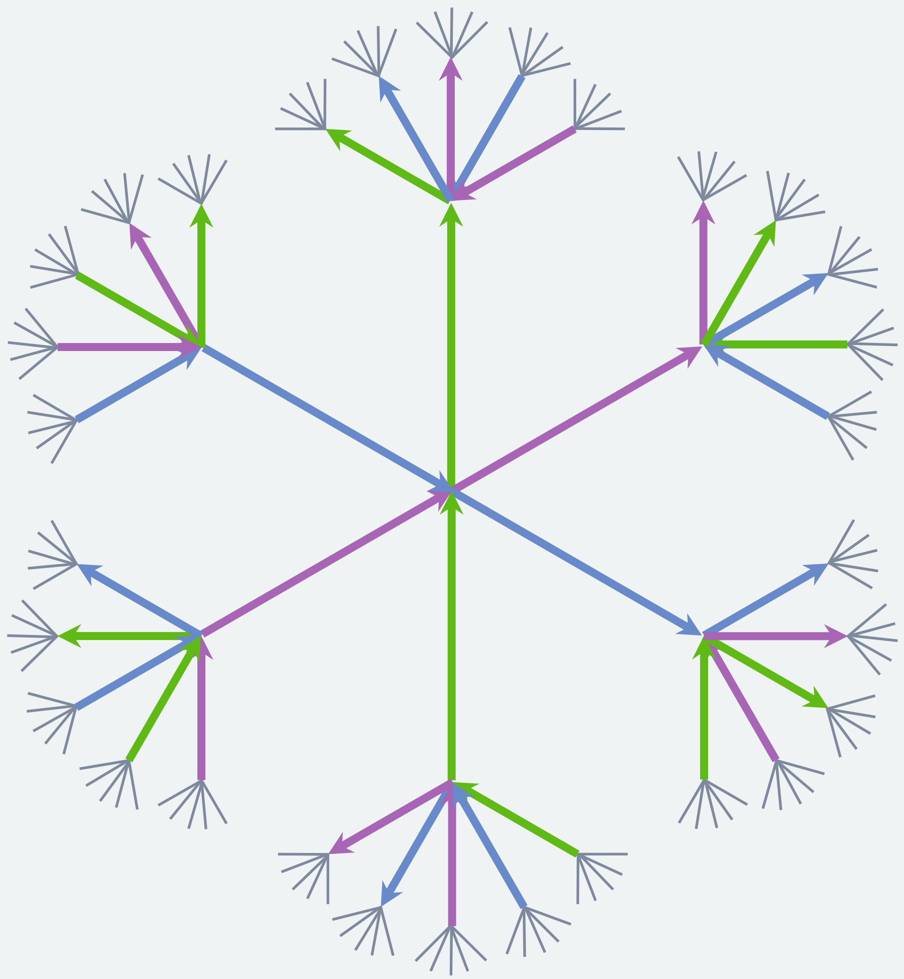
First step: algebraic recipe to describe infinite graphs



