Some regularity results in geometric measure theory

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> "*É necessário sempre acreditar que um sonho é possível. Que o céu é o limite e você, truta, é imbatível.*"
>
> *Racionais MC's*

To my wife and parents, I dedicate the following words (in portuguese, so they can read them).

Aos meus pais eu devo tudo que tenho na vida! Nada disso seria possível sem os anos em que meu pai trabalhou em dois empregos (sem um dia de folga) para conseguir pagar meu curso pré-vestibular, sem os anos que minha mãe dedicou como dona de casa e cuidando da minha educação. Eu agradeço muito a eles por sempre terem confiado em mim e nesse caminho que escolhi trilhar. O fato deles nunca terem frequentado sequer o ensino fundamental (e, portanto, não entenderem bem como o estudo pode gerar frutos no longo prazo) nunca os fez colocar empecilhos em minhas escolhas, sempre me apoiaram e sacrificaram muitas coisas na vida deles para que esses objetivos tenham sido alcançados.

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Abstract

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During the author's PhD, the following single-named contributions were made: [RdO22] and [Res23]. Moreover the following articles were written together with his collaborators: [ACN+22], [NR22], and [RR23]. Since the topics of these works are substantially different, we will present this PhD thesis as a composition of some regularity theorems in geometric measure theory. Specifically, this thesis is an exposition of the results in [NR22], [Res23] and [RR23]. Such papers are unlike in nature, considerably different techniques and theories are used in each of them, however they share the same goal: proving regularity for measure-theoretic objects solving variational problems.

Keywords: geometric measure theory, regularity theory, varifolds, currents, finite perimeter sets, Plateau problem, area minimizing currents, stationary varifolds, anisotropic functionals.

Contents

Part A

Introduction

Chapter 1

Some words about the general development of this thesis

All the results attained in this work lies within the vast field of geometric measure theory (GMT). Since the techniques used and the problems sought are substantially different in nature, our first step is to give a brief introduction of what GMT is. After that, a little introduction and explanation about the work evolved by the author in

> [RdO22]: R. Resende. (2022). On clusters and the multi-isoperimetric profile in Riemannian manifolds with bounded geometry. Journal of Dynamical and Control Systems, 1-23,

and, together with collaborators, in

> [ACN+22]: J. H. Andrade, J. Conrado, S. Nardulli, P. Piccione, R. Resende. (2022). Multiplicity of solutions to the multiphasic Allen-Cahn-Hilliard system with a small volume constraint on closed parallelizable manifolds. arXiv preprint arXiv:2203.05034.,

will be given. Unfortunately, these two articles will be left out of this thesis to preserve its homogeneity of topics and its 'reasonable size'. This thesis will encompass the achievements of the following articles:

> [NR22]: S. Nardulli, R. Resende. (2022). Density of the boundary regular set of 2d area minimizing currents with arbitrary codimension and multiplicity. arXiv preprint arXiv:2204.11947.

> [RR23]: A. De Rosa and R. Resende. (2023). Boundary regularity for anisotropic minimal Lipschitz graphs. arXiv preprint arXiv:2305.11258.

> [Res23]: R. Resende. (2023). Lipschitz approximation for general almost area minimizing currents. Submitted to arXiv.

The last three cited works are directly related to regularity questions in GMT. By the author's choice, the regularity theory shall constitute the main topic of this thesis.

Chapter 2

Geometric measure theory

Quoting Frank Morgan's book ([Mor09]):

> *"Singular geometry governs the physical universe: soap bubble clusters meeting along singular curves, black holes, defects in materials, chaotic turbulence, crystal growth. The governing principle is often some kind of energy minimization. Geometric measure theory provides a general framework for understanding such minimal shapes,* a priori *allowing any imaginable singularity and then proving that only certain kinds of structures occur."*

Historically, GMT was born out of the desire to solve Plateau's problem (named after the physicist Joseph Plateau) which asks if, for every smooth closed curve in $\mathbb{R}^3$, there exists a surface of least area among all surfaces whose boundary equals the given curve.

Plateau experimented the following: when one dips a piece of wire with Jordan's curve shape in bucket containing a soap solution, the shape of the film that forms after pulling out the wire is the surface that minimizes either locally or absolutely the superficial tension of the film; neglecting the gravitational effects (taking into account the fact that the mass of the film is tiny), this is equivalent to minimize the area of the film; thus the referred surface has to be minimal. A solution of the problem is understood as an object S such that:

$$\text{Area}(S) = \inf \{\text{Area}(S_0) : \partial S_0 = \partial S\},$$

where, of course, we have to make precise: what is the class of competitors S_0, the notion of Area being considered, and which boundary operator ∂ we are selecting.

Great mathematicians studied this problem during the last two centuries. However, the first rigorous solution came up only in 1930 in the independent works of Jesse Douglas and Tibor Radò. Where they proved that, for all Jordan curve γ in $\mathbb{R}^3$, there is a surface with the topological type of a disc which has γ as its boundary and minimizes area. The solution given by Douglas impressed the mathematical community with its simplicity and beauty, using the direct method of the Calculus of Variations and reducing the problem of minimizing the area to the problem of minimizing the Dirichlet integral. However, the solution of Douglas/Radò minimizes area only on the universe of surfaces with the topological type of a disc.

Afterward, in 1956, W.H. Fleming built-in $\mathbb{R}^3$ a Jordan's curve that is not the boundary of any surface of a finite topological type of minimal area. This situation cannot be adequately treated with the classical methods developed by Douglas/Radò, involving comparisons among surfaces with the same topological type. Among other facts, this motivated the emergence of weak definitions of ”surface”, in a nonclassical sense, and new approaches to the problem, involving classical differential geometry combined with the theory of distributions and measure theory, which is nowadays called Geometric Measure Theory. Such methods were developed by many authors starting in the 60s; they were developed in three branches:

- **Finite perimeter sets.** De Giorgi with the perimeter theory [DG54, DG55, DG61]. This theory is appropriate to study geometric variational problems in codimension 1, defining a surface as the boundary of a finite perimeter set, also called Caccioppoli sets;
- **Currents.** Federer and Fleming [FF60] with the theory of integral currents, designed to work in higher codimension, density, and multiplicity. The theory of current is extensively studied in Ferderer's masterpiece book [Fed69];
- **Varifolds.** In [Alm68] and [All72b] the authors defined a k-dimensional surface to be a k-dimensional varifold, i.e., a Radon measure in $\mathbb{R}^n \times G(n;k)$, where $G(n;k)$ denotes the Grassmann manifold of k-dimensional planes in $\mathbb{R}^n$. Varifolds are more general than currents due to the fact that currents are coupled with an orientation while varifolds are not.

We mention that one can also consider general sets, several results are available in this direction, however, since none of the works in this text use this level of generality, we will use the three definitions in the items above.

There are two main topics in which a satisfactory variational theory, such as the minimization problems, in particular the Plateau's problem, should cover: existence and regularity. It is clear that these are not the only important aspects of the theory, they are essential though.

- **Existence.** With the notions of surface, spanning and k-dimensional volume fixed in the formulation of the problem, can we prove the existence of minimizers for the oriented Plateau problem?;
- **Regularity.** What can we infer about the regularity of such minimizers of the oriented Plateau problem? For instance: can we approximate them by classical smooth submanifolds? Are themselves classical surfaces, i.e. C^1? If there exists a set where the minimizer is not regular, can we estimate the size of this set or prove some weak regularity property for it?

One of the main accomplishments of GMT's theory is that, under fairly general assumptions, one can prove good convergence, compactness, and existence, results. Meaning that geometric variational problems have solutions in the space being considered (space of Caccioppoli sets, currents, or varifolds). These results are commonly derived by the well-known direct method of the calculus of variations. Such features dramatically fails in the classical space of smooth manifolds.

Since existence and convergence results are, by now, well-established due to several works, see for instance [All72a], [FF60], [Giu84]. For a detailed far-reaching exposition on this topics, we refer the reader to [Mag12], [Mor09], [Fed69], and the references therein.

Chapter 3

A few words about isoperimetry

The goal of this chapter is to present **in a nutshell** the works evolved in [RdO22] and [ACN+22].

The isoperimetric problem is a long-standing problem in mathematics. By an isoperimetric problem one means a geometric variational problem in which one tries to find a hypersurface that encloses a fixed volume and has least perimeter. Similarly to the Plateau problem, as explained in chapter 2, GMT plays a major role in the study of solution for the isoperimetric problem, the so-called isoperimetric sets. Since existence of isoperimetric sets in the space of differentiable manifolds is not generally true, one has to consider spaces where, at least, existence in known.

Denoting by $\mathrm{P}(E)$ the perimeter of a Caccioppoli set $E \subset \mathbb{R}^n$ and $|E|$ its n-dimensional Lebesgue measure, one says that E is an isoperimetric set if the following holds true

$$\mathrm{P}(E) = \inf \{\mathrm{P}(E_0) : |E_0| = |E| \text{ where } E_0 \text{ is a Caccioppoli set}\}.$$

Satisfactory solutions for such isoperimetric problems in $\mathbb{R}^n$ were derived using methods of GMT developed by Almgren, Federer, Fleming, De Giorgi, and Reifenberg. It is also well-known that in the Euclidean space, balls are the only isoperimetric sets.

The problem of existence of isoperimetric regions has been widely considered in Riemannian manifolds, instead of in the Euclidean space. Sutdying this problem in general Riemannian manifolds is much more involved, in fact, one can define Caccioppoli sets in a Riemannian manifold (M, g). Of course, the notion of perimeter and volume will depend on the metric g chosen. The problem is posed as follows:

$$\mathrm{P}_g(E) = \inf \{\mathrm{P}_g(E_0) : \mathrm{Vol}_g(E_0) = \mathrm{Vol}_g(E) \text{ where } E_0 \text{ is a Caccioppoli set w.r.t } g\}.$$

Classical compactness results of GMT ensure existence in **compact** manifolds (M, g), we have existence of isoperimetric sets, it is nowadays well-known, see [Mor09].

The existence of isoperimetric regions in noncompact Riemannian manifolds (M, g) is not an easy task. However, we can find papers in this directions which give pretty good answers in some specific types of manifolds. For an example, in [GR12], the authors proved the existence of isoperimetric regions to the case of noncompact sub-Riemannian manifolds with cocompact isometry group. For the Riemannian setting, we refer the reader to [Mor03], [Rit01], [RR04], [CR08], [Nar18], [Nar09], [MnFN19] and [Nar14]. For more details on regularity theory see either [Mor03] or [Mor09]. According to these references, we see that we need some condition on the geometry of the manifold (M, g)

in order to prove existence of isoperimetric sets. Since in [NP18], it is provided a counterexample, i.e., a manifold which does not satisfy the **bounded geometry conditions** and hence does not contain isoperimetric regions for some volumes. A Riemannian manifold is said to have **bounded geometry** if there exists a constant $k \in \mathbb{R}$, such that $Ric_g \geq k(n-1)$ (i.e., $Ric_g \geq k(n-1)g$ in the sense of quadratic forms) and $\mathrm{Vol}_g(B_g(p, inj_M)) \geq v_0$ for some positive constant v_0.

