

Supergames with States*

Markus Baldauf[†] Kenneth L. Judd [‡] Joshua Mollner[§]
Sevin Yeltekin[¶]

March 11, 2015

Abstract

This paper studies the pure subgame perfect equilibrium outcomes of supergames with finite states and perfect monitoring. First, we define an order-preserving function, which returns payoffs that are one-period incentive compatible. We then use Tarski's fixed point theorem and the one-stage deviation principle to characterize the average discounted pure subgame perfect equilibrium payoffs as the largest fixed point of this function. We also demonstrate that these equilibrium payoffs can be obtained through iterative application of this function.

1 Introduction

Dynamic considerations are a prevalent feature of strategic decision making in many settings. The idea that today's actions can affect tomorrow's payoffs has been studied in the context of collusive and competitive interaction in virtually every applied discipline of economics. However, the notion that today's actions can change the state of the world that the game is played in has received relatively less attention in applied work. This is mainly a consequence of the increased complexity that arises from the inclusion of a state variable in a dynamic game.

*We thank Tim Bresnahan, Liran Einav, Matthew Jackson and Pietro Tebaldi for useful comments. All remaining errors are ours.

[†]Stanford Univ. Dept of Economics, 579 Serra Mall, Stanford, CA 94305. baldauf@stanford.edu.

[‡]Hoover Institution, 434 Galvez Mall, Stanford, CA 94305. kennethjudd@mac.com.

[§]Stanford Univ. Dept of Economics, 579 Serra Mall, Stanford, CA 94305. jmollner@stanford.edu.

[¶]Tepper School of Business, 5000 Forbes Avenue, Carnegie Mellon University, Pittsburgh, PA 15213. sevin@andrew.cmu.edu.

Traditionally, economists have reverted to extreme assumptions to tackle this complexity. Simplifying techniques utilized include focusing only on a repeated game (without states), assuming symmetry across players, or merely analyzing single-agent decision making over time. Furthermore, in conjunction with these techniques, it is commonplace to simply focus on the Pareto frontier of games by assuming that the discount factor is large enough.

This paper shows how the collection of pure subgame perfect equilibrium payoffs of a large class of dynamic games can be obtained by repeated application of a simple function that eliminates one-shot dominated strategies. Specifically, we consider games with an infinite horizon and finite action space with perfect monitoring where all agents can observe the entire history of the actions of all players. Furthermore, there is a finite number of states that are allowed to change stochastically as the game unfolds. Such games arise naturally in industrial organization; oligopoly settings with research and development, advertising, learning by doing, capacity expansion, entry and exit, among others. Dynamic games also have numerous applications in macroeconomics including, but not limited to, models of time consistent optimal policy and robust policy design, models of sovereign debt strategic default and international trade.

Methodologically our paper builds on the pioneering work of Abreu, Pearce and Stacchetti (Abreu et al., 1986, 1990); we analyze the game recursively in payoff space rather than in action or strategy space. Furthermore, we extend Cronshaw and Luenberger (1990, 1994) by introducing states. In terms of the proof strategy we employ Tarski (1955) to show that there exists a fixed point of a correspondence-valued function, and we invoke the one-stage deviation principle to establish that this fixed point corresponds to the pure subgame perfect equilibrium (SPE) payoffs.¹ Our work is also related to Cole and Kocherlakota (2001) who study a model that is in some respects more general than ours (for example, they allow for private states and imperfect monitoring), but less general in other respects. In particular, while we allow for arbitrary compact action spaces, they restrict attention to the discrete case.

2 Setting

We consider dynamic games with fully observable actions that are played repeatedly by a set $I = \{1, \dots, N\}$ of infinitely-lived players. The game unfolds with simultaneous moves at each stage. In each period, player i chooses an action from some set A_i . An action profile $a = (a_1, \dots, a_N)$

¹This principle, which is essentially the principle of optimality for discounted dynamic programming, is found in many places in the literature, for example Fudenberg and Tirole (1991, Theorem 4.2).

is an element of the set $A = \times_{i=1}^N A_i$.

The state evolves stochastically according to the probability mass function $p : A \times X \rightarrow \Delta(X)$, where X is a finite set with typical element x .