6-regular infinite tree $\mathbb{T}_6$



$\mathbb{T}_6$ as a *color-regular* graph

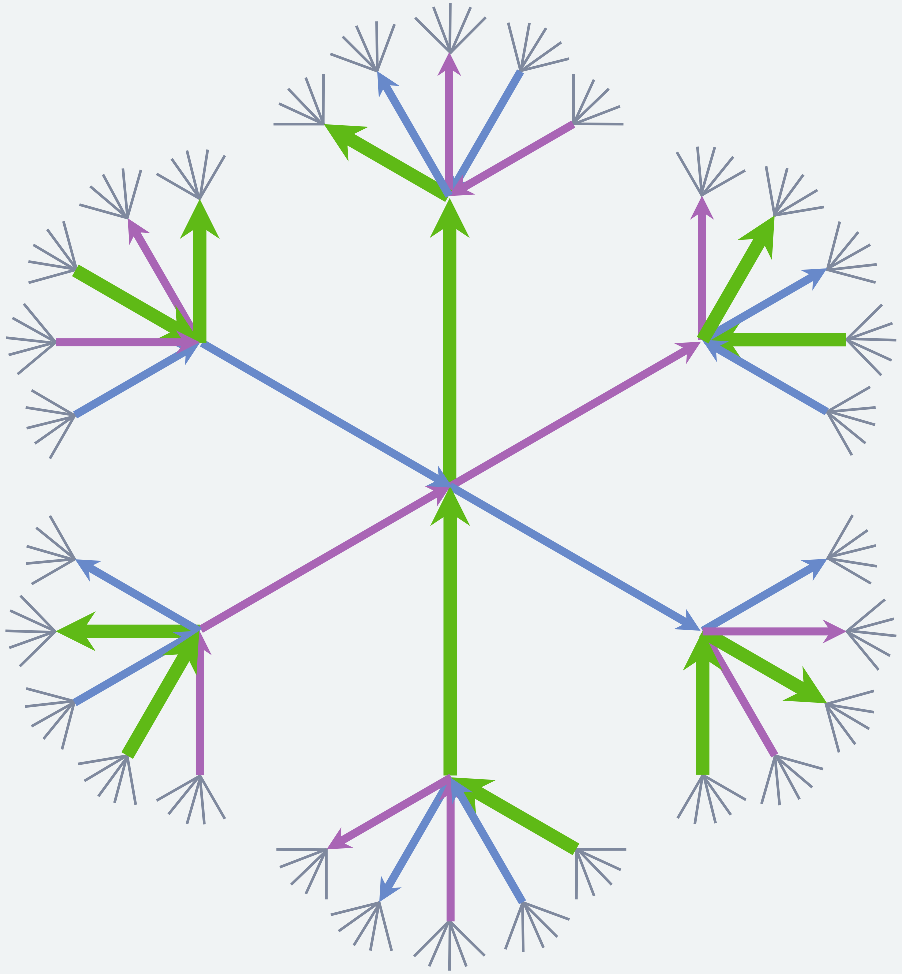


All green arrows = permutation of tree

Represent $\mathbb{T}_6$ as sum of 3 infinite permutations

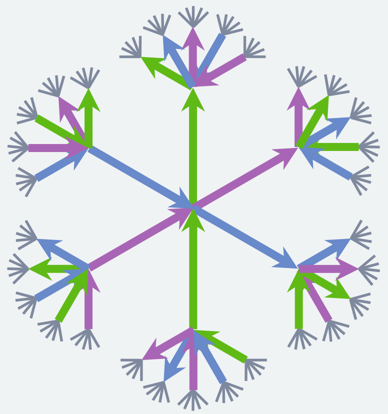
$g_1 + g_2 + g_3 +$
$g_1^{-1} + g_2^{-1} + g_3^{-1}$