Furthermore, we have seen a rapidly development of the theory of metric measure spaces, more specifically, the RCD-spaces, where RCD stands for Riemannian curvature-dimension condition. Several results concerning existence of isoperimetric sets were generalized to these RCD-spaces that encompass the class of Riemannian manifolds with bounded geometry. The interested reader can consult the following survey [Poz23] and the references therein, or [APP22] and [ANP22].

3.1 Partitioning problem

Following the lines of the isoperimetric problem, one can ask whether or not it is possible to solve the problem of partitioning a space into several pieces with a pescribed fixed volume. This is a celebrated and long-standing question for which existence in Euclidean spaces and **compact** Riemannian manifolds is well-kown (for the same reasons explained in chapter 3). To be more precise, we set the following definition.

> Let (M^n, g) be a Riemannian manifold of dimension n. An **N-cluster** $\mathcal{E}$ in (M^n, g) is a finite family of Caccioppoli sets $\mathcal{E} := \{\mathcal{E}(h)\}_{h=1}^N$ with $0 < \mathrm{Vol}_g(\mathcal{E}(h)) < +\infty, 1 \leq h \leq N$, and $\mathrm{Vol}_g(\mathcal{E}(h) \cap \mathcal{E}(k)) = 0, 1 \leq h < k \leq N$.

Every N-cluster is a **partition** of M if we consider it united with its 'exterior chamber', i.e., $\mathcal{E}(0) = M \setminus \bigcup_{h=1}^N \mathcal{E}(h)$. Thus, we can state the **multi-isoperimetric problem** as the minimization problem:

$$\frac{1}{2}\sum_{h=1}^N \mathrm{P}_g(\mathcal{E}(h)) = \inf\left\{\frac{1}{2}\sum_{h=1}^N \mathrm{P}_g(\mathcal{E}'(h)) : \mathcal{E}' \text{ is an N-cluster with } \mathbf{v}(\mathcal{E}') = \mathbf{v}\right\},$$

where $\mathbf{v}(\mathcal{E}') := (\mathrm{Vol}_g(\mathcal{E}'(1)), \ldots, \mathrm{Vol}_g(\mathcal{E}'(N)))$ and $\mathbf{v} \in \mathbb{R}_+^N$ is a fixed vector-volume.

In [RdO22], among other results, I prove the existence of isoperimetric clusters for prescribed vector-volume, i.e. minimizers of variational problem above, in a complete Riemannian manifold, assuming the bounded geometry condition. This is a generalization of all results aforementioned, since, for $N = 1$, the notion of isoperimetric cluster and isoperimetric set coincide.

3.2 Multi-phasic Allen-Cahn-Hilliard system

It is well known that the Allen-Cahn-Hilliard (ACH) **equation** is related (when the temperature goes to zero) to the isoperimetric problem. In [ACN+22], we explore this relation to generate solutions for the ACH **system**. Let us be more precise.

Given a **compact** Riemannian manifold (M, g), we study the existence and multiplicity of vectorial m-map solutions $\mathbf{u} = (\mathrm{u}_1, \ldots, \mathrm{u}_m) \in C^\infty_g(M, \mathbb{R}^m)$ to the following ACH system

$$\begin{cases} d\boldsymbol{\mathcal{E}}_{\varepsilon,\boldsymbol{W}}(\mathbf{u}) = \boldsymbol{\Lambda} \text{ on } M, \\ \boldsymbol{\mathcal{V}}_g(\mathbf{u}) = \mathbf{v}, \end{cases} \qquad \boldsymbol{\mathcal{E}}_{\varepsilon,\boldsymbol{W}}(\mathbf{u}) := \begin{cases} \int_M \left(\varepsilon|\nabla_g \mathbf{u}|^2 + \varepsilon^{-1}\boldsymbol{W}(\mathbf{u})\right) d\mathrm{Vol}_g, & \text{if } \mathbf{u} \in \mathfrak{M}_{\mathbf{v}} \\ \infty, & \text{if } \mathbf{u} \in L^1_g(M, \mathbb{R}^m) \setminus \mathfrak{M}_{\mathbf{v}}. \end{cases}$$

The functional $\boldsymbol{\mathcal{E}}_{\varepsilon,\boldsymbol{W}} : L^1_g(M, \mathbb{R}^m) \to \overline{\mathbb{R}}$ is called the **vectorial ACH energy**, $\boldsymbol{\Lambda}\mathbb{R}^m$ is a Lagrange multiplier, $\boldsymbol{\mathcal{V}}_g$ is the volume functional given by

$$\boldsymbol{\mathcal{V}}_g(\mathbf{u}) = \left(\int_M \mathrm{u}_1 d\mathrm{Vol}_g, \ldots, \int_M \mathrm{u}_m d\mathrm{Vol}_g\right),$$

$\mathbf{v} = (\mathrm{v}_1, \ldots, \mathrm{v}_m) \in \mathbb{R}^m_+$ with $\mathrm{v} = |\mathbf{v}| := \sum_{i=1}^m \mathrm{v}_i \ll 1$, and $\nabla_g \mathbf{u} = (\nabla_g \mathrm{u}_1, \ldots, \nabla_g \mathrm{u}_m)$, $0 < \varepsilon \ll 1$ is the **temperature** parameter, $\boldsymbol{W} \in C^\infty(\mathbb{R}^m)$ is a multi-well (multiphasic) potential vanishing at a finite set of (global) minima points $\mathcal{Z} \subset \mathbb{R}^m_+$ containing the origin and such that $\#\mathcal{Z} = N$, and

$$\mathfrak{M}_{\mathbf{v}} = \left\{\mathbf{u} \in W^{1,2}_g(M, \mathbb{R}^m) : \boldsymbol{\mathcal{V}}_g(\mathbf{u}) = \mathbf{v}\right\}.$$

For the case of the ACH *equation*, it is well-known that the ACH energy Γ-converges to the perimeter (introduced in the sense of chapter 3) as the temperature ε goes to 0. This is the cornerstone to relate the ACH energy with the ancient problem of finding minimal (in our case CMC (constant mean curvature)!) hypersurfaces in a Riemannian manifold. Let us explain better this connection.

An ancient problem in this differential geometry is to determine the number of critical points of the perimeter functional P_g on a general Riemannian manifold. Indeed, Yau conjectured that any closed Riemannian manifold contains infinitely many critical points of the perimeter. This conjecture was recently solved affirmatively for generic metrics [MNS19, Li19] and for the remaining cases when $3 \leqslant n \leqslant 7$ [Son23]. Recall that critical points of the perimeter are called minimal hypersurfaces and they must have zero mean curvature, meanwhile isoperimetric sets are CMC hypersurfaces, or CMC boundaries.

The convergence of the ACH energy to the perimeter functional was recently used to construct minimal hypersurfaces in any closed Riemannian manifold [Gua18, GG18] as an alternative approach to min-max methods [Alm65, Pit81]. For CMC boundaries, it is only known the min-max construction in [Dey19, Dey22, ZZ19]. It is not known whether or not Yau's conjecture also holds true for CMC hypersurfaces.

In [ACN+22], instead of considering the ACH equation, we consider the ACH *system* and we explore the relations between it and the multi-isoperimetric problem (partitioning problem in section 3.1). In fact, we can also prove the Γ-convergence of the energy $\boldsymbol{\mathcal{E}}_{\varepsilon,\boldsymbol{W}}(\mathbf{u})$ to the **weighted** perimeter of a cluster as ε goes to 0. Several new difficulties are overcome in this paper, since the convergence is to a weighted perimeter which *does* not have crucial properties satisfied by the non-weighted perimeter. Among other results, we prove a lower bound for the number of solutions of the ACH system under suitable conditions on M and the multiphasic potential $\boldsymbol{W}$.

Chapter 4

Brief introduction to Part B

4.1 Stationary varifolds

This chapter's main goal is to present a historical overview of the results concerning stationary varifolds (more generally, with p-integrable generalized mean curvature) where the notion of stationarity also varies depending on the functional being taken. This topic relates directly to the results achieved in [RR23] described in part B. Even though the results there are for varifolds with p-integrable generalized mean curvature, we will focus on stationary varifolds in this introduction.

As mentioned in chapter 2, a m-varifold $\mathbf{V}$ is a Radon measure on $\mathbb{R}^{m+n} \times G(m+n, m)$ where $G(m+n, m)$ denote the Grassmann manifold of m-dimensional planes in $\mathbb{R}^{m+n}$. The varifolds are a natural generalization of **non-oriented** manifolds, for a nice, detailed, and elucidative introduction we refer the reader to [All87] and [Alm65].

Given $U \subset \mathbb{R}^{m+n}$, for every diffeomorphism $\psi \in C^1_c(U, \mathbb{R}^{m+n})$, the push-forward $\psi^{\#}\mathbf{V}$ of $\mathbf{V}$ with respect to ψ is also an m-varifold if defined as

$$\int_{\mathbf{Gr}(U)} \Phi(x,\pi) d(\psi^{\#}\mathbf{V})(x,\pi) = \int_{\mathbf{Gr}(U)} \Phi(\psi(x), d_x\psi(\pi)) J\psi(x,\pi) d\mathbf{V}(x,\pi), \ \forall \Phi \in C^0_c(\mathbf{Gr}(U)).$$

Here $d_x\psi(\pi)$ denotes the image of π under the map $d_x\psi(x)$ and $J\psi(x,\pi)$ is the m-Jacobian determinant of the differential $d_x\psi$ restricted to π, see [Sim14, Chapter 8].

We define the **first variation of V** as follows. Denote $\|\mathbf{V}\|$ the total variation of the Radon measure $\mathbf{V}$, given $g \in C^1_c(U, \mathbb{R}^n)$, we define

$$\delta \mathbf{V}(g) := \frac{\mathrm{d}}{\mathrm{d}t} \|\phi_t^{\#}\mathbf{V}\|(U)\Big|_{t=0}, \text{ where } \phi_t(x) := x + tg(x).$$

A varifold $\mathbf{V}$ is said to be **stationary** if $\delta\mathbf{V} \equiv 0$. This is clearly a generalization of the definition of *smooth* minimal manifolds.

The varifolds can be pretty wild in nature. Nonetheless, when we impose a variational obstruction on them (for instance, stationarity), we can control their behaviour in a regularity viewpoint. Indeed, denote the density of the varifold $\mathbf{V}$ at p by $\Theta^m(\mathbf{V}, p) := \lim_{r\to 0} \frac{\|\mathbf{V}\|(\mathbf{B}(p,r))}{\omega_m r^m}$, in [All72a], Allard proves his celebrated ε-regularity theorem which we loosely (to avoid technicalities) state below:

> Given a *stationary* integral m-varifold $\mathbf{V}$, $r > 0$, and $p \in \mathbf{V}$, satisfying $1 \leq \Theta^m(\mathbf{V}, q)$ for $\|\mathbf{V}\|$-a.e. $q \in \mathbf{B}(p, r)$ and $\|\mathbf{V}\|(\mathbf{B}(p, r)) \leq (1 + \varepsilon)\omega_m r^m$, there exists $r_0 > 0$ such that $\operatorname{spt}(\|\mathbf{V}\|) \cap \mathbf{B}(p, r_0)$ is a C^1-submanifold.

