33-234 Quantum Physics $\qquad\qquad$ Spring 2014

Gaussian Integrals

The following treatment can be found in many textbooks, for example, C. Kittel, *Thermal Physics*. We need these integrals in order to calculate expectation values of quantities for the quantum harmonic oscillator. The generalization to the complex plane allows construction of the Gaussian, minimum uncertainty wavepacket.

Integrals of the form

$$I_p(\lambda) = \int_{-\infty}^{\infty} dy\ y^p\ e^{-\lambda y^2}, \tag{1}$$

where p is an integer, occur in many context in physics (classical and quantum wavepackets, thermal physics, ...). For $p = 2l$, l any positive integer or zero, the integrand is even and I_p is twice the integral from 0 to ∞. For $p = 2l + 1$, $I_p(\lambda) = 0$ since the integrand is odd. For general p, $\int_0^\infty$ is related to the gamma function, $\Gamma(z)$.

We start with I_0 which is just the integral of a Gaussian (which is related to the normal distribution encountered in statistics). This can be evaluated by first considering the square of the integral:

$$\begin{aligned} I_0^2 &= \int dy\ e^{-\lambda y^2} \int dx\ e^{-\lambda x^2} && (2) \\ &= \int dy \int dx\ e^{-\lambda(x^2+y^2)}. && (3) \end{aligned}$$

I_0^2 is an integral over the plane which we can re-cast in polar coordinates, (ρ, θ). The area element is $dx\ dy = \rho\ d\theta\ d\rho$. We also let $z = \lambda(x^2 + y^2) = \lambda\rho^2$ and use the relation $dz = d(\lambda\rho^2) = 2\lambda\ \rho\ d\rho$ to evaluate the integral as

$$\begin{aligned} I_0^2 &= \int_0^{2\pi} d\theta \int_0^\infty d\rho\ \rho\ e^{-\lambda\rho^2} && (4) \\ &= 2\pi \int_0^\infty dz\ \frac{1}{2\lambda}\ e^{-z} && (5) \\ &= \frac{\pi}{\lambda}. && (6) \end{aligned}$$

Finally, we have

$$I_0(\lambda) = \sqrt{\frac{\pi}{\lambda}} = \sqrt{\pi}\ \lambda^{-1/2}. \tag{7}$$

A normalized Gaussian probability distribution is then $\sqrt{\frac{\lambda}{\pi}}\ e^{-\lambda y^2}$.

Now we turn to non-zero, even powers, p. The integrals are related to derivatives of I_0 with respect to λ:

$$I_2 = \int_{-\infty}^{\infty} dy\ y^2 e^{-\lambda y^2} = -\frac{dI_0}{d\lambda}. \tag{8}$$

In taking the derivative of I_0, the derivative can be taken inside the integral where it simply brings down a factor of $-y^2$, just what we need for I_2. Since this has to give the same thing as having the derivative outside the integral, we can just differentiate the evaluated form, Eq. 7, to get

$$I_2 = -\frac{d}{d\lambda}\sqrt{\pi}\ \lambda^{-1/2} = \frac{1}{2}\sqrt{\pi}\ \lambda^{-3/2}. \tag{9}$$

Further,

$$I_4 = \frac{d^2}{d\lambda^2} I_0 = \left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\sqrt{\pi}\ \lambda^{-5/2} = \frac{3}{4}\sqrt{\pi}\ \lambda^{-5/2}, \tag{10}$$

and this process can be continued to obtain any $I_p(\lambda)$ needed.