Formally: generators of $\mathbb{Z} * \mathbb{Z} * \mathbb{Z}$ or $\mathbb{F}_3$



Adjacency matrix

$$A = P_{g_1} + P_{g_2} + P_{g_3} + P_{g_1}^* + P_{g_2}^* + P_{g_3}^*$$



$(P_{g_1}^* = P_{g_1}^{-1})$

Express as a polynomial

$$p(X_1, X_2, X_3) = X_1 + X_2 + X_3 + X_1^* + X_2^* + X_3^*$$

Then $A = p(P_{g_1}, P_{g_2}, P_{g_3})$

Also written $p(g_1, g_2, g_3)$

Add inverse perms ➔ undirected graph

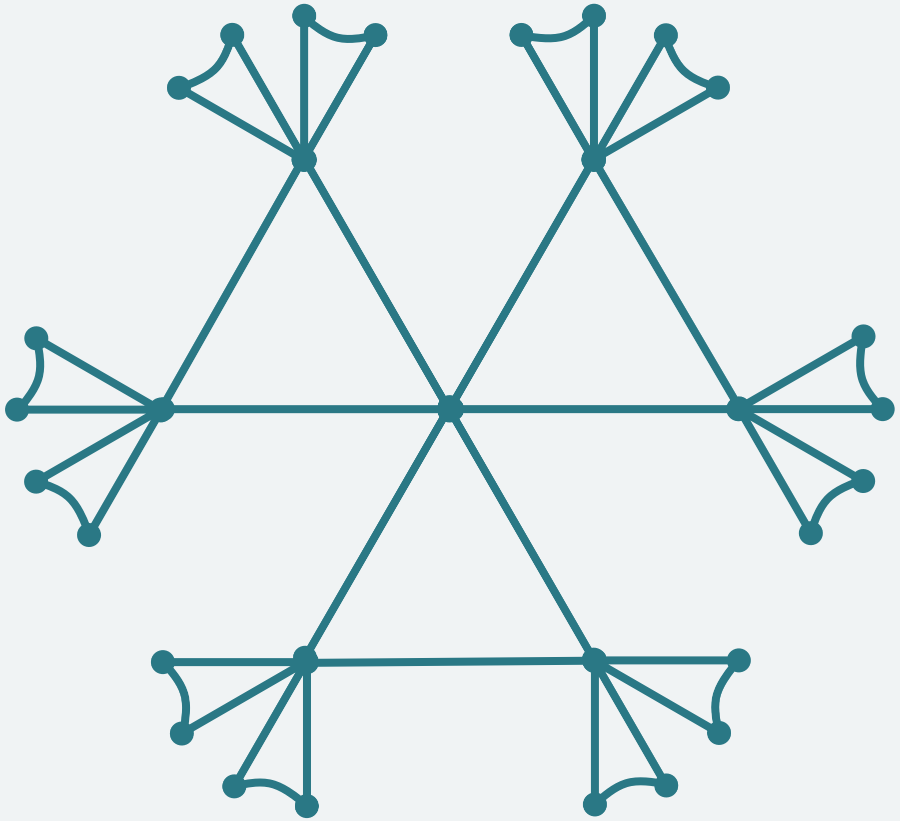
Consider the polynomial

$$q(X_1, X_2) = X_1 + X_2 + X_1 X_2 + X_1^* + X_2^* + X_2^* X_1^*$$

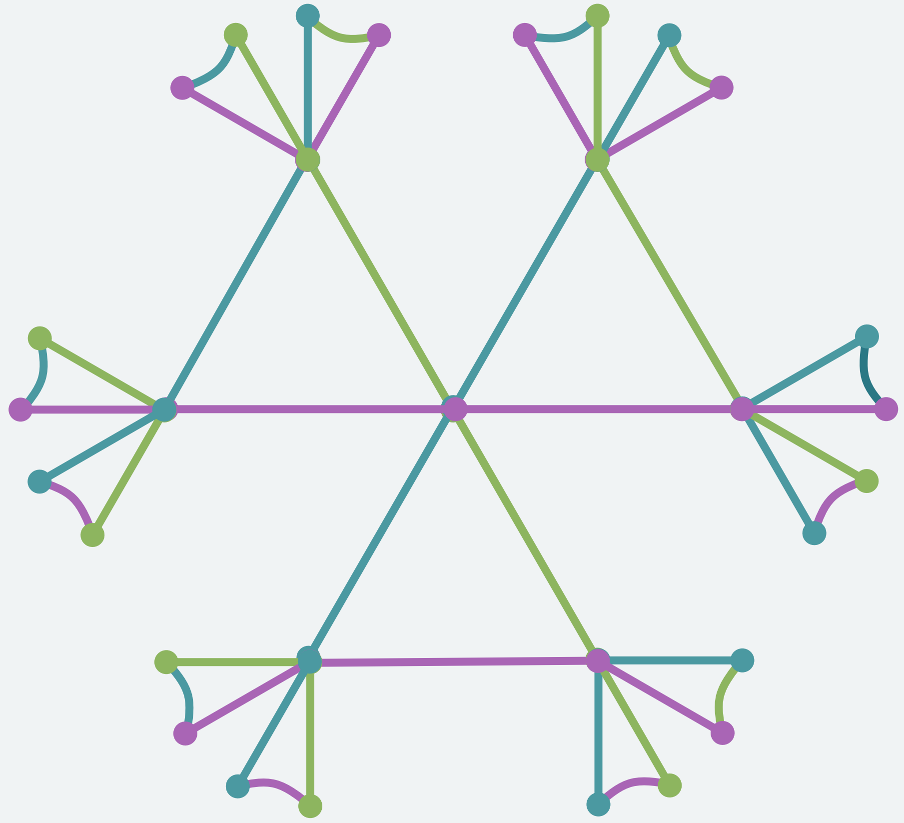
What is $q(g_1, g_2)$?