Player i realizes period payoffs according to the function $\Pi_i : A \times X \rightarrow \mathfrak{R}$. The action space for the dynamic game is A^∞ . Agent i 's average discounted payoffs from a specific sequence of states and action profiles are:

$$U_i(a^\infty, x^\infty) = (1 - \delta) \sum_{t=0}^{\infty} \delta^t \Pi_i(a_t, x_t)$$

where $\delta \in (0, 1)$ is the common discount factor across agents.^{2,3} A t -period history, h^t , is a pair of sequences $(\{a_s\}_{s=0}^{t-1}, \{x_s\}_{s=0}^t)$. Let H^t denote the set of t -period histories. A pure strategy for player i is a sequence of functions $\{\sigma_{i,t}\}_{t=0}^{\infty}$ with $\sigma_{i,t} : H^t \rightarrow A_i$.

The results that follow rely on the following three assumptions:

Assumption 1. $\forall i \in I, A_i$ is compact.

Assumption 2. $\forall i \in I, \Pi_i(a, x)$ is continuous in a .

Assumption 3. $p(x'|a, x)$ is continuous in a .

3 Characterization of Equilibrium Payoffs

In this section we follow Cronshaw and Luenberger (1990) and apply Tarski's theorem to repeated games with perfect monitoring and discrete states. We then show that the correspondence that maps the current state into the set of average discounted payoffs that can be sustained in pure SPE can be obtained as the limit of an iterative procedure.

3.1 Tarski's Theorem

A lattice consists of a non-empty set \mathcal{L} and a partial order \leq on \mathcal{L} . The system $\langle \mathcal{L}, \leq \rangle$ is a lattice if for any two elements $a, b \in \mathcal{L}$ there is a least upper bound $a \vee b \in \mathcal{L}$ and a greatest lower bound $a \wedge b \in \mathcal{L}$. The lattice $\langle \mathcal{L}, \leq \rangle$ is complete if every subset L of \mathcal{L} has a least upper bound $\sup L$ and a greatest lower bound $\inf L$.

²For compactness of notation, we consider the case of a common discount factor. However, the arguments can be easily modified to accommodate heterogeneous discount factors.

³Note that average discounted payoffs can be decomposed into a convex combination of current period payoffs, with weight $1 - \delta$, and the average discounted payoffs for the rest of the game.

A function $f : \mathcal{L} \rightarrow \mathcal{L}$ is said to be order-preserving if for all $a, b \in \mathcal{L}$, $a \leq b$ implies that $f(a) \leq f(b)$. We now state a fixed point theorem (Tarski, 1955).

Theorem 1 (Tarski). *Let $\langle \mathcal{L}, \leq \rangle$ be a complete lattice, and let $f : \mathcal{L} \rightarrow \mathcal{L}$ be order-preserving. Let P be the set of fixed points of f . Then $\langle P, \leq \rangle$ is a complete lattice. In particular, there exists a greatest fixed point, $\sup P \in P$, and a smallest fixed point, $\inf P \in P$. Moreover, for all $a \in \mathcal{L}$, $a \leq f(a)$ implies $a \leq \sup P$, and $a \geq f(a)$ implies $a \geq \inf P$.*

3.2 Application of Tarski's theorem to dynamic games

For player i the minimal and maximal payoffs are bounded by the scalars:

$$\begin{aligned}\bar{\Pi}_i &= \max_{(a,x) \in A \times X} \Pi_i(a,x) \\ \underline{\Pi}_i &= \min_{(a,x) \in A \times X} \Pi_i(a,x)\end{aligned}$$

Define \mathcal{W} as the following set of correspondences:

$$\mathcal{W} = \{W : X \rightrightarrows \mathfrak{R}^N \mid \forall x \in X, W(x) \subseteq \times_{i=1}^N [\underline{\Pi}_i, \bar{\Pi}_i]\}$$

We define a partial order \leq on \mathcal{W} as follows. Consider two elements $W, W' \in \mathcal{W}$. We say $W \leq W'$ if $\forall x \in X, W(x) \subseteq W'(x)$.