Moreover, in [All75], Allard gave a notion of boundary of varifolds based on the definition of first variation. Such notion surely encompasses the classical notion of a manifold spanning a boundary in the sense of smooth geometry. For this notion of boundary, he also obtained an ε-regularity theorem which we state below:

> Let $\mathbf{V}$ be a *stationary* integral m-varifold, $\alpha \in (0, 1)$, $r > 0$, $p \in \Gamma$, Γ a $C^{1,\alpha}$-'boundary' for $\mathbf{V}$. Assume that $1 \leq \Theta^m(\mathbf{V}, q)$ for $\|\mathbf{V}\|$-a.e. $q \in \mathbf{B}(p, r)$ and $\|\mathbf{V}\|(\mathbf{B}(p, r)) \leq (\frac{1}{2} + \varepsilon)\omega_m r^m$, then there exists $r_0 > 0$ such that $\operatorname{spt}(\|\mathbf{V}\|) \cap \mathbf{B}(p, r_0)$ is a C^1-submanifold.

A remark on the statement above, Allard originally proved the statement for $\Gamma \in C^{1,1}$, a few decades later, it was generalized for $\Gamma \in C^{1,\alpha}$ in [Bou16].

The majority of the regularity results in GMT, for instance both quoted above, take advantage of well-known techniques as blowing-up, excess decays, Caccioppoli-type inequalities ($L^2 - L^\infty$ inequalities), etc. Several of these tools already appear in PDEs and in the calculus of variations. Even though, these techniques are very powerful in GMT, they deeply rely on the so-called monotonicity formula. It states that if $\mathbf{V}$ is stationary then:

$$\frac{\|\mathbf{V}\|(\mathbf{B}(p, r))}{\omega_m r^m} - \Theta^m(\mathbf{V}, p)\omega_m = \int_{\mathbf{B}(p,r) \times G(m+n,m)} \frac{|\mathrm{proj}_{\pi^\perp}(q - p)|^2}{|p - q|^{m+2}} d\mathbf{V}(q, \pi).$$

This is a poor-man version of the monotonicity formula and the consequences that we will mention are just some instances of the extremely strong implications one can derive from it.

One of the main consequences of the monotonicity formula is that the blowups of these stationary varifolds are **cones**, i.e., varifolds that are invariant under homotheties!!! Now, working with blowups become such a powerful tool, since we have way easier objects to study at the limit. Another great implication of the monotonicity formula is that the density Θ^m is an upper semicontinuous function.

4.2 Anisotropic stationary varifolds

In section 4.1, we considered the first variation related to the area functional. However, in real world applications, this is not always the case. In fact, the so-called anisotropies (functionals that can be other than the area functional) are widespread in real world problems since anisotropic objects frequently appear in physics, chemistry, biology, and other fields [Vir18, Tay78, GS86]. Moreover, we can pose the very same questions of section 4.1 to varifolds satisfying a variational obstruction related to anisotropic functionals. Results on regularity in this direction are at a very early stage compared to those for the area functional. Let us dig into that.

We consider an **anisotropic integrand** to be a C^2 function $\mathcal{F} : U \times G(m + n, m) \to (0, +\infty)$ and we define the **anisotropic energy** of $\mathbf{V}$ with respect to the anisotropic integrand $\mathcal{F}$ in $A \subset \mathbb{R}^{m+n}$ as

$$\mathcal{E}_{\mathbf{V}}(A) := \int_{\mathbf{Gr}(A)} \mathcal{F}(y, \pi)\, \mathrm{d}\mathbf{V}(y, \pi).$$

Note that the area integrand is recovered when we consider $\mathcal{F} \equiv 1$. We define the notion of **anisotropic first variation** of an m-varifold $\mathbf{V}$ as the distribution that acts on each $g \in C_c^1(U, \mathbb{R}^n)$ as follows

$$\delta_{\mathcal{F}}\mathbf{V}(g) := \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_{(\phi_t^{\#}\mathbf{V})}(U)\Big|_{t=0}, \text{ where } \phi_t(x) := x + tg(x).$$

We say that $\mathbf{V}$ is **anisotropic stationary** if $\delta_{\mathcal{F}}\mathbf{V} \equiv 0$.

In trying to answer the same questions of section 4.1, one faces several problems. For instance, there is no monotonicity formula for this general integrands. In fact, in [All74], it is shown that a monotonicity formula essentially holds for anisotropic stationary varifolds if, and only if, the functional is a linear transformation of the area. It means that the very first step of proving that the blowup procedure for a varifold delivers a cone dramatically fails in this generality.

A new approach is needed to handle anisotropic functional and its variational obstructions on varifolds (or even currents and finite perimeter sets!). Even though there are some regularity results available for *minimizers* of anisotropic energies, a far-reaching list of open problems remains unsolved. It even includes open problems in very restrictive cases, for example, in codimension $n = 1$. Furthermore, the setting of minimizers is richer in tools than the setting of stationary objects, since we have a way to construct competitors to the minimality. In short, very little is known for stationary objects w.r.t. anisotropic energies!

Groundbreaking achievements were made in [DRT22], the authors, among other results, have proven a much more general version of the following:

> Let $\mathcal{F} \in C^2(\mathbb{R}^{m+n} \times G(m+n, m), (0, \infty))$ be a functional satisfying suitable conditions, consider an open, bounded set $\Omega \subset \mathbb{R}^m$, $u \in \operatorname{Lip}(\Omega, \mathbb{R}^n)$ be a map whose graph Γ_u induces an *anisotropic stationary* m-varifold. Then there exists $\alpha > 0$ and an open set Ω_0 of full measure in Ω such that $u \in C^{1,\alpha}(\Omega_0, \mathbb{R}^n)$.

In [RR23], we prove an ε-regularity theorem for boundary points similar to Allard's result mentioned above where he consider the area integrand. In fact, we have proven a more general version of the statement below:

> Let $\mathcal{F} \in C^2(\mathbb{R}^{m+n} \times G(m+n, m), (0, \infty))$ satisfying suitable conditions, $\alpha \in (0, 1)$, $r > 0$, $\Omega \subset \mathbb{R}^m$, $u \in \operatorname{Lip}(\Omega, \mathbb{R}^n)$, $p \in \Gamma$, and Γ be a $C^{1,\alpha}$-'boundary' for the varifold Γ_u induced by u. Assume that Γ_u is *anisotropic stationary*, then there exists $\delta > 0$ with the following property. If $\varepsilon < \delta$ and $\|\Gamma_u\|(\mathbf{B}(x, \rho)) \leq (\frac{1}{2} + \varepsilon)\omega_m \rho^m$, for all $\rho \in (0, r)$, then there exist $r_0 > 0$ and $\alpha_0 \in (0, 1)$ such that $u \in C^{1,\alpha_0}(\mathrm{B}(x, r_0))$.

Chapter 5

Brief introduction to Part C and D

This introduction is focused on the work developed in [Res23] and [NR22]. We aim at giving a historical overview on the regularity theory for area minimizing currents. The success Federer and Fleming's theory of currents is due to the vastness of applications arising from the existence result in any dimension and codimension. In another words, under fairly general assumptions, we always get a (integral) current that is a solution of the Plateau problem, we call them **area minimizing currents**.

Thanks to the effort of many great mathematicians, an actual satisfactory regularity theory was reached in the 70s in **codimension** 1, both the interior regularity, see for instance [Giu84] and [Mag12], and the boundary regularity, see Hardt and Simon [HS79]. The main regularity results obtained in codimension 1 is the full regularity at the boundary, i.e. it is a classical C^1 surface, and that singularities could exist on the interior, although the singular set can be of dimension at most $m-7$, where m denotes the dimension of the area minimizing current.

In higher codimension, i.e., for an area minimizing current of dimension m in $\mathbb{R}^{m+n}$ and with $n \geq 2$, Almgren proved in his masterpiece that the interior singular set has Hausdorff dimension at most $m-2$. This is known as Almgren's Big regularity paper [Alm00], since its length is indeed big (970 pages!), De Lellis and Spadaro revisited the theory and gave a much shorter and accessible proof in the series of articles [DS11, DS15, DS14, DS16a, DS16b]. Chang proved in [Cha88] that such set is discrete when $m = 2$. Actually, in the paper of Chang, a substantial part of the proof is missing, but it has been completed recently by De Lellis, Spadaro, and Spolaor in a series of joint works, see [DSS17a, DLSS18, DSS17b, DLSS20].

In codimension 2 and higher, the theory gets a completely new prospect which is substantially more complicated. In the aforementioned work, Almgren have introduced a lot of brand new ideas which were vital for the development of the regularity theory in this setting. De Lellis and Spadaro sought for a simplified proof and constructions for Almgren's theory, they have done it using modern techniques from geometric analysis and nonlinear analysis. Moreover, De Lellis, De Philippis, Hirsch, Massaccesi ([DDHM18]), De Lellis, Nardulli, Steinbruechel ([DLNS21],[DLNS23]), and Nardulli and I ([NR22]), have adapted Almgren's tools to handle the boundary case which was not covered by Almgren's theory in [Alm00]. There are still several open questions in this high codimension case.

In order to state the results available, we need to set a few definitions. Consider a complete Riemannian manifold $\Sigma \subset \mathbb{R}^{m+n}$ of dimension $m + \bar{n}$ and an oriented submanifold $\Gamma \subset \Sigma$ of dimension $m-1$. We will denote by $[\![M]\!]$ the natural multiplicity one current associated with any rectifiable set $M \subset \Sigma$. From now on, assume that T is an integer rectifiable integral current of dimension m

in Σ, one can consult [Fed69] for the precise definition. If $p \in \operatorname{spt} T \setminus \operatorname{spt} \partial T$, we say that p is an interior point, if $p \in \operatorname{spt} \partial T$, we call p a boundary point.