Vertices = vertices infinite 4-regular tree

$$q(X_1, X_2) = X_1 + X_2 + X_1X_2 + X_1^* + X_2^* + X_2^*X_1^*$$



$$q(X_1, X_2) = X_1 + X_2 + X_1X_2 + X_1^* + X_2^* + X_2^*X_1^*$$

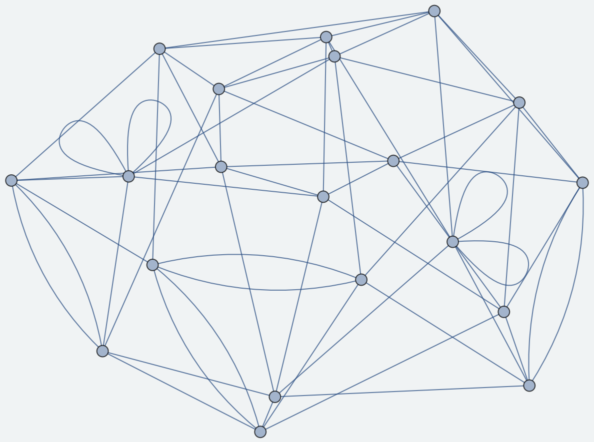


Approximate spectrum and structure of graphs described by polynomials

Structural approximation

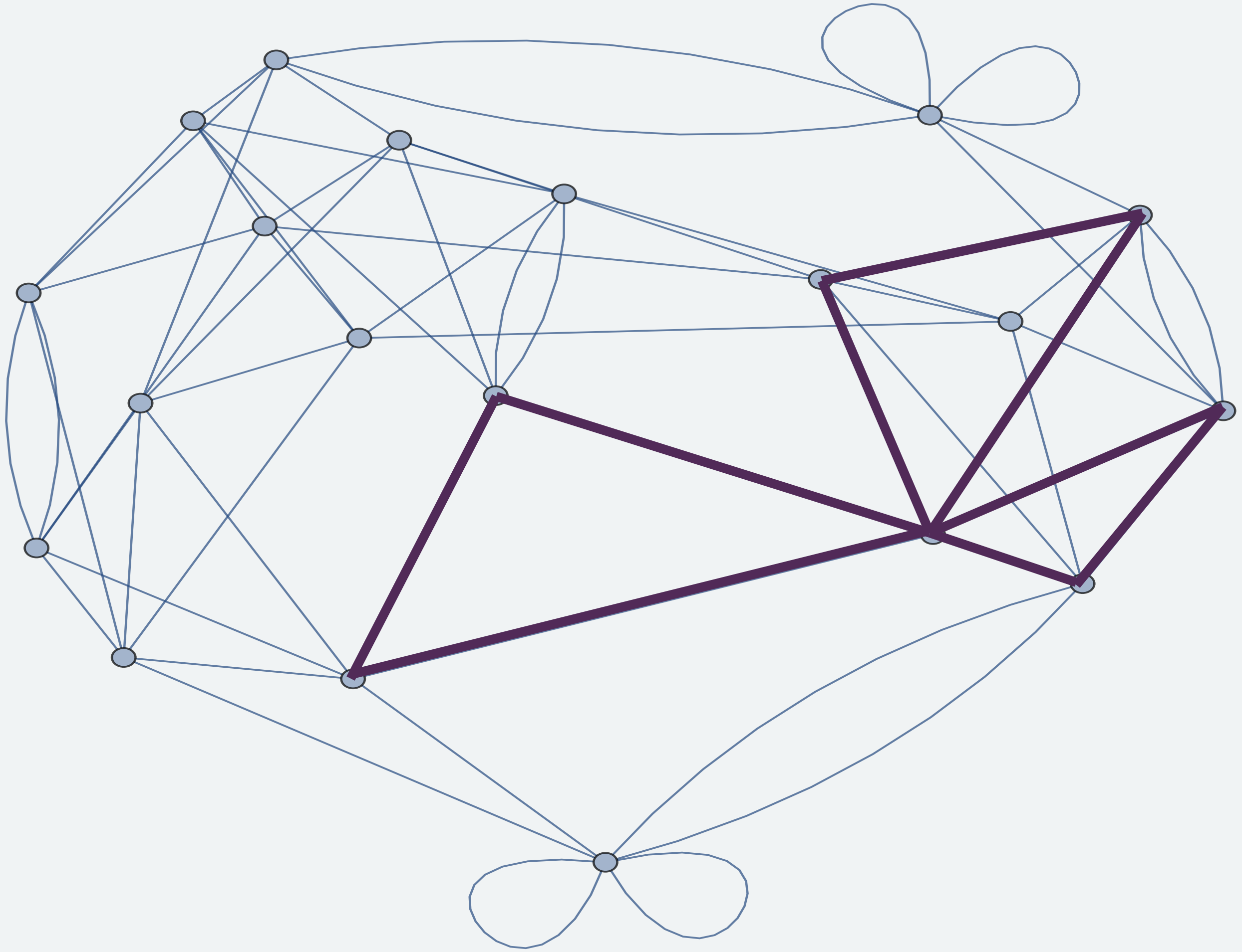
Approach: finite permutations

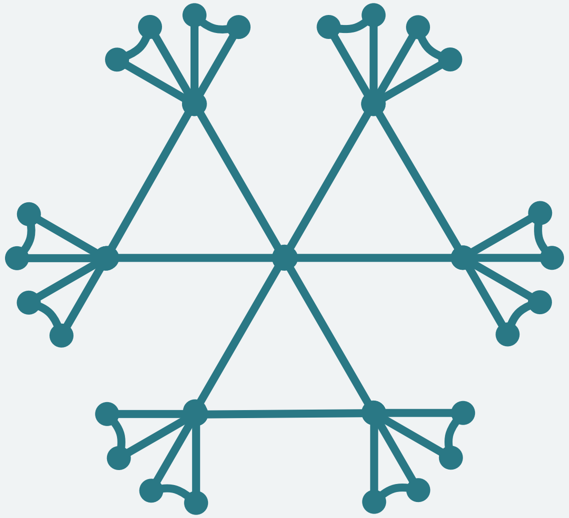
- Replace infinite permutations with finite permutations on $[n]$: $P_{\sigma_1}, P_{\sigma_2}, P_{\sigma_3}$
- $p(X_1, X_2, X_3) = X_1 + X_2 + X_3 + X_1^* + X_2^* + X_3^*$
- $p(\sigma_1, \sigma_2, \sigma_3)$ is adj matrix of 6-regular graph on $[n]$



$$q(X_1, X_2) = X_1 + X_2 + X_1 X_2 + X_1^* + X_2^* + X_2^* X_1^*$$

Applied to random permutations:





Compare to infinite graph

What does it mean to approximate?

- We say that G *covers* H if there's a surjection from G to H which is a local bijection
  - $q(g_1, g_2)$ covers $q(\sigma_1, \sigma_2)$
- For any base graph H there is a unique (usually infinite) tree (the *universal covering tree*) that covers H

Approximating using random permutations

- Random permutations actually form a good approximation
- Friedman's theorem: random d-regular graph structurally + spectrally approximates the infinite d-regular tree (is almost Ramanujan)

Spectral approximation

Closeness in spectrum

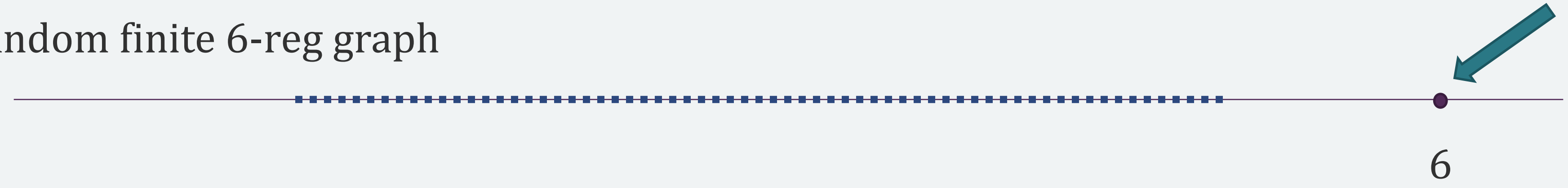
Spectrum:
$\{\lambda : (\lambda I - A) \text{ is not invertible}\}$

Sequence of permutations such that

$$\mathrm{spec}\left(p\left(\sigma_{1,n}, \sigma_{2,n}, \sigma_{3,n}\right)\right) \to \mathrm{spec}\left(p\left(g_1, g_2, g_3\right)\right)$$

Trivial eigenvalues

Spectrum of random finite 6-reg graph



- Eigenvalues of all 1s vector
- All 1s is not an eigenvector of infinite graph since it's not bounded
- How to find these? They are the eigenvalues of $p(1,1,1)$
  - p applied to identity permutation on $[1]$
  - Or, degree of the graph

Spectral approximation

Infinite 6-reg tree $\mathbb{T}_6$

Nontrivial spectrum of finite graph G

Notion of convergence:

For n large enough, every point in $\mathrm{spec}(G)$ is within ε of a point of $\mathrm{spec}(\mathbb{T}_6)$ and vice versa

“Convergence in Hausdorff distance”

Ramanujan graphs

A — adj matrix of n-vertex, d-regular graph

Eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$

Trivial eigenvalue $\lambda_1 = d$

Alon–Boppana theorem:

$$\max(\lambda_2, |\lambda_n|) \geq \underbrace{2\sqrt{d-1}}_{\text{Spectral radius of infinite d-regular tree}} - o_n(1)$$