Lemma 1. *$\langle \mathcal{W}, \leq \rangle$ is a complete lattice.*

Proof. Let $\{W^j \mid j \in \mathcal{J}\}$ be a family of elements of \mathcal{W} . Then the family has a greatest lower bound and least upper bound, defined as $(\inf_{j \in \mathcal{J}} W^j)(x) = \bigcap_{j \in \mathcal{J}} W^j(x)$ and $(\sup_{j \in \mathcal{J}} W^j)(x) = \bigcup_{j \in \mathcal{J}} W^j(x)$, both of which are elements of \mathcal{W} . \square

Definition 1. $B : \mathcal{W} \rightarrow \mathcal{W}$ is the function

$$B(W)(x) = \left\{ (1 - \delta)\Pi(a, x) + \delta \sum_{x' \in X} p(x'|a, x)w(x') \mid w \in W, a \in A, \forall i \in I, IC_i \geq 0 \right\}$$

where

$$\begin{aligned}IC_i &= (1 - \delta)\Pi_i(a, x) + \delta \sum_{x' \in X} p(x'|a, x)w_i(x') \\ &\quad - \max_{\tilde{a} \in A_i} \left[(1 - \delta)\Pi_i(\tilde{a}, a_{-i}, x) + \delta \sum_{x' \in X} p(x'|\tilde{a}, a_{-i}, x) \inf \{w_i(x') \mid w \in W\} \right]\end{aligned}$$

In words, $B(W)(x)$ is the set of possible average discounted payoffs consistent with optimal play in state x today, where continuation values are drawn from W .

Lemma 2. *B is order-preserving.*

Proof. Suppose that $W, W' \in \mathcal{W}$, where $W \leq W'$, and let $z \in B(W)(x)$. Then there exists a pair $(a, w) \in A \times W$ such that

$$z = (1 - \delta)\Pi(a, x) + \delta \sum_{x' \in X} p(x'|a, x)w(x')$$

and $\forall i \in I$

$$z_i \geq \max_{\tilde{a} \in A_i} \left[(1 - \delta)\Pi_i(\tilde{a}, a_{-i}, x) + \delta \sum_{x' \in X} p(x'|\tilde{a}, a_{-i}, x) \inf\{w_i(x') \mid w \in W\} \right].$$

Since $W \leq W'$ it follows that $w \in W'$, and also that $\forall i \in I, \forall x' \in X, \inf\{w_i(x') \mid w \in W'\} \leq \inf\{w_i(x') \mid w \in W\}$. Therefore, $z \in B(W')(x)$. We conclude that $B(W) \leq B(W')$. \square

Lemma 3. *If $W \in \mathcal{W}$ is compact-valued, then $B(W)$ is compact-valued.*

Proof. Let $W \in \mathcal{W}$ be compact-valued, and let $x \in X$. We show that $B(W)(x)$ is (i) bounded, and (ii) closed.

Boundedness. By the definition of B , and the fact that $W \leq \sup \mathcal{W}$,

$$B(W)(x) \subseteq \left\{ (1 - \delta)\Pi(a, x) + \delta \sum_{x' \in X} p(x'|a, x)w(x') \mid w \in \sup \mathcal{W} \right\}.$$

Since $\forall (a, x) \in A \times X, \Pi(a, x) \in \times_{i=1}^N [\underline{\Pi}_i, \overline{\Pi}_i]$, and since $\forall x' \in X, \sup \mathcal{W}(x') = \times_{i=1}^N [\underline{\Pi}_i, \overline{\Pi}_i]$, we obtain $B(W)(x) \subseteq \times_{i=1}^N [\underline{\Pi}_i, \overline{\Pi}_i]$.