- Let $p \in \mathbb{R}^{m+n}$, then we define the **density of T at p** as $\Theta(T, p) := \lim_{r \downarrow 0} \frac{\|T\|(\mathbf{B}_r(p))}{\omega_m r^m}$. We say that **the boundary is taken with multiplicity** $Q^\star$ if $\partial T = Q^\star [\![\Gamma]\!]$;
- We say that $x \in \operatorname{spt}(T) \setminus \operatorname{spt}(\partial T)$ is an **interior regular point** if there is a neighbourhood U of x and m-dimensional smooth submanifold $\Xi \subset U \cap \Sigma$ and a positive integer Q such that $T \llcorner \mathbf{B}_r(x) = Q [\![\Xi]\!]$. The set of interior regular points, which of course is relatively open in $\operatorname{spt}(T) \setminus \operatorname{spt}(\partial T)$, is denoted by $\operatorname{Reg}_i(T)$. Its complement $\operatorname{spt}(T) \setminus (\operatorname{Reg}_i(T) \cup \operatorname{spt}(\partial T))$ is the **interior singular set** of T and will be denoted by $\operatorname{Sing}_i(T)$;
- If $\partial T = Q [\![\Gamma]\!]$ for some $Q \in \mathbb{N} \setminus \{0\}$, we say that the point $p \in \Gamma$ is a **boundary regular point** for T if there are a neighbourhood $U \ni p$ and an m-dimensional regular submanifold $\Xi \subset U \cap \Sigma$ (without boundary in U) such that $\operatorname{spt}(T) \cap U \subset \Xi$. Such points are denoted $\operatorname{Reg}_b(T)$ and its complementary set in Γ is denoted $\operatorname{Sing}_b(T)$;
- T will be called **area minimizing current** at $x \in \operatorname{spt}(T)$ at scale $r_0 > 0$, if

  $$\|T\| (B_r(x)) \leq \|T + \partial Q\| (B_r(x)) \,,$$

  for all $0 < r < r_0$ and for all $(m+1)$-dimensional integral currents Q in Σ with support in $B_r(x)$. A current T is called area minimizing in the open set U, if the current T is area minimizing in each $x \in \operatorname{spt}(T) \cap U$.

5.1 Interior regularity in codimension 1

We now treat the interior case, i.e., $\partial T = 0$. In a nutshell, the results available in the interior regularity theory for area minimizing currents are:

De Giorgi, Simon, Federer, Almgren and Fleming have stated in several different works that the dimension of the singular set is at most $m - 7$, where m is the dimension of the current, which is an optimal result taking into consideration the famous example, the so-called Simons' cone, given by Simons in [Sim68] of an area minimizing 7 dimensional current S in $\mathbb{R}^8$. In fact, in [Sim68], it is proven that S is stationary and stable current. Afterwards, Bombieri, De Giorgi, and Giusti proved in [BGG69] that S is indeed an area minimizing 7-current. Having in mind this counterexample, the result below in optimal:

Theorem A. *Let $\Omega \subset \mathbb{R}^{m+n}$ be open, Σ smooth as defined above and T an area minimizing current in $\Omega \cap \Sigma$ and $\bar{n} = 1$. Then:*

- *For $m \leq 6$, $\operatorname{Sing}_i(T) \cap \Omega$ is empty* (Fleming and De Giorgi [DG61, De 65, Fle62] for $m = 2$, Almgren [Alm66] for $m = 3$, and Simon [Sim68] for $4 \leq m \leq 6$, see also the works of Reifenberg [Rei64] and Triscari [Tri63]),
- *If $m = 7$, $\operatorname{Sing}_i(T) \cap \Omega$ consists of isolated points* (Federer in [Fed80]),
- *For $m \geq 8$, $\operatorname{Sing}_i(T) \cap \Omega$ has Hausdorff dimension at most $m - 7$* (Federer in [Fed80]) *and is countably $(m-7)$-rectifiable* (Simon in [Sim95]),

- *For every $m \geq 7$, there are area minimizing currents T in the euclidean space $\mathbb{R}^{m+1}$ for which* $\mathrm{Sing}_i(T)$ *has positive $(m-7)$-dimensional Hausdorff measure* (Bombieri-De Giorgi-Giusti in [BGG69]).

5.2 Interior regularity in arbitrary codimension

Almgren have proved in [Alm00] that in general dimension and codimension we have that

$$\dim\left(\mathrm{Sing}_i\left(T\right)\right) \leq m-2.$$

One notes that this is weaker than the result in codimension 1, nevertheless it is also an optimal result for $\bar{n} > 1$, this is assured by the famous Federer's examples of complex varieties. In fact, in [Fed65], Federer shows that complex varietes induce area minimizing currents, thus if one considers the following

$$F := \left\{(z,w) \in \mathbb{C}^2 : z^2 = w^3\right\},$$

one gets an example of a 2 dimensional *singular* area minimizing current in $\mathbb{R}^4$. In short, the regularity result for interior points follows.

Theorem B. *Let $\Omega \subset \mathbb{R}^{m+n}$ be open, Σ as defined above and T an area minimizing current in $\Omega \cap \Sigma$ and $\bar{n} \geq 2$. Then:*

- *For $m = 1$,* $\mathrm{Sing}_i(T) \cap \Omega$ *is empty;*
- *For $m = 2$,* $\mathrm{Sing}_i(T) \cap \Omega$ *is discrete, see* [Cha88] *and* [DSS17a, DLSS18, DSS17b, DLSS20]*;*
- *For $m \geq 2$,* $\mathrm{Sing}_i(T) \cap \Omega$ *has Hausdorff dimension at most $m-2$, by Almgren, see* [Alm00], *for $\Sigma \in C^5$, and by De Lellis-Spadaro, see* [DS11, DS15, DS14, DS16a, DS16b]*, for $\Sigma \in C^{3,\alpha}$;*
- *For every $m \geq 2$, there are area minizing currents T in $\mathbb{R}^{m+2}$ for which* $\mathrm{Sing}_i(T)$ *has positive hausdorff $(m-2)$-dimensional Hausdorff measure* (Federer in [Fed65]).

We will not introduce in this text the theories, definition, and notions that were used to overcome the difficulties of increasing the codimension, for the precise definitions and details, one can consult the original work of Almgren ([Alm00]) and the other works aforementioned. One can also see part D and part D where we use all these techniques to approach our results. Let us at least name these important concepts: Almgren's Q-valued maps defined on the Euclidean space, Lipschitz approximations by Q-maps for integer rectifiable currents and for area minimizing currents, excess decays with extremely crucial power laws, frequency functions, center manifolds, Lipschitz approximation by Q-maps defined on submanifolds, blowup arguments, these theories are the backbone of the framework created by Almgren and developed by the outstanding mathematicians aforementioned.

5.3 Boundary regularity in codimension 1

As mentioned before, Hardt and Simon ([HS79]) stated the $C^{1,\alpha}$ regularity of the boundary of an area minimizing current in Euclidean spaces. A central problem that Hardt and Simon faced was that their conditions allowed the existence of boundary **two-sided** points:

The generalization of Hardt-Simon work to the Riemannian setting is due to Steinbruechel in [Ste22] where the following statement is proven.

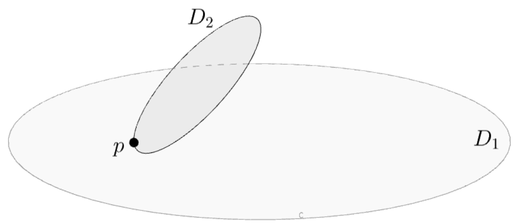


Figure 5.1: *Let $T = [\![D_1]\!] + [\![D_2]\!]$ and $p \in \partial D_2 \cap \operatorname{int}(D_1)$. We say that p is a two-sided boundary point for T.*

Theorem C. *If $\Omega \subset \mathbb{R}^{m+n}$ is open, Σ smooth as before with $\bar{n} = 1$, $\alpha \in (0, 1)$ and T is an area minimizing m-current in $\Omega \cap \Sigma$ and ∂T is an oriented embedded C^2 $(m-1)$-submanifold of Ω ($C^{1,\alpha}$ when $\Sigma = \mathbb{R}^{m+1}$, [HS79]). Then for any point $x \in \operatorname{spt}(\partial T)$, there is a neighbourhood V of x in Ω satisfying that $V \cap \operatorname{spt} T$ is an embedded $C^{1,\frac{1}{4}}$ m-submanifold with boundary ($C^{1,\alpha}$ when $\Sigma = \mathbb{R}^{m+1}$, [HS79]).*

As it was noticed in [GMT86, Problem 4.19], this result can be extended under the same assumptions but with arbitrary boundary multiplicity, i.e., $\partial T = Q^\star [\![\Gamma]\!], Q^\star \geq 1$, using a decomposition argument provided by White in [Whi79]. Unfortunately, it is known that this kind of decomposition argument is a specific feature of the codimension 1 setting.

5.4 Boundary regularity in arbitrary codimension

In the arbitrary codimension setting, the multiplicities play a crucial role and there is no such simple reductions.

5.4.1 Boundary being taken with multiplicity 1.

In [All75, All69], Allard has proven that the boundary of an area minimizing current taking the boundary with multiplicity 1 is regular, however, he needed to impose a crucial condition on spt (∂T) which is that $\operatorname{spt}(\partial T)$ is cointained in the boundary of a uniformly convex set, we call this condition **convex barrier**. Indeed, the strong result that Allard proved is the following theorem.

Theorem D. *Let T be an area minimizing integral current, $p \in \operatorname{spt}(\partial T), U$ an open neighbourhood of p and assume that $\operatorname{spt}(\partial T) \cap U$ is a C^k oriented $(m-1)$-submanifold of $\mathbb{R}^{m+n}$. Then, if $\Theta(T, p) = \frac{1}{2}$, there exists V an open neighbourhood of p such that $\operatorname{spt}(T) \cap U$ is a C^{k-1} oriented m-submanifold of $\mathbb{R}^{m+n}$.*

With a convex barrier assumption over the boundary of T, Allard also proved that $\Theta(T, p) = \frac{1}{2}$ for every p as in the theorem above, see [All75, Section 5.2] .

Removing the additional geometric restriction (convex barrier assumption), there were no results about even existence of boundary regular points. De Lellis, De Philippis, Hirsch and Massaccesi in

[DDHM18] proved the density of $\mathrm{Reg}_b(T)$ in $\mathrm{spt}(T)$ where T is an area minimizing current with boundary multiplicity 1 in a Riemannian manifold of class $C^{3,\alpha}$.

To achieve the following result the authors have adapted the constructions and trailblazing ideas introduced by Almgren to the boundary case which is even much more involved and required highly nontrivial new ideas.

Theorem E. *Assuming that Σ is a $C^{3,\alpha}$ submanifold of $\mathbb{R}^{m+n}$ with $\alpha \in (0,1)$, Γ is a $C^{3,\alpha}$ oriented submanifold of Σ and T area minimizing in $\Sigma \cap \mathbf{B}_2(0)$ with $\partial T \llcorner \mathbf{B}_2(0) = [\![\Gamma \cap \mathbf{B}_2(0)]\!]$. Then $\mathrm{Reg}_b(T)$ is a relatively open dense set in $\Gamma \cap \mathbf{B}_2(0)$.*

We cannot hope to prove an estimate for the size of the singular set, as we have seen for the interior singular set. This is due to the fact that, [DDHM18, Thm 1.8], there exist a smooth curve Γ and an area minimizing current T with $\partial T = [\![\Gamma]\!]$ such that the Hausdorff dimension of $\mathrm{Sing}_b(T)$ is 1, i.e., the same Hausdorff dimension of Γ.