Ramanujan graphs

Alon–Boppana theorem:

$$\max(\lambda_2, |\lambda_n|) \geq 2\sqrt{d-1} - o_n(1)$$

Ramanujan graphs:

$$\max(\lambda_2, |\lambda_n|) \leq 2\sqrt{d-1}$$

Explicit constructions:

- [Margulis ’88], [Lubotsky–Phillips–Sarnak ’89], [Morgenstern ’94] ($d-1$ prime power)
- [Marcus–Spielman–Srivastava ’15] (bipartite)

Friedman's theorem

Random graphs are almost Ramanujan

For any $\varepsilon > 0$, a random d-reg graph whp has

$$\max(\lambda_2, |\lambda_n|) \leq 2\sqrt{d-1} + o_n(1)$$

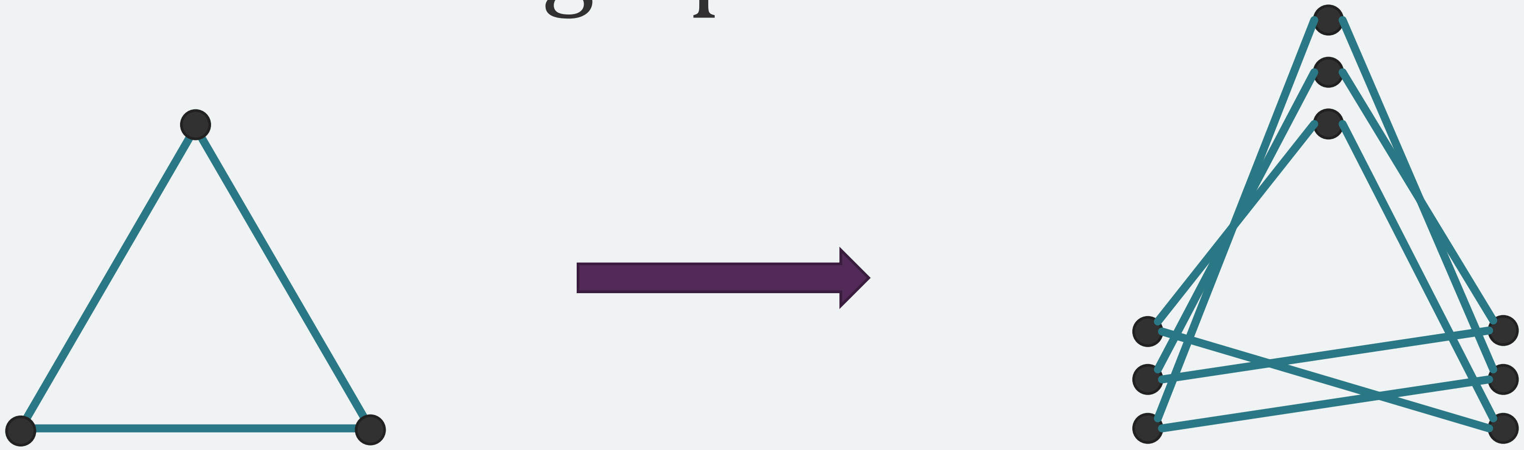
Explicit construction in recent work of Mohanty–O'Donnell–Paredes '20

Beyond Friedman’s theorem

- Generalizing Friedman’s theorem to other algebraically-described graphs
- Concept: random lifts and universal covering

Lifts

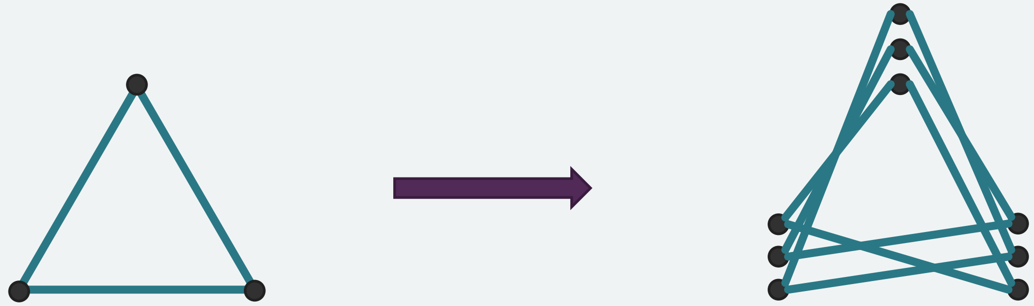
Start with a base graph G



Replace each vertex with n vertices
Replace each edge with a matching

} n-lift

Lifts



- n-lift G_n covers the base graph G
- Limiting object (∞-lift) is the universal covering tree
- Spectrum: Trivial eigenvalues are spec(G)
  - Consider newly added eigenvalues
- *Random* n-lift: iid uniform random matchings for edges

X-Ramanujan graphs

A sequence of graphs $\{G_n\}$ such that

1. Nontrivial spectrum of G_n is ε-close to spec(X) in Hausdorff distance
2. X covers G_n

Generalized Friedman's conj.