Closedness. Let $\{z^n\}_{n=1}^\infty$ be a sequence of points in $B(W)(x)$ converging to some z^∞ . Then $\forall n \in \mathbb{N}, \exists (a^n, w^n) \in A \times W$ such that

$$z^n = (1 - \delta)\Pi(a^n, x) + \delta \sum_{x' \in X} p(x'|a^n, x)w^n(x'),$$

and $\forall i \in I$

$$z_i^n \geq \max_{\tilde{a} \in A_i} \left[(1 - \delta)\Pi_i(\tilde{a}, a_{-i}^n, x) + \delta \sum_{x' \in X} p(x'|\tilde{a}, a_{-i}^n, x) \inf\{w_i(x') \mid w \in W\} \right].$$

Since A is compact and W is a compact-valued correspondence on a finite domain, $\{(a^n, w^n)\}_{n=1}^\infty$ contains a subsequence that converges to some $(a^\infty, w^\infty) \in A \times W$. Since Π and p are both continuous in a , it follows that

$$z^\infty = (1 - \delta)\Pi(a^\infty, x) + \delta \sum_{x' \in X} p(x'|a^\infty, x)w^\infty(x')$$

and $\forall i \in I$

$$z_i^\infty \geq \max_{\tilde{a} \in A_i} \left[(1 - \delta)\Pi_i(\tilde{a}, a_{-i}^\infty, x) + \delta \sum_{x' \in X} p(x'|\tilde{a}, a_{-i}^\infty, x) \inf\{w_i(x') \mid w \in W\} \right].$$

Therefore $z^\infty \in B(W)(x)$, as required. \square

Definition 2. Let $V = \sup \{W \in \mathcal{W} \mid B(W) = W\}$.

In words, V is the supremum of all fixed points of the function B . It follows from Tarski's theorem that V is itself a fixed point (and therefore the largest fixed point) of B .

Lemma 4. Let $W^n = B^n(\sup \mathcal{W})$ for all $n \in \mathbb{N} \cup \{0\}$. The sequence $\{W^n\}_{n=0}^\infty$ (i) is nested, i.e. for all $n \in \mathbb{N}$, $W^n \leq W^{n-1}$, and (ii) converges to V in the limit.

Proof. Nestedness. We first prove the result for $n = 1$. Following the boundedness portion of the proof of lemma 3, we can show that $\forall x \in X$

$$W^1(x) = B(W^0)(x) \subseteq \times_{i=1}^N [\underline{\Pi}_i, \overline{\Pi}_i] = W^0(x).$$

We therefore have that $W^1 \leq W^0$. The result then follows from induction and lemma 2.

Convergence. By inductively applying lemma 3, we can show that each W^n is compact-valued. Therefore, $\{W^n\}_{n=0}^\infty$ is a nested sequence of compact-valued correspondences, which thus converge to the correspondence $W^\infty(x) = \bigcap_{n=0}^\infty W^n(x)$.

Next, we show that $V \leq W^\infty$. Obviously $V \leq \sup \mathcal{W} = W^0$. Using induction and the fact that B is order-preserving (lemma 2), we obtain that for all $n \in \mathbb{N}$, $V = B(V) \leq B(W^n) = W^{n+1}$. Since this holds for all $n \in \mathbb{N}$, it is also true at the limit.

We complete the proof by showing that $W^\infty \leq V$. By Tarski (theorem 1), it suffices to demonstrate that $\forall x \in X$, $W^\infty(x) \subseteq B(W^\infty)(x)$. There is nothing to show if $W^\infty(x) = \emptyset$, so assume otherwise. Let $z \in W^\infty(x)$. Then $\forall n \in \mathbb{N}$, $z \in B(W^n)(x)$ and so $\exists (a^n, w^n) \in A \times W^n$ such that

$$z = (1 - \delta)\Pi(a^n, x) + \delta \sum_{x' \in X} p(x'|a^n, x)w^n(x')$$

and $\forall i \in I$,

$$z_i \geq \max_{\tilde{a} \in A_i} \left[(1 - \delta)\Pi_i(\tilde{a}, a_{-i}^n, x) + \delta \sum_{x' \in X} p(x'|\tilde{a}, a_{-i}^n, x)\underline{w}_i^n(x') \right],$$

where $\underline{w}_i^n(x') = \inf \{w_i(x') \mid w \in W^n\}$. Since A is compact, and because $\{W^n\}_{n=0}^\infty$ is a nested sequence of compact-valued correspondences on a finite domain, $\{(a^n, w^n)\}_{n=1}^\infty$ contains some subsequence $\{(a^{n_k}, w^{n_k})\}_{k=1}^\infty$ with limit $(a^\infty, w^\infty) \in A \times W^0$. In fact it can be shown that $w^\infty \in W^\infty$.⁴ Furthermore, since $\forall x' \in X$ $\{\underline{w}_i^n(x')\}_{n=1}^\infty$ is a monotone non-decreasing

