Problem 6.10 Harmonic oscillator solution using raising and lowering operators.

The operators given in the problem statement are in terms of displacement x but can be transformed into a simpler form in terms of the dimensionless parameter $s = x/x_0$ where $x_0 = \frac{\hbar^{1/2}}{(Km)^{1/4}}$: calling the dimensioned operator $\tilde{a}_+$ we can write

$$\begin{aligned}
\tilde{a}_+ &= \sqrt{\frac{K}{2}}x - \frac{\hbar}{\sqrt{2m}}\frac{\partial}{\partial x} && (1)\\
&= \sqrt{\frac{\hbar}{2}}\left(\frac{K}{m}\right)^{1/4} s - \sqrt{\frac{\hbar}{2}}\left(\frac{K}{m}\right)^{1/4}\frac{\partial}{\partial s} && (2)\\
&= \left(\frac{1}{2}\hbar\omega_0\right)^{1/2}\left[s - \frac{\partial}{\partial s}\right], && (3)
\end{aligned}$$

and similarly for $\tilde{a}_-$ with a $+$ sign instead of the $-$. So, we define the dimensionless operators, $a_\pm$ as

$$a_\pm = s \mp \frac{\partial}{\partial s}. \tag{4}$$

This means that energies are being measured in units of $\frac{1}{2}\hbar\omega_0$.

In terms of dimensionless quantities, the Schrodinger equation for the harmonic oscillator is written as

$$-\frac{\partial^2\psi(s)}{\partial s^2} + s^2\psi(s) = \lambda\psi(s), \tag{5}$$

where λ is the dimensionless eigenvalue from which the energy is obtained: $E_n = \lambda\frac{1}{2}\hbar\omega_0$. The Hamiltonian operator is then $H = -\frac{\partial^2}{\partial s^2} + s^2$.

(a) Show that $[H, a_\pm] = \pm\hbar\omega_0 a_\pm$ (note that this is of the form of the commutator considered in problem 6.9).

$$\begin{aligned}
[H, a_\pm] &= \left[\left(-\frac{\partial^2}{\partial s^2} + s^2\right), \left(s \mp \frac{\partial}{\partial s}\right)\right] && (6)\\
&= -\left[\frac{\partial^2}{\partial s^2}, s\right] - \left[\frac{\partial^2}{\partial s^2}, \mp\frac{\partial}{\partial s}\right] + \left[s^2, s\right] \mp \left[s^2, \frac{\partial}{\partial s}\right]. && (7)
\end{aligned}$$

The middle two commutators are zero and we only have to evaluate the first and last:

$$\left[\frac{\partial^2}{\partial s^2}, s\right]\psi = \frac{\partial^2}{\partial s^2}\left(s\psi(s)\right) - s\frac{\partial^2\psi}{\partial s^2} = 2\frac{\partial\psi}{\partial s} \tag{8}$$

and

$$\left[s^2, \frac{\partial}{\partial s}\right]\psi = s^2\frac{\partial\psi}{\partial s} - \frac{\partial}{\partial s}\left(s^2\psi\right) = -2s\psi \tag{9}$$

So,

$$[H, a_\pm] = -2\frac{\partial}{\partial s} \pm 2s = \pm 2a_\pm \tag{10}$$

Using the result from problem 6.9, we can now expect that $a_\pm$ will raise or lower the eigenvalue (in this case, the dimensionless quantity, λ) by 2. Since $E = \frac{1}{2}\hbar\omega_0\lambda$, this corresponds to changing the energy by $\pm\hbar\omega_0$. And, of course, using $a_\pm$ on an eigenfunction, $\psi_n(s)$ will give the corresponding new eigenfunction, $\psi_{n\pm1}(s)$ (recall that n is defined through $\lambda = 2n+1$, so changing λ by 2 changes n by 1).

(b) Show that $\phi = a_\pm\psi$ is an energy eigenfunction with eigenvalue $E \pm \hbar\omega_0$ if ψ is an eigenfunction with eigenvalue E.

This follows from problem 6.9. In dimensionless terms, we want to show that eigenvalue λ shifts by 2 by applying $a_\pm$.

From part (a), we have $[H, a_\pm]\psi = \pm 2a_\pm\psi = \pm 2\phi$ and

$$H\phi = Ha_\pm\psi = \pm 2\phi + a_\pm H\psi = \pm 2\phi + a_\pm E\psi = \pm 2\phi + E\phi = (E \pm 2)\phi \quad (11)$$

as we want.

(c) Show that $a_+a_- = H - 1$ (in dimensioned units, the '1' would be $\frac{1}{2}\hbar\omega_0$).

$$\begin{aligned}
a_+a_-\psi &= \left[s - \frac{\partial}{\partial s}\right]\left[s + \frac{\partial}{\partial s}\right]\psi & (12)\\
&= \left[s - \frac{\partial}{\partial s}\right]\left(s\psi + \frac{\partial\psi}{\partial s}\right) & (13)\\
&= s^2\psi + s\frac{\partial\psi}{\partial s} - s\frac{\partial\psi}{\partial s} - \psi - \frac{\partial^2\psi}{\partial s^2} & (14)\\
&= -\frac{\partial^2\psi}{\partial s^2} + s^2\psi - \psi, & (15)
\end{aligned}$$

so, $a_+a_- = -\frac{\partial^2}{\partial s^2} + s^2 - 1 = H - 1$.

(d) If $\psi_0(s)$ is the ground state wavefunction, then $a_-\psi_0(s) = 0$. What is the ground state energy eigenvalue?

$$a_+a_-\psi_0 = (H-1)\psi_0 = 0, \quad (16)$$

so, $H\psi_0 = \psi_0$ and $\lambda_0 = 1$. This means that the ground state energy is $E_0 = \frac{1}{2}\hbar\omega_0$.

(e) $a_-\psi_0(s) = 0$ is a differential equation for $\psi_0(s)$; what is the ground state wavefunction?

The differential equation is

$$a_-\psi_0 = \left(s + \frac{\partial}{\partial s}\right)\psi_0 = 0 \quad (17)$$

or

$$\frac{\partial \psi_0}{\partial s} = -s\psi_0. \tag{18}$$

This is a first order differential equation with the solution

$$\psi_0(s) = Ae^{-s^2/2} \tag{19}$$

as we found previously by solving a significantly more complicated second order differential equation. The solution can be verified by substitution.

Once we have the ground state, we can obtain all others by successive applications of a_+. For example,

$$\begin{aligned}
\psi_1(s) &= a_+\psi_0(s) = \left(s - \frac{\partial}{\partial s}\right)\psi_0(s) = 2s\psi_0(s) = 2se^{-s^2/2} && (20)\\
\psi_2(s) &= a_+\psi_1(s) = \left(s - \frac{\partial}{\partial s}\right)2se^{-s^2/2} && (21)\\
&= (2s^2 + 2s^2 - 2)e^{-s^2/2} = (4s^2 - 2)e^{-s^2/2} && (22)
\end{aligned}$$

Note that we even get the Hermite polynomials in the standard form with 2^n as the coefficient of the highest power term, s^n, in each.