A random n-lift G_n of a base graph H has nontrivial spectrum $\mathrm{spec}(G_n) \setminus \mathrm{spec}(H)$ ε-close to $\mathrm{spec}(G_\infty)$ with high probability

Proved by Bordenave–Collins '19

More general result: stated in terms of polynomials

Our result

- Explicit form of Bordenave–Collins' theorem
- Consequently:
  Given a base graph H and $\varepsilon > 0$, we have a poly(n)-time algorithm which constructs an n-lift with nontrivial spectrum ε-close to that of the universal covering tree of H.
- Includes more infinite graphs too

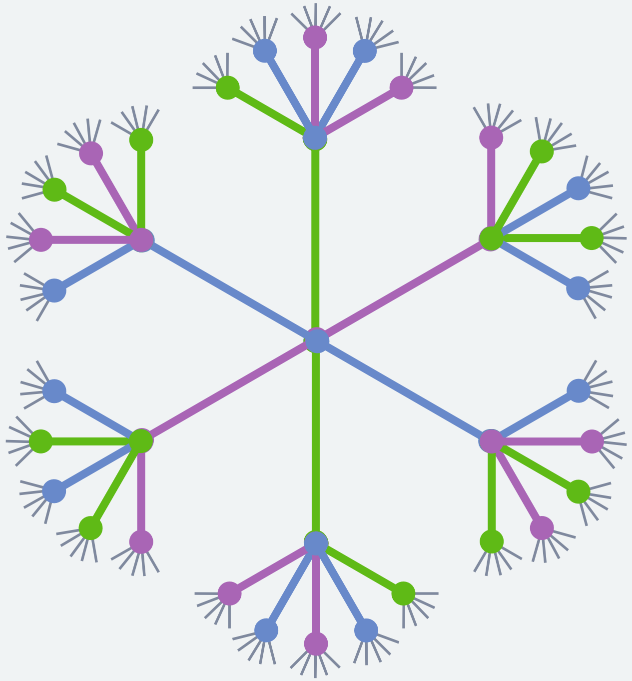
Up next

- What graphs can we describe with polynomials?
- Make some natural algebraic generalizations of polynomials