⁴To see this, note that $\forall m \in \mathbb{N}$, $\exists l \in \mathbb{N}$ such that $n_l \geq m$. By nestedness, $\forall k \geq l$, $w^{n_k}(x) \in W^m(x)$. Thus, all but finitely many terms of $\{w^{n_k}(x)\}_{k=1}^\infty$ are in $W^m(x)$. Since $W^m(x)$ is closed, $w^\infty(x) \in W^m(x)$. Since we can repeat this argument $\forall m \in \mathbb{N}$, $w^\infty(x) \in \bigcap_{m=1}^\infty W^m(x) = W^\infty(x)$.

sequence, it also converges, to $\underline{w}_i^\infty(x') = \inf\{w_i(x')|w \in W^\infty\}$. Since Π and p are both continuous in a , it follows that

$$z = (1 - \delta)\Pi(a^\infty, x) + \delta \sum_{x' \in X} p(x'|a^\infty, x)w^\infty(x')$$

and $\forall i \in I$,

$$z_i \geq \max_{\tilde{a} \in A_i} \left[(1 - \delta)\Pi_i(\tilde{a}, a_{-i}^\infty, x) + \delta \sum_{x' \in X} p(x'|\tilde{a}, a_{-i}^\infty, x)\underline{w}_i^\infty(x') \right].$$

Therefore $z \in B(W^\infty)(x)$, as required. \square

Lemma 5. *V is compact-valued.*

Proof. For all $x \in X$, $V(x) \subseteq \times_{i=1}^N [\underline{\Pi}_i, \overline{\Pi}_i]$ and is therefore bounded. Therefore it only remains to show that V is closed-valued. Since V is a fixed point of B , and since B is order-preserving (by lemma 2), $V = B(V) \leq B(\text{cl}(V))$, where $\text{cl}(\cdot)$ denotes component-wise closure. Applying this closure operator to both the left and right hand sides of the previous equation, $\text{cl}(V) \leq \text{cl}(B(\text{cl}(V)))$. We have seen that $\forall x \in X$ $V(x)$ is bounded; $\text{cl}(V)$ is therefore compact-valued. Applying lemma 3, $\text{cl}(B(\text{cl}(V))) = B(\text{cl}(V))$. Therefore $\text{cl}(V) \leq B(\text{cl}(V))$. Then by Tarski's theorem and the definition of V , $\text{cl}(V) \leq V$, which implies that V is closed-valued. \square

3.3 Game theoretic interpretation of V

We use the one-stage deviation principle for infinite horizon games, which provides a useful characterization of SPE. This principle applies to games that are continuous at infinity, or games in which events in the distant future are vanishingly unimportant. This condition is satisfied, for example, in games where overall payoffs are a discounted sum of uniformly bounded stage payoffs, as is the case in our setting. The following statement appears in Fudenberg and Tirole (1991, Theorem 4.2).⁵

Theorem 2. *In an infinite-horizon multi-stage game with observed actions that is continuous at infinity, profile s is subgame perfect if and only if there is no player i and strategy \hat{s}_i that agrees with s_i except at a single t and h^t , and such that \hat{s}_i is a better response to s_{-i} than s_i conditional on history h^t being reached.*

Definition 3. Let V^* denote the correspondence that maps the current state into the set of expected average discounted payoffs that can be sustained in pure SPE.

⁵The games we study may feature stochastic state transitions. These can be accommodated within the formulation of Fudenberg and Tirole (1991) by thinking of nature as a player who is indifferent among all outcomes.

An immediate consequence of the one-stage deviation principle is that V^* is equivalent to the correspondence that maps the current state into the set of expected average discounted payoffs that can be sustained by strategy profiles in which no player has a profitable one-stage deviation.

Furthermore, $V^* \in \mathcal{W}$, as period payoffs are contained within $\times_{i=1}^N [\underline{\Pi}_i, \overline{\Pi}_i]$, and therefore average discounted payoffs are as well.