However, it still leaves some open questions. For example, it is not known whether or not $\mathrm{Sing}_b(T)$ is a $\mathcal{H}^{m-1}$-null set. Moreover, the counterexample given above provides 'fake' singularities, in the sense that, around a singular point, the area minimizing current is given by 'unions' of regular manifolds (see section 5.3). Furthermore, one can indeed divide the singular set into 'fake' singularities (also called crossing-type singularities) and 'genuine' singularities (also called branch points). In [DDHM18], the authors propose some conjectures about the Hausdorff dimension of the set of 'genuine' singularities.

5.4.2 Boundary being taken with arbitrary multiplicity.

As noticed, in codimension 1, the multiplicity of the boundary can be handled with a decomposition argument, it means that we can focus on proving theorems for the simpler case that $Q^\star = 1$. This is not true in higher multiplicity cases, hence requiring different proofs/approaches.

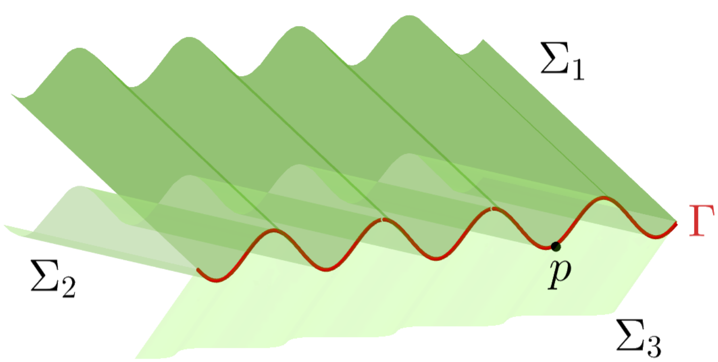


Figure 5.2: *While in Allard's setting, [All75], only one sheet is allowed, in [DLNS23] several sheets taking the same boundary as in the picture is permitted.*

One can ask if the similar results for multiplicity equal to 1 can be stated for arbitrary boundary multiplicity with/without this convex barrier condition. It is much trickier to achieve such type of result, the works by De Lellis, Nardulli and Steinbruechel, [DLNS21, DLNS23], answer this question positively for 2-dimensional currents with the convex barrier condition.

Theorem F. *Let T be an 2-dimensional area minimizing integral current, $p \in \mathrm{spt}(\partial T), U$ an open neighbourhood of p and assume that $\mathrm{spt}(\partial T) \cap U$ is a $C^{3,\alpha}$ curve in $\mathbb{R}^{2+n}$. Then if $\Theta(T,p) = \frac{Q^\star}{2}$, for some $Q^\star \in \mathbb{N} \setminus \{0\}$, there exists V an open neighbourhood of p such that $\mathrm{spt}(T) \cap U$ is a $C^{3,\alpha}$ oriented surface in $\mathbb{R}^{2+n}$.*

Removing the convex barrier assumption, Nardulli and I ([NR22]) have proven the following result:

Theorem G. *Let T be an area minimizing 2-dimensional current in $\mathbf{B}_2(0)$ with $\partial T \llcorner \mathbf{B}_2(0) = Q^\star [\![\Gamma \cap \mathbf{B}_2(0)]\!]$, for some $Q^\star \in \mathbb{N} \setminus \{0\}$, $\alpha \in (0,1)$, and Γ is a $C^{3,\alpha}$ curve in $\mathbb{R}^{2+n}$. Then $\mathrm{Reg}_b(T)$ is a relatively open dense set in $\Gamma \cap \mathbf{B}_2(0)$.*

Part B

Boundary regularity for anisotropic minimal Lipschitz graphs

Chapter 6

Introduction

6.1 Regularity theorems for the area functional

In his seminal work [All72a], Allard developed the regularity theory for varifolds with bounded first variation. He first obtained a rectifiability theorem, proving that, for every m-varifold $\mathbf{V}$,

$$\text{if } \sup_{\|X\|_\infty \le 1} \delta \mathbf{V}(X) \le 1, \text{then } \mathbf{V} \llcorner \{x \in \mathbb{R}^{m+n} : \Theta^*_m(\mathbf{V}, x) > 0\} \text{ is a rectifiable varifold.} \tag{R}$$

Additionally, he proved a celebrated ε-regularity theorem, which guarantees, for every m-varifold $\mathbf{V}$ with generalized mean curvature in $\mathrm{L}^p(\mathcal{H}^m)$, $p > m$, and $\mathcal{H}^m(\mathrm{spt}(\|\mathbf{V}\|) \cap \mathbf{B}(x,r))$ close to $\omega_m r^m$, that $\mathrm{spt}(\|\mathbf{V}\|)$ is $C^{1,\eta}$ locally around x for some $\eta \in (0,1)$.

Afterwards, in [All75], Allard extended this regularity result to varifolds with $C^{1,1}$ boundary. Here the boundary is intended as a $C^{1,1}$ submanifold Γ with dimension $m-1$ such that the first variation of the varifold is bounded away from Γ.

One of the reasons why Allard considered a $C^{1,1}$ boundary is that for each point $x \in \Gamma$ there is a neighborhood of x in Γ such that the distance function $y \mapsto \mathrm{dist}(y, \Gamma)$ is differentiable in a tubular neighborhood of Γ. For more details, we refer the reader to [GS22], where the authors explore Federer's notion of reach of Γ to prove that Γ is $C^{1,1}$ if, and only if, the reach is strictly positive. Bourni [Bou16] generalized Allard's boundary regularity theorem to $C^{1,\alpha}$ boundaries, for $\alpha \in (0,1)$, using a Whitney partition argument to overcome the non-differentiability of the distance function above around Γ.

6.2 Anisotropic functionals

A natural question is whether or not the regularity theorems mentioned in Section 6.1 still hold if the first variation is not computed with respect to the area functional, but rather with respect to more general anisotropic functionals $\mathcal{F} : \mathbb{R}^{m+n} \times \mathbf{Gr}(m,n) \to (0, +\infty)$.

Anisotropic functionals, together with their minimizers and critical points, have been extensively studied, and several results available for the area functional have been extended to the anisotropic setting. This is typically not an easy task, as several basic properties of isotropic minimal surfaces dramatically fail for anisotropic minimal surfaces. More precisely, Allard's proof of the aforementioned regularity theorems strongly rely on the well-known **monotonicity formula**. However, in [All74], Allard showed that the monotonicity formula holds only for linear transformations of the area functional. The lack of a monotonicity formula for general anisotropic functionals gives rise to numerous technical issues in the theory, since the majority of the isotropic results **deeply** rely on it.

De Philippis, De Rosa and Ghiraldin proved in [DPDRG18] that, if $\mathcal{F}$ is of class C^1 and satisfies the so called **atomic condition (AC)**, the rectifiability criterium (R) holds also for the anisotropic first variation $\delta_{\mathcal{F}}$ in place of δ. This result found applications, among others, in the solution of the anisotropic Plateau problem [DPDRG20, DD22] and the anisotropic min-max theory [DR18]. In the case of an autonomous anisotropy $\mathcal{F}$, i.e., $\mathcal{F}$ does not depend on the variable in $\mathbb{R}^{m+n}$, the authors in [DPDRG18] showed that the validity of (R) is actually equivalent to AC. We refer the interested reader to the following works for further developments of the theory: [HP17, DL06, DPDRG20, DL22, Tio21, DLDRG19]. In codimension $n = 1$ and in dimension $m = 1$, AC is equivalent to **strict convexity** of $\mathcal{F}$. In [DRK20], De Rosa and Kolansinski have proven that the atomic condition implies the Almgren's strict ellipticity condition. We refer the reader to the following works about this type of functionals in higher codimension, where basic questions remain open to date: [PT04, BES62, BES63].

Several important regularity theorems have been obtained for anisotropic minimizers. In particular, Almgren [Alm68] proved regularity for sets minimizing an elliptic anisotropic energy in any dimension and codimension; Duzaar and Steffen, [DS02], exhibited how to obtain interior and boundary regularity for integer rectifiable currents in any dimension and codimension that almost minimize an elliptic anisotropic energy. Schoen, Simon and Almgren [SSA77] proved that, in codimension 1, anisotropic energy minimizers in the sense of currents have singular set of Hausdorff codimension at least 2; De Philippis and Maggi in [DM15] proved regularity for free boundary Caccioppoli sets that minimize an elliptic anisotropic energy. Figalli in [Fig17] focused on the proof of regularity for almost minimal integral rectifiable currents, in codimension 1 and with density 1, under weak conditions on the anisotropic functional: namely $C^{1,1}$ anisotropies rather than the usual C^2 assumption. We also refer the reader to [Har77, LIN85, DDH19] for the boundary regularity of anisotropic energy (almost) minimizers and stable surfaces.

However, the regularity theory of stationary points for anisotropic integrands is much less understood, due to the number of nontrivial difficulties caused by the lack of a monotonicity formula and of mass ratio bounds. For codimension 1 varifolds, Allard proved regularity under a **density lower bound assumption** [All86, The basic regularity Lemma, Assumption (1)]. De Lellis, De Philippis, Kirchheim, and Tione presented in an expository fashion several open questions in the theory, see [DDKT21]. To the best of our knowledge, for codimension bigger than or equal to 2, the only regularity result for varifolds that are stationary for an anisotropic energy is proved by De Rosa and Tione in [DRT22] for varifolds induced by Lipschitz graphs.

6.3 Main result

The aim of this work is to prove the anisotropic counterpart of Allard's boundary regularity theorem [All75]. To this aim, we will consider the anisotropic integrands introduced in [DRT22, Definition 3.3] satisfying the uniformly scalar atomic condition (USAC), c.f. definition 7.3.1.

Our main result is the following. For a more precise and detailed statement, we refer to theorem 11.0.2.

Theorem H. *Let* $m, n \geq 2$, $\mathcal{F}$ *be an integrand of class* C^2 *satisfying USAC,* $\Gamma \subset \mathbb{R}^{m+n}$ *be an* $(m-1)$*-submanifold of class* $C^{1,\alpha}$, $\Omega \subset \mathbb{R}^m$, $u \in Lip(\Omega, \mathbb{R}^n)$, *and* $\partial \mathrm{graph}(u) = \Gamma$. *Assume that the anisotropic mean curvature of* u *is in* L^p *for* $p > m$. *Then there exists* $\delta = \delta(m, n, p, \mathcal{F}, \|u\|_{\mathrm{Lip}}, \Gamma) > 0$ *with the following property. If* $\sigma < \delta$, $x \in \Gamma$ *and* $r_0 > 0$ *are such that*

$$\frac{\|\mathrm{graph}(u)\|(\mathbf{B}(x,r))}{\omega_m r^m} \leq \frac{1}{2} + \sigma \qquad \forall r \in (0, r_0),$$

then there exist $\rho > 0$ *and* $\eta \in (0,1)$ *depending only on* $m, n, p, \mathcal{F}, \|u\|_{\mathrm{Lip}}, \Gamma$ *such that*

$$u \in C^{1,\eta}(\mathrm{B}\,(x, \rho)).$$

Following Allard's paper [All75], an interesting and direct application of this results is for minimizing currents satisfying a **convex barrier** assumption. In this scenario, we are able to prove that all boundary points satisfy the mass ratio bound, thus we would have full regularity for the boundary. In fact, this was already shown by Hardt in [Har77].