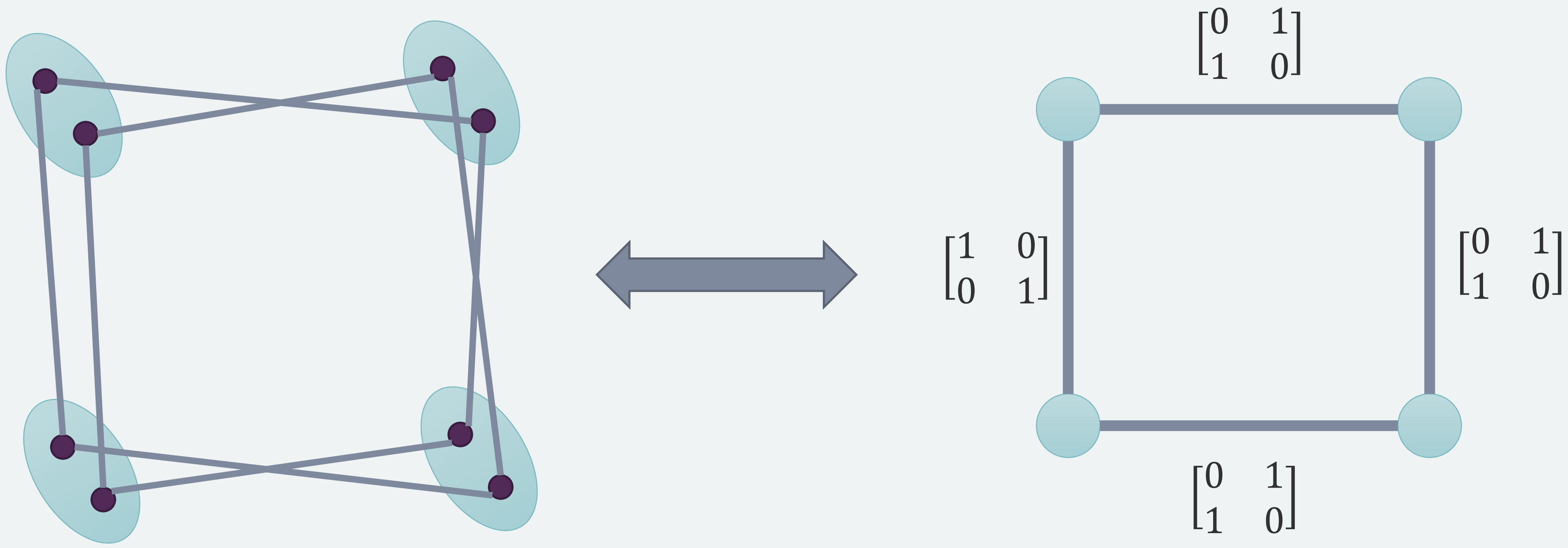
Weighted graphs

$$p(X_1, X_2, X_3) = 0.15X_1 + 0.25X_2 + 0.1X_3 + \cdots$$

Interpretation: weighted random walk on tree

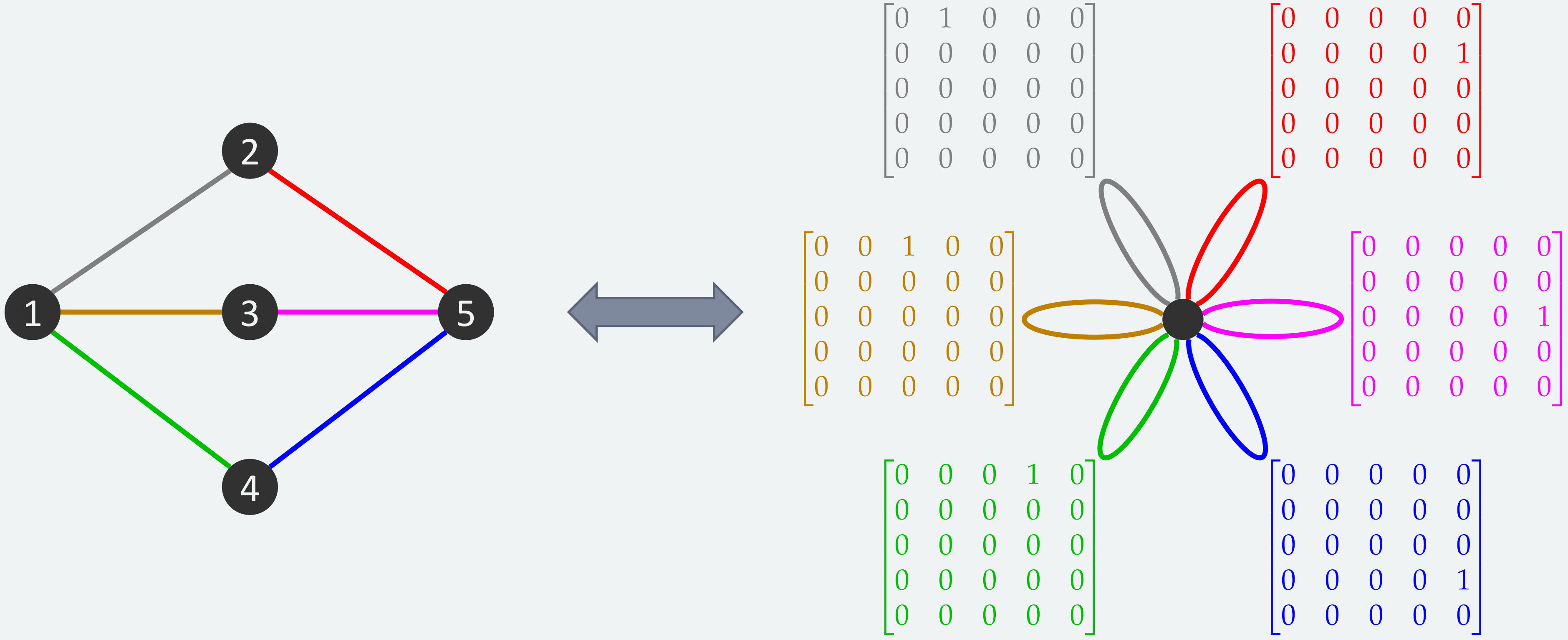


Matrix-weighted graphs



Matrix weighted graph with same adjacency matrix

Matrix-weighted graphs

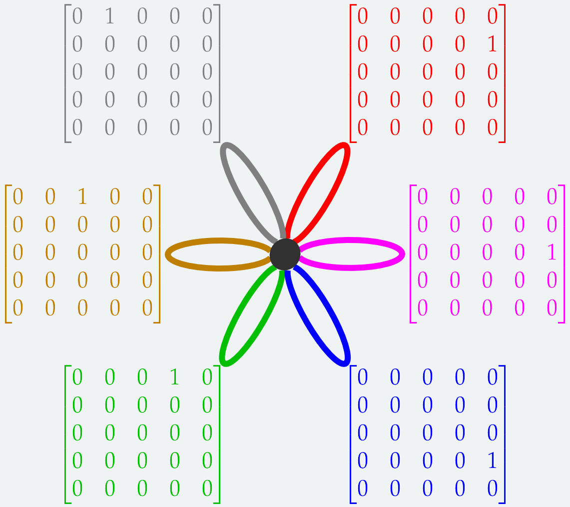


Matrix polynomials

$$p(X_1, \dots, X_6) = a_1 X_1 + \cdots + a_6 X_6 + a_1^* X_1^* + \cdots$$

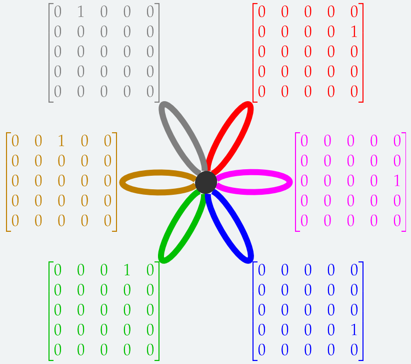
Evaluate as

$$p(\sigma_1, \dots, \sigma_6) = \sum_{i=1..6} P_{\sigma_i} \otimes a_i + P_{\sigma_i}^* \otimes a_i^*$$



is $p(1, \dots, 1)$

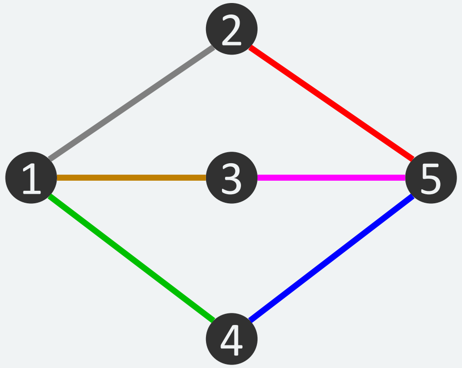
Matrix polynomials



$$p(\sigma_1, \ldots, \sigma_6) = \sum_{i=1..6} P_{\sigma_i} \otimes a_i + P_{\sigma_i}^* \otimes a_i^*$$

$\sigma_1, \ldots, \sigma_6$ are permutations

is a lift of



$p(g_1, \ldots, g_6)$ is the *universal covering tree*

$g_1, \ldots, g_6$ are infinite cyclic permutations/generators of $\mathbb{Z}$

Bordenave and Collins' work

For any matrix polynomial $p(X_1, \ldots, X_d)$, iid unif random permutations $\{\sigma_{1,n}, \ldots, \sigma_{d,n}\}$ satisfy

$$\mathrm{spec}\left(p\left(\sigma_{1,n}, \ldots, \sigma_{d,n}\right)\right) \to \mathrm{spec}(p(g_1, \ldots, g_3))$$