Lemma 6. $V^* \leq B(V^*)$.

Proof. Suppose that $z \in V^*(x)$. Then there exists some pure SPE, σ , such that when the initial state is x , the expected average discounted payoffs are z . Let $a \in A$ be the action profile prescribed by σ when the initial state is x . Let $v(x')$ denote the expected continuation values the players receive in equilibrium if the state transitions to x' in the next period. Then

$$z = (1 - \delta)\Pi(a, x) + \delta \sum_{x' \in X} p(x'|a, x)v(x').$$

Subgame perfection requires that $\forall x' \in X$, $v(x') \in V^*(x')$. It also requires that continuation values be drawn from V^* if some player i deviates from a_i to \tilde{a} . Since no such deviation from equilibrium may be profitable, we must have

$$z_i \geq \max_{\tilde{a} \in A_i} \left[(1 - \delta)\Pi_i(\tilde{a}, a_{-i}, x) + \delta \sum_{x' \in X} p(x'|\tilde{a}, a_{-i}, x) \inf\{w_i(x') \mid w \in V^*\} \right].$$

It is therefore the case that $z \in B(V^*)(x)$. \square

Lemma 7. $V \leq V^*$.

Proof. Because V is a fixed point of B , $\forall (u, x) \in V \times X$, there exist $a(u, x) \in A$ and $v(u, x) \in V$ such that the following hold

$$u(x) = (1 - \delta)\Pi(a(u, x), x) + \delta \sum_{x' \in X} p(x'|a(u, x), x)v(u, x)(x') \quad (*)$$

and $\forall i \in I$,

$$u_i(x) \geq \max_{\tilde{a} \in A_i} \left[(1 - \delta)\Pi_i(\tilde{a}, a_{-i}(u, x), x) + \delta \sum_{x' \in X} p(x'|\tilde{a}, a_{-i}(u, x), x) \inf\{v_i(x') \mid v \in V\} \right] \quad (**)$$

Let $u_0 \in V$. We recursively construct a pure strategy profile σ in the following way. First, $\forall i \in I$ let $\sigma_{i,0}(h_0) = a_i(u_0, x_0)$. Second, for $t \geq 1$, we say that a player j deviated at history h_{t-1} if $a_{j,t-1} \neq \sigma_{j,t-1}(h_{t-1})$. Let $u_t(h^t) = v(u_{t-1}(h_{t-1}), x_{t-1})$ if there were either zero or multiple deviations at history h_{t-1} . If only player j deviated at history h_{t-1} ,

then $\forall x \in X$ let $u_t(h^t)(x) \in \arg \min\{v_j(x)|v \in V\}$. Since V is compact-valued (by lemma 5), this is well-defined. Then $\forall i \in I$ let $\sigma_{i,t}(h^t) = a_i(u_t(h^t), x_t)$.

We conclude the proof by showing that that under σ , (i) the expected equilibrium average discounted payoffs are given by u_0 , and (ii) no player has a profitable one-stage deviation. This will demonstrate that the payoffs in V can be sustained by strategy profiles in which no player has a profitable one-stage deviation, which combined with the one-stage deviation principle, will imply that $V \leq V^*$.

Proof of (i). For any initial state $x_0 \in X$, define $\hat{u}(t)$ as the profile of expected equilibrium continuation values at time t and $\hat{\Pi}(t)$ as the profile of expected equilibrium stage payoffs at time t , both conditional on x_0 :

$$\hat{u}(t) = \begin{cases} u_0(x_0) & \text{if } t = 0 \\ \mathbb{E}[v(v(\dots v(v(u_0, x_0), x_1)\dots), x_{t-1})(x_t)|x_0] & \text{if } t \geq 1 \end{cases}$$

$$\hat{\Pi}(t) = \begin{cases} \Pi(a(u_0, x_0), x_0) & \text{if } t = 0 \\ \mathbb{E}[\Pi(a(v(v(\dots v(v(u_0, x_0), x_1)\dots), x_{t-1}), x_t), x_t)|x_0] & \text{if } t \geq 1 \end{cases}$$