When either one does not have the convex barrier condition or the condition on the density, the problem is much more subtle even for minimizers of the **area** integrand. For some results in this direction we refer to [DDHM18, DLNS21, NR22, HS79].

Chapter 7

Notation and preliminaries

We fix integers $m, n \geq 1$ and denote $\mathbb{R}_+ := \{t \in \mathbb{R} : t \geq 0\}$. We denote by U an open subset of $\mathbb{R}^{m+n}$, $\mathbf{B}(x,r) := \{y \in \mathbb{R}^{m+n} : |x-y| < r\}$, $\mathbf{B}_r := \mathbf{B}(0,r)$. If π is a linear subspace of $\mathbb{R}^{m+n}$, we denote $\mathrm{B}_\pi(x,r) := \mathbf{B}(x,r) \cap (x+\pi)$, and we also denote $\mathbf{p}_\pi$ the orthogonal projection from $\mathbb{R}^{m+n}$ onto π. When $\pi = \mathbb{R}^m \times \{0\}$, we omit π in the preceding notations.

For $s \geq 0$, $\mathcal{H}^s$ denotes the s-dimensional Hausdorff measure induced by the Euclidean metric in $\mathbb{R}^{m+n}$, and $\omega_s := \mathcal{H}^s(\mathrm{B}_\pi(0,1))$ where π is an s-dimensional subspace. We denote the inner product of vectors by $\langle , \rangle : \mathbb{R}^{m+n} \times \mathbb{R}^{m+n} \to \mathbb{R}$, the product of matrices by $\cdot$ where to any $A = (a_{ij})_{i=1,\dots,h}^{j=1,\dots,r}$ and $B = (b_{ij})_{i=1,\dots,r}^{j=1,\dots,s}$ it assigns $A \cdot B = (\sum_{k=1}^r a_{ik} b_{kj})_{i=1,\dots,h}^{j=1,\dots,s}$, and $A : B = \mathrm{tr}(A^t \cdot B)$.

For the basic theory that we will assume, we refer the reader to [Fed69], [Sim14], [All72a], and the references therein.

7.1 Measures, rectifiability and Grassmannian

We denote by $\mathfrak{M}(U, \mathbb{R}^m)$ the set of $\mathbb{R}^m$-valued Radon measures on U, when $m = 1$, we denote with $\mathfrak{M}_+(U)$ the set of nonnegative Radon measures on U. Given $\mu \in \mathfrak{M}(U, \mathbb{R}^m)$, we set:

- for a Borel set $A \subset U$, $\mu \llcorner A(E) := \mu(E \cap A)$ as the **restriction of μ to A**;
- $\|\mu\| \in \mathfrak{M}_+(U)$ to be the **total variation of μ**. Recall that, for any open set $A \subset U$,

$$\|\mu\|(A) := \sup \left\{ \int \langle g(x), \mathrm{d}\mu(x) \rangle : g \in C_c^\infty(A, \mathbb{R}^m), \|g\|_\infty \leq 1 \right\},$$

  where $\langle g(x), \mathrm{d}\mu(x) \rangle := \sum_{i=1}^{m+n} g_i(x) \mathrm{d}\mu_i(x)$;
- the **upper and lower s-dimensional density of μ at x**, respectively, as

$$\Theta_s^*(\mu, x) := \limsup_{r \to 0^+} \frac{\|\mu\|(\mathbf{B}(x,r))}{\mathcal{H}^s(\mathbf{B}(x,r))}, \quad \Theta_*^s(\mu, x) := \liminf_{r \to 0^+} \frac{\|\mu\|(\mathbf{B}(x,r))}{\mathcal{H}^s(\mathbf{B}(p,r))}.$$

  In case $\Theta_s^*(\mu, x) = \Theta_*^s(\mu, x)$, we call this number the **density of μ at x** and denote it by $s^{(\mu)}x$;

- for a Borel function $g : U \to \mathbb{R}^n$, the **push-forward of μ through g** as $g_\sharp \mu = \mu \circ g^{-1}$.

Let $M \subset U \subset \mathbb{R}^{m+n}$, we say that M is **s-rectifiable** if there exist a sequence of Lipschitz maps $\{g_j : \mathbb{R}^s \to U\}_{j=1}^{+\infty}$ and an $\mathcal{H}^s$-null set M_0 such that

$$M = M_0 \cup \left(\bigcup_{j=1}^{+\infty} g_j(M_j) \right).$$

In [Sim14, Lemma 1.2, Chapter 3], it is shown that M is s-rectifiable if, and only if, M can be covered, up to a $\mathcal{H}^s$-null set, by countably many s-dimensional submanifolds of U of class C^1. A nonnegative Radon measure $\mu \in \mathfrak{M}_+(U)$ is said to be **s-rectifiable**, if there is an s-rectifiable set $M \subset U$ and a nonnegative Borel function $\Theta : U \to \mathbb{R}_+$ such that $\mu = \Theta \mathcal{H}^s \llcorner M$.

The Grassmannian of s-dimensional linear subspaces of $\mathbb{R}^{m+n}$ is denoted by $\mathbf{Gr}(m+n, s)$, we will often call $\pi \in \mathbf{Gr}(m+n, s)$ as an s-plane in $\mathbb{R}^{m+n}$. We endow $\mathbf{Gr}(m+n, s)$ with the metric

$$\|\pi - \tilde{\pi}\| := \sqrt{\sum_{i,j=1}^{m+n} \left(\langle \mathbf{e}_i, \mathbf{p}_\pi(\mathbf{e}_j)\rangle - \langle \mathbf{e}_i, \mathbf{p}_{\tilde{\pi}}(\mathbf{e}_j)\rangle\right)^2}, \quad \forall \pi, \tilde{\pi} \in \mathbf{Gr}(m+n, s),$$

where $\mathbf{p}_\pi$ and $\mathbf{p}_{\tilde{\pi}}$ denote the orthogonal projections of $\mathbb{R}^{m+n}$ on π and $\tilde{\pi}$, respectively, and $\{\mathbf{e}_i\}_{i=1}^{m+n}$ is the canonical orthonormal basis of $\mathbb{R}^{m+n}$. We also fix the notation

$$\mathbf{Gr}(A, m+n, s) := A \times \mathbf{Gr}(m+n, s), \quad \forall A \subset U \subset \mathbb{R}^{m+n},$$

and $\mathbf{Gr}(A) := \mathbf{Gr}(A, m+n, m)$.

7.2 Varifolds

We say that $\mathbf{V}$ is an **m-varifold on U** if $\mathbf{V}$ is a nonnegative Radon measure defined on $\mathbf{Gr}(U)$. The space of all m-varifolds on U is denoted by $\mathbf{V}_m(U)$. For every $\mathbf{V} \in \mathbf{V}_m(U)$ we can define the measure $\|\mathbf{V}\| \in \mathfrak{M}_+(U)$, which is often called **weight of $\mathbf{V}$**, by the relation

$$\|\mathbf{V}\|(A) = \mathbf{V}(\mathrm{proj}^{-1}(A)), \quad \forall A \subset U,$$

where henceforth proj denote the canonical projection of $\mathbf{Gr}(U)$ on U. Hence, we define

$$\Theta^*_m(\mathbf{V}, x) := \Theta^*_m(\|\mathbf{V}\|, x), \qquad \Theta^m_*(\mathbf{V}, x) := \Theta^m_*(\|\mathbf{V}\|, x),$$

and, when $\Theta^m(\|\mathbf{V}\|, x)$ exists,

$$\Theta^m(\mathbf{V}, x) := \Theta^m(\|\mathbf{V}\|, x).$$

Of particular interest are rectifiable varifolds, which enjoy a richer structure than general varifolds, see [Sim14, Chapter 4 and 9]. In fact, we say that $\mathbf{V} \in \mathbf{V}_m(U)$ is an **m-rectifiable varifold** if, there exists an m-rectifiable set M in U and a positive locally $\mathcal{H}^m$-integrable function Θ on M with $\Theta \equiv 0$ on $\mathbb{R}^n \setminus M$ such that

$$\mathbf{V}(A) = \int_{\mathrm{proj}(A) \cap M} \Theta(y) \, \mathrm{d}\mathcal{H}^m(y), \quad \forall A \subset \mathbf{Gr}(U).$$

In this case, we use the notation $\mathbf{V} = \mathbf{v}(M, \Theta)$.

For every diffeomorphism $\psi \in C^1_c(U, \mathbb{R}^{m+n})$, the push-forward $\psi^\# \mathbf{V} \in \mathbf{V}_m(U)$ of $\mathbf{V} \in \mathbf{V}_m(U)$ with respect to ψ is defined as

$$\int_{\mathbf{Gr}(U)} \Phi(x,\pi) d(\psi^\# \mathbf{V})(x,\pi) = \int_{\mathbf{Gr}(U)} \Phi(\psi(x), d_x\psi(\pi)) J\psi(x,\pi) d\mathbf{V}(x,\pi), \ \forall \Phi \in C^0_c(\mathbf{Gr}(U)).$$

Here $d_x\psi(\pi)$ denotes the image of π under the map $d_x\psi(x)$ and

$$J\psi(x,\pi) := \sqrt{\det\left(\left(d_x\psi\big|_\pi\right)^* \circ d_x\psi\big|_\pi\right)}$$

is the m-Jacobian determinant of the differential $d_x\psi$ restricted to π, see [Sim14, Chapter 8].

We consider an **anisotropic integrand** to be a C^1 function $\mathcal{F} : \mathbf{Gr}(U) \to (0, +\infty)$ and we define the anisotropic energy of $\mathbf{V}$ with respect to the anisotropic integrand $\mathcal{F}$ in A as

$$\mathcal{E}_{\mathbf{V}}(A) := \int_{\mathbf{Gr}(A)} \mathcal{F}(y,\pi) \, \mathrm{d}\mathbf{V}(y,\pi).$$

Note that the area integrand is recovered when we consider $\mathcal{F} \equiv 1$.

We define the notion of **anisotropic first variation** or **$\mathcal{F}$-first variation** of an m-varifold $\mathbf{V}$ as the distribution that acts on each $g \in C^1_c(U, \mathbb{R}^n)$ as follows

$$\delta_{\mathcal{F}}\mathbf{V}(g) := \frac{\mathrm{d}}{\mathrm{d}t} \mathcal{E}_{(\phi_t^\# \mathbf{V})}(U)\bigg|_{t=0},$$

where $\phi_t(x) := x + tg(x)$. If $\delta_{\mathcal{F}}\mathbf{V} \equiv 0$, we say that $\mathbf{V}$ is **anisotropically stationary** or **$\mathcal{F}$-stationary**.