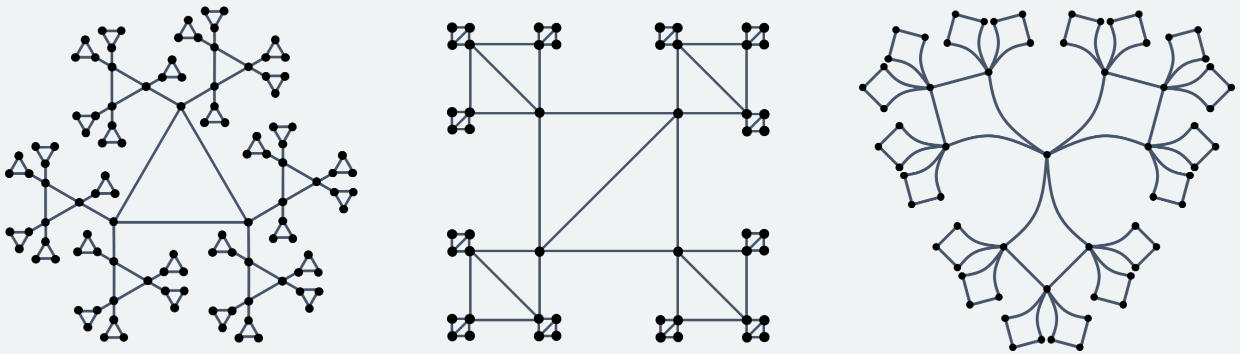
(minus trivial eigenvalues)

whp in Hausdorff distance

Graphs from matrix polys

We show that they include:

- Free products of finite vertex transitive graphs/rooted graphs
- “Additive products” [Mohanty–O’Donnell ’20]
- “Amalgamated free products” [Vargas–Kulkarni ’20]
- And others



Mohanty, Sidhanth, O’Donnell, Ryan. X-Ramanujan graphs. In *Proceedings of the 31st Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 1226–1243, 2020.

Our results

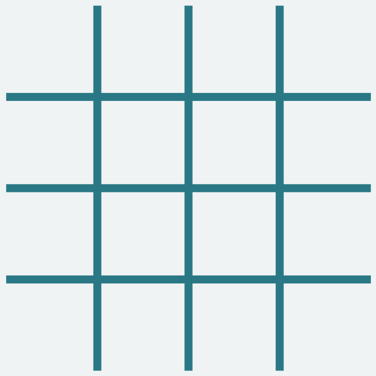
Given $X, \varepsilon > 0$, we give a poly(n) time algo which produces a graph G on $n' \sim n$ vertices

- G is covered by X
- G's nontrivial spectrum is ε-close in Hausdorff distance to X's spectrum

Not covered

Some non-examples:

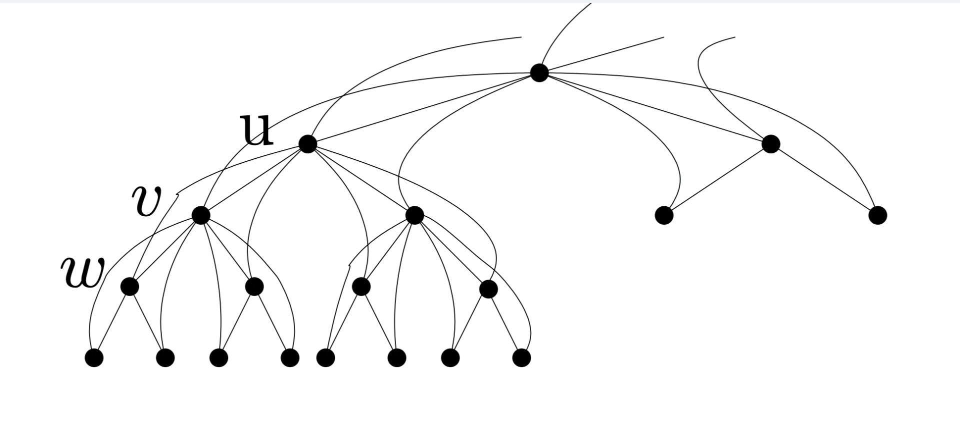
- Grids — matrix polynomial graphs have finite treewidth and are hyperbolic



Graphs from matrix polys

Some non-examples:

- “Grandparent graph” — matrix polynomial graphs are unimodular



Peres, Yuval; **Pete, Gábor**; **Scolnicov, Ariel**. Critical percolation on certain nonunimodular graphs. *New York J. Math.* 12 (2006), 1--18.

Proof steps

1. Linearization
   - Reduce to proving theorem for only linear polys
2. Matrix-valued Ihara–Bass formula
   - Relate spectra of adjacency operator and *non-backtracking operator*
   - Norm bounds for NB operator ➔ Hausdorff distance bounds on adj operator
3. Prove norm bounds on NB operator
   - Trace method with matrix weights

Conclusion

- Non-commutative polynomials
  - Recipe for constructing many infinite graphs and finite graphs covered by them
- Explicit constructions of finite graphs spectrally close to infinite graphs
- Open question: is there a similar recipe for other combinatorial objects, e.g. high dimensional expanders?

Proof – norm bounds

- Further reductions following [MOP ’20] (Ideas from [Bilu–Linial ’06])
- Main technical difficulty:
  construct 2-lifts which all new eigenvalues of nonbacktracking matrix are bounded
- Key step: trace method with matrix weights
  - Matching walks on the finite lift with walks on the infinite graph