Applying the law of iterated expectations to (\star) , we obtain $\hat{u}(t) = (1 - \delta)\hat{\Pi}(t) + \delta\hat{u}(t + 1)$. Multiplying both sides by δ^t , summing from 0 to T , and canceling terms appearing on both sides,

$$\hat{u}(0) = (1 - \delta) \sum_{t=0}^T \delta^t \hat{\Pi}(t) + \delta^{T+1} \hat{u}(T + 1).$$

Then taking the limit as $T \rightarrow \infty$ gives

$$u_0(x_0) = (1 - \delta) \sum_{t=0}^{\infty} \delta^t \hat{\Pi}(t),$$

which establishes that the expected equilibrium average discounted payoffs are given by u_0 .

Proof of (ii). Suppose that some player j makes a one-stage deviation at history h^t to the action \tilde{a} . Then, letting $\Pr(h^t)$ denote the probability that history h^t is reached on the equilibrium path, his change in average discounted expected utility is

$$\Pr(h^t)\delta^t [u_{j,t}(h^t) - \Pi_j(\tilde{a}, a_{-j}(u_t(h^t), x_t), x_t) - \delta \sum_{x' \in X} p(x'|(\tilde{a}, a_{-j}(u_t(h^t), x_t), x_t)) \inf\{v_j(x')|v \in V\}].$$

which is nonpositive by $(\star\star)$. □

Now we state the main result of this paper.

Theorem 3. V^* is the largest fixed point of the function B . Furthermore, $V^* = \lim_{n \rightarrow \infty} B^n(\sup \mathcal{W})$.

Proof. $V^* \leq V$ follows from lemma 6 and Tarski's theorem. Together with lemma 7, we obtain $V = V^*$. The result then follows immediately from the definition of V and lemma 4(ii). \square

4 Conclusions

In this paper we have shown how to characterize the payoff correspondence of pure SPE of infinitely repeated dynamic games with discrete states and perfect monitoring. Such games have been applied extensively in many fields, particularly in industrial organization. Many of the basic oligopoly settings enjoy natural formulations as dynamic games, including research and development, strategic pricing over time, advertising, network externalities, consumer switching costs, learning by doing, capacity expansion, firm-level capital accumulation, among others. In recent years, dynamic (stochastic) games have been used to model strategic industry interactions with entry and exit.

Macroeconomics has also been a fertile area for the application of the theory of dynamic games (in both discrete time and continuous time) as many important questions in macroeconomics can be recast as recursive dynamic games. These include but are not limited to dynamic general equilibrium models of time consistent optimal policy and robust policy design, models of sovereign debt and debt renegotiation with strategic default, international trade and environmental macroeconomics. In fact, since all policy design is part of a strategic game and economic activity takes place over time, dynamic game theory tools are needed for virtually all policy analyses. This paper lays the foundation for new recursive methods for analyzing such policies and dynamic interactions. In future work, we plan to extend the setting to incorporate continuous state variables and imperfect monitoring.

References

- Abreu, Dilip, David Pearce, and Ennio Stacchetti**, “Optimal cartel equilibria with imperfect monitoring,” *Journal of Economic Theory*, 1986, 39 (1), 251–269.
- , —, and —, “Toward a theory of discounted repeated games with imperfect monitoring,” *Econometrica: Journal of the Econometric Society*, 1990, pp. 1041–1063.
- Cole, Harold L and Narayana Kocherlakota**, “Dynamic games with hidden actions and hidden states,” *Journal of Economic Theory*, 2001, 98 (1), 114–126.
- Cronshaw, Mark and David G Luenberger**, “Subgame Perfect Equilibria in Infinitely Repeated Games with Perfect Monitoring and Discounting,” Technical Report, University of Colorado Working Paper 1990.
- and —, “Strongly symmetric subgame perfect equilibria in infinitely repeated games with perfect monitoring and discounting,” *Games and Economic Behavior*, 1994, 6 (2), 220–237.
- Fudenberg, Drew and Jean Tirole**, *Game Theory*, MIT Press, 1991.
- Tarski, Alfred**, “A lattice-theoretical fixpoint theorem and its applications,” *Pacific journal of Mathematics*, 1955, 5 (2), 285–309.