We recall the following formula for the anisotropic first variation of a varifold:

Proposition 7.2.1 (Lemma A.2, [DPDRG18]). *Let $\mathcal{F} \in C^1(\mathbf{Gr}(U))$ and $\mathbf{V} \in \mathbf{V}_m(U)$, then for every $g \in C^1_c(U, \mathbb{R}^{m+n})$ we have*

$$\delta_{\mathcal{F}}\mathbf{V}(g) = \int_{\mathbf{Gr}(U)} \Big[\langle D_x\mathcal{F}(x,\pi), g(x)\rangle + \mathcal{B}_{\mathcal{F}}(x,\pi) : Dg(x)\Big] \mathrm{d}\mathbf{V}(x,\pi),$$

where the matrix $\mathcal{B}_{\mathcal{F}}(x,\pi) \in \mathbb{R}^{m+n} \otimes \mathbb{R}^{m+n}$ is uniquely defined by

$$\mathcal{B}_{\mathcal{F}}(x,\pi) : L := \mathcal{F}(x,\pi)(\pi : L) + \Big\langle D_\pi \mathcal{F}(x,\pi), \pi^\perp \circ L \circ \pi + \Big(\pi^\perp \circ L \circ \pi\Big)^* \Big\rangle, \tag{7.2.1}$$

for all $L \in \mathbb{R}^{m+n} \otimes \mathbb{R}^{m+n}$.

If we assume that $\delta_{\mathcal{F}}\mathbf{V}$ is a Radon measure on $\mathbf{B}_{r_0} \setminus \Gamma$, there exists a $\|\mathbf{V}\|$-measurable function $\mathcal{H}_{\mathcal{F}} : \mathbf{B}_{r_0} \setminus \Gamma \to \mathbb{R}^{m+n}$ called either **anisotropic mean curvature vector** or **$\mathcal{F}$-mean curvature vector** such that

$$\delta_{\mathcal{F}}\mathbf{V}(g) = -\int_{\mathbf{B}_{r_0}\setminus\Gamma} \langle \mathcal{H}_{\mathcal{F}}, g\rangle \, \mathrm{d}\|\mathbf{V}\|, \quad \forall g \in C^1(\mathbf{B}_{r_0}) \text{ s.t. } g|_\Gamma \equiv 0, \tag{7.2.2}$$

$$|\mathcal{H}_{\mathcal{F}}(x)| = D_{\|\mathbf{V}\|}\|\delta_{\mathcal{F}}\mathbf{V}\|(x), \quad \forall x \in \mathbf{B}_{r_0} \setminus \Gamma,$$

where $D_{\|\mathbf{V}\|}\|\delta_{\mathcal{F}}\mathbf{V}\|$ denoted the Radon-Nykodim derivative.

7.3 Assumptions on the anisotropic integrand

As we briefly mentioned in the introduction, there are several ellipticity conditions which one might impose on $\mathcal{F}$. We refer the reader to the references in Section 6.2. We will just recall the ellipticity condition that we will use in this paper, i.e. the **uniformly scalar atomic condition**, introduced in [DRT22, Definition 3.3].

To this aim, we denote the dual function of $\mathcal{F}$ by $\mathcal{F}^*$ which is defined on $\mathbf{Gr}(U, m+n, n)$ as $\mathcal{F}^*(x,\pi) := \mathcal{F}(x,\pi^\perp)$.

Definition 7.3.1 (Uniformly scalar atomic condition)**.** Given an anisotropic integrand $\mathcal{F} \in C^1(\mathbf{Gr}(U))$, $\mathcal{F}$ satisfies the uniformly scalar atomic condition (USAC) if for every $x \in U$ there exists a constant $K_{\mathcal{F},x} > 0$ such that

$$\mathcal{B}_{\mathcal{F}}(x,\pi_0) : \mathcal{B}_{\mathcal{F}^*}(x,\pi_1^\perp) \geq K_{\mathcal{F},x}\|\pi_0 - \pi_1\|^2, \quad \forall \pi_0, \pi_1 \in \mathbf{Gr}(m+n,m).$$

Remark 7.3.2. We recall that De Rosa and Tione proved in [DRT22, Proposition 3.5] that USAC implies the so-called atomic condition. The atomic condition was in turn introduced in [DPDRG18, Definition 1.1] to prove the Rectifiability Theorem ((R) with respect to the anisotropic first variation $\delta_{\mathcal{F}}$). Hence, the Rectifiability Theorem (R) holds assuming that the anisotropic integrand satisfies USAC.

Chapter 8

Anisotropic first variation at boundary points

We isolate here the assumptions under which we work in this section.

Assumption 1. We set the boundary, varifold and anisotropy assumptions as follows:

> **(Boundary)** Let Γ be a closed $(m-1)$-dimensional submanifold of class $C^{1,\alpha}$ for some $\alpha \in (0,1]$. Assume that $0 \in \Gamma$, the radius $r_0 > 0$ is such that $\Gamma \cap \mathbf{B}_{r_0}$ is a graph of a $C^{1,\alpha}$ function over $T_0\Gamma$ and $\kappa \geq 0$ is a constant which satisfies
>
> $$|\mathbf{p}_{N_x\Gamma}(x-y)| \leq \kappa|x-y|^{1+\alpha}, \quad \|\mathbf{p}_{N_x\Gamma} - \mathbf{p}_{N_y\Gamma}\| \leq \kappa|x-y|^{\alpha} \quad \text{and} \quad c\kappa r_0^{\alpha} < \frac{1}{2}, \tag{8.0.1}$$
>
> for all $x, y \in \Gamma \cap \mathbf{B}_{r_0}$;
>
> **(Varifold)** Let $\mathbf{V} \in \mathbf{V}_m(\mathbf{B}_{r_0})$ satisfying $0 \in \operatorname{spt}(\mathbf{V})$ and $\Theta(x) \geq 1$ for $\|\mathbf{V}\|$-almost every $x \in \mathbf{B}_{r_0}$. We assume that $\delta_{\mathcal{F}}\mathbf{V}$ is a Radon measure when restricted to $\mathbf{B}_{r_0} \setminus \Gamma$, and the $\mathcal{F}$-mean curvature $\mathcal{H}_{\mathcal{F}}$ of $\mathbf{V}$ belongs to $\mathrm{L}^1(\mathbf{B}_{r_0} \setminus \Gamma, \mathbf{V})$;
>
> **(Anisotropy)** Let $\mathcal{F} \in C^1(\mathbf{Gr}(\mathbf{B}_{r_0}))$.

8.1 A good distance function

If Γ were of class $C^{1,1}$ we would have that Γ has strictly positive reach and the distance function $\mathrm{d}(x, \Gamma)$ is differentiable (not necessarily of class C^1) in a tubular neighborhood of thickness of the reach. However, for a $C^{1,\alpha}$ boundary Γ, the distance function is not necessarily differentiable and thus we need to “smoothen it". Bourni in [Bou16, Section 3] showed how to properly construct this smooth distance function and we briefly recall the main properties that we are going to use in our work.

Following the scheme of [KP99, Definition 5.3.2 and 5.3.9], let $\mathcal{W}$ be a Whitney decomposition of $\mathbf{B}_{r_0} \setminus \Gamma$ into nontrivial closed $(m+n)$-cubes such that, for every $C \in \mathcal{W}$, we have that

$$\operatorname{diam}(C) \leq \mathrm{d}(C, \Gamma) \leq 3\operatorname{diam}(C).$$

We will fix the following notations: x_C is the center of the cube C, p_C is a point in Γ that satisfies $|x_C - p_C| = \mathrm{d}(x_C, \Gamma)$ and $\{\varphi_C\}_{C\in\mathcal{W}}$ is a Whitney partition of the unity associated to $\mathcal{W}$ as in [KP99, Definition 5.3.9] such that

$$|D\varphi_C(x)| \le \frac{c}{\mathrm{d}(x,\Gamma)}, \tag{8.1.1}$$

where $c \ge 2$ is a dimensional constant. Since by construction $\sum_{C\in\mathcal{W}} \varphi_C \equiv 1$, and for every x there exists $C_x \in \mathcal{W}$ such that $\varphi_{C_x}(x) > 0$, therefore

$$\sum_{C\in\mathcal{W}} \varphi_C^2(x) \ge C(m,n,r_0) > 0. \tag{8.1.2}$$

We recall the following lemma:

Lemma 8.1.1 ([Bou16])**.** *If we assume that $c\kappa r_0^\alpha < 1/2$, there exists $\rho : \mathbf{B}_{r_0} \to \mathbb{R}_+$ such that*

(i) *ρ is a positive function of class C^1 with $|D\rho(x)| \le 1 + c\kappa\rho(x)^\alpha$;*

(ii) *the following equality holds*

$$\rho(x)D\rho(x) = \sum_{C\in\mathcal{W}} \varphi_C(x)\mathbf{p}_{N_{p_C}\Gamma}(x - p_C) + Y(x),$$

where $|Y(x)| \le c\kappa \mathrm{d}(x,\Gamma)^{1+\alpha} \le c\kappa\rho(x)^{1+\alpha}$;

(iii) *we have that*

$$\frac{\mathrm{d}(x,\Gamma)}{2} \le (1 - c\kappa\mathrm{d}(x,\Gamma)^\alpha)\,\mathrm{d}(x,\Gamma) \le \rho(x) \le (1 + c\kappa\mathrm{d}(x,\Gamma)^\alpha)\,\mathrm{d}(x,\Gamma) \le \frac{3\mathrm{d}(x,\Gamma)}{2}.$$

Remark 8.1.2. Notice that, the constructions in this subsection do work if we replace Γ by any k-manifold of class $C^{1,\alpha}$ with $k < m + n$.

8.2 First variation formula

We state the formula for the anisotropic first variation at boundary points in the following proposition. First, following Allard's framework, we show that, under assumption 1, the anisotropic first variation is a Radon measure in the whole ball $\mathbf{B}_{r_0}$, i.e., including the boundary Γ.

Proposition 8.2.1. *Under assumption 1, $\delta_{\mathcal{F}}\mathbf{V}$ is a Radon measure on $\mathbf{B}_{r_0}$. Moreover, there exists a $\|\delta_{\mathcal{F}}\mathbf{V}\|$-measurable function $\mathcal{N}_{\mathcal{F}}$ defined on Γ such that $\mathcal{N}_{\mathcal{F}}(p) \in N_p\Gamma, \forall p \in \Gamma$, and*

$$\delta_{\mathcal{F}}\mathbf{V}(g) = -\int_{\mathbf{B}_{r_0}\setminus\Gamma} \langle \mathcal{H}_{\mathcal{F}}, g\rangle \,\mathrm{d}\|\mathbf{V}\| + \int_\Gamma \langle \mathcal{N}_{\mathcal{F}}, g\rangle \,\mathrm{d}\|\delta_{\mathcal{F}}\mathbf{V}\|_{\mathrm{sing}}, \quad \forall g \in C^1(\mathbf{B}_{r_0}).$$

Remark 8.2.2. Thanks to proposition 8.2.1, under assumption 1, $\delta_{\mathcal{F}}\mathbf{V}$ is a Radon measure on the whole ball $\mathbf{B}_{r_0}$ and $\Theta \ge 1$, $\|\mathbf{V}\|$-a.e. in $\mathbf{B}_{r_0}$. Hence, if $\mathcal{F}$ satisfies USAC, by the Rectifiability criterium [DPDRG18, Theorem 1] and remark 7.3.2, the varifold $\mathbf{V}$ shall be m-rectifiable.

Proof. We want to show that for any compact subset $W \subset \mathbf{B}_{r_0}$ and g of class C^1 with support in W, we have $\delta_{\mathcal{F}}\mathbf{V}(g) \le C\sup_{x\in\mathbf{B}_{r_0}} |g(x)|$. To that end, we cannot directly apply (7.2.2), since g

does not need to vanish on Γ. We thus define the family of smooth functions $f_h : \mathbb{R} \to \mathbb{R}$ such that $h \in]0,1[$,

$$f_h(t) = \begin{cases} 1, & \text{if } t \le h/2, \\ 0, & \text{if } t \ge h, \end{cases}, \quad f_h'(t) \le 0, \quad |f_h'(t)| \le 3/h.$$

Recalling the definition of ρ in lemma 8.1.1, by proposition 7.2.1, we obtain that

$$\begin{aligned} \delta_{\mathcal{F}}\mathbf{V}(g) &= \int_{\mathbf{Gr}(\mathbf{B}_{r_0}\setminus\Gamma)} \Big[\langle D_x\mathcal{F}, g\rangle + \mathcal{B}_{\mathcal{F}} : Dg\Big] \mathrm{d}\mathbf{V} \\ &= \overbrace{\int_{\mathbf{Gr}(\mathbf{B}_{r_0}\setminus\Gamma)} \langle D_x\mathcal{F}, g\rangle \,\mathrm{d}\mathbf{V}}^{(*)} + \int_{\mathbf{Gr}(\mathbf{B}_{r_0}\setminus\Gamma)} \mathcal{B}_{\mathcal{F}} : D\left(g + (f_h\circ\rho)g - (f_h\circ\rho)g\right) \mathrm{d}\mathbf{V}. \end{aligned}$$

Notice that $(*)$ is controlled by $C_{\mathcal{F},W}\sup_{\mathbf{B}_{r_0}}|g|$, thus it remains to bound

$$\int_{\mathbf{Gr}(\mathbf{B}_{r_0}\setminus\Gamma)} \mathcal{B}_{\mathcal{F}} : \Big[\overbrace{D\left((1-f_h\circ\rho)\,g\right)}^{T_1} + \overbrace{(f_h\circ\rho)\,Dg}^{T_2} + \overbrace{f_h'\circ\rho(\nabla\rho)^t\cdot g}^{T_3}\Big] \mathrm{d}\mathbf{V}. \tag{8.2.1}$$

Using that $\mathcal{F}$ is of class C^1 and g has support in W, by the definition of $\mathcal{B}_{\mathcal{F}}$ in (7.2.1), we can bound the modulus of (8.2.1) by $C|T_1 + T_2 + T_3|$, where the constant is such that $C = C(\mathcal{F}, W) > 0$.

Since $(1 - f_h\circ\rho)\,g$ vanishes on Γ, by (7.2.2), we have that

$$\int_{\mathbf{Gr}(\mathbf{B}_{r_0}\setminus\Gamma)} \mathcal{B}_{\mathcal{F}} : D((1-f_h\circ\rho)\,g)\mathrm{d}\mathbf{V} = -\int_{\mathbf{Gr}(\mathbf{B}_{r_0}\setminus\Gamma)} \langle (1-f_h\circ\rho)\,g, \mathcal{H}_{\mathcal{F}} + D_x\mathcal{F}\rangle \,\mathrm{d}\mathbf{V}. \tag{8.2.2}$$

We notice that $f_h\circ\rho \to 0$ as $h \to 0$, which together with (8.2.2) ensures the estimate $|T_1| + |T_2| \le C_1(\mathcal{F}, W)\sup|g|$. It remains to bound the last summand T_3 by $C_2(\mathcal{F}, W)\sup|g|$, which is done by precisely the same proof provided in [Bou16, Equation 3.10]. Therefore we have that $\delta_{\mathcal{F}}\mathbf{V}(g) \le C\sup_{\mathbf{B}_{r_0}}|g|$ which guarantees that $\delta_{\mathcal{F}}\mathbf{V}$ is a Radon measure on $\mathbf{B}_{r_0}$. The moreover part can be proved as in [Bou16, Theorem 3.1], hence we omit the details here. □

Chapter 9

Caccioppoli inequality at boundary points

An usual step in the proof of regularity theorems is proving an estimate where the excess is controlled by the height, mean curvature, and an 'error' in case of 'boundary points'. This is the so-called Caccioppoli-type inequality. To the best of our knowledge, there is no such result for boundary points of m-rectifiable varifolds with L^2-integrable anisotropic mean curvature.

Allard did prove a Caccioppoli-type inequality in [All75, Lemma 4.5] for the area functional. Unfortunately, the techniques used in the isotropic case do not work in the anisotropic case due to the lack of a monotonicity formula. We also have another difficulty compared to Allard's work: our boundary Γ has regularity $C^{1,\alpha}$ while the setting of [All75] requires a boundary Γ of class $C^{1,1}$, as explained in the introductory section.

We aim to achieve a Caccioppoli-type inequality (proposition 9.0.2) in the sense of [All75, Lemma 4.5], [DRT22, Proposition 4.3], and [Bou16, Lemma 4.10].

Assumption 2. We assume assumption 1. We further impose that the anisotropic functional $\mathcal{F}$ satisfies USAC, defined in definition 7.3.1, and $\mathcal{H}_{\mathcal{F}} \in \mathrm{L}^2(\mathbf{B}_{r_0})$.

Under such assumptions, by remark 8.2.2, the varifold $\mathbf{V}$ is m-rectifiable. So, henceforth we might use the following notation $\mathbf{V} = \mathbf{v}(M, \Theta)$. We define the classical notions of excess and height for varifolds as follows.

Definition 9.0.1. Let $\mathbf{V} = \mathbf{v}(M, \Theta)$ be a rectifiable m-varifold and $\pi \in \mathbf{Gr}(m+n, m)$. We define **the tilt excess of V with respect to π in $\mathbf{B}(x, r)$** as the number

$$\mathbf{E}_{\mathbf{V}}(\pi, x, r) := \frac{1}{r^m} \int_{\mathbf{B}(x,r)} \|\pi - T_y M\|^2 \mathrm{d}\|\mathbf{V}\|(y).$$

We also define **the height excess of V with respect to π in $\mathbf{B}(x, r)$** to be the number

$$\mathbf{H}_{\mathbf{V}}(z, \pi, x, r) := \frac{1}{r^m} \int_{\mathbf{B}(x,r)} \mathrm{d}(y - z, \pi)^2 \mathrm{d}\|\mathbf{V}\|(y).$$

We usually hide the subscripts whenever it is clear from the context.

We now state the Caccioppoli-type inequality in this context.

Proposition 9.0.2 (Caccioppoli-type inequality)**.** *Under assumption 2, there exists a constant* $C = C(m, n, \|\mathcal{F}\|_{C^2}, K_{\mathcal{F}}, \Gamma) > 0$ *such that*

$$C\mathbf{E}(\pi, 0, r/2) \leq \frac{1}{r^2}\mathbf{H}(z, \pi, 0, r) + r^{2-m}\|\mathcal{H}_{\mathcal{F}}\|^2_{\mathrm{L}^2(\mathbf{B}_r)} + \kappa^2 r^{2\alpha}, \tag{9.0.1}$$

for all $z \in \mathbb{R}^{m+n}, 4r < r_0, \pi \in \mathbf{Gr}(m+n, m)$ *with* $T_0\Gamma \subset \pi$.

When the varifold $\mathbf{V}$ is induced by the graph of a Lipschitz function, the next corollary states that the quantities in proposition 9.0.2 can be replaced by integrations on balls of the subspace $\mathbb{R}^m$, while in proposition 9.0.2 they are quantities/integrations over balls of the ambient space $\mathbb{R}^{m+n}$.

Given an open bounded set $\Omega \subset \mathrm{B}_{r_0} \subset \mathbb{R}^m$ and a Lipschitz function $u : \Omega \to \mathbb{R}^n$, we will denote by

$$\mathbf{V}[u] := \mathbf{v}(\mathrm{graph}(u), 1)$$

the m-varifold induced by $\mathrm{graph}(u) \subset \mathbb{R}^{m+n}$ and by $\mathcal{H}_{\mathcal{F}}$ its anisotropic mean curvature. Let also $\mathcal{H}[u] : \Omega \to \mathbb{R}^n$ denote the function $\mathcal{H}[u](x) := \mathcal{H}_{\mathcal{F}}(x, u(x))$ and, for any $R > 0, z \in \mathrm{B}_R, s < \mathrm{d}(z, \partial\mathrm{B}_R)$, and $f : \mathrm{B}_R \subset \mathbb{R}^m \to \mathbb{R}^n$ measurable function, we set

$$(f)_{x,s} := \frac{1}{\mathcal{H}^m\left(\mathrm{B}\left(x, s\right) \cap s\Omega\right)} \int_{\mathrm{B}(x,s)\cap s\Omega} f(y)\mathrm{d}y \quad \text{and} \quad (f)_s := (f)_{0,s}.$$

For the reader's convenience, we recall that $\mathrm{B}\left(x, s\right) := \mathbf{B}((x, 0), s) \cap (\mathbb{R}^m \times \{0\})$.

Corollary 9.0.3 (Caccioppoli-type inequality)**.** *Assume that* $\mathbf{V}[u]$ *and* $\mathcal{H}[u]$ *satisfy assumption 2. There exists a constant* $C_c = C_c(m, n, \|\mathcal{F}\|_{C^2}, K_{\mathcal{F}}, \Gamma, \|u\|_{\mathrm{Lip}}) > 0$ *and* $C_u := 2 + 2\|u\|_{\mathrm{Lip}}$ *such that*

$$\begin{aligned} C_c \fint_{\mathrm{B}_r\cap\Omega} \|Du(y) - L\|^2 \mathrm{d}y \leq \frac{1}{r^2}\fint_{\mathrm{B}_{C_u r}\cap\Omega} |u(y) - (u)_{C_u r} - L(y)|^2 \mathrm{d}y \\ + r^2 \fint_{\mathrm{B}_{C_u r}\cap\Omega} |\mathcal{H}[u](y)|^2 \mathrm{d}y + \kappa^2 r^{2\alpha}, \end{aligned}$$

for all $L \in \mathbb{R}^m \otimes \mathbb{R}^n$ *such that* $T_0\Gamma \subset \mathrm{im}(h(L))$ *and* $\|L\| \leq 2\|u\|_{\mathrm{Lip}}$ *and all* $r \in (0, 4^{-1}r_0)$.

Remark 9.0.4. The function h stands for one of the canonical charts of the Grassmannian, we make it precise defining $h : \mathbb{R}^m \otimes \mathbb{R}^n \to \mathbb{R}^{m+n} \otimes \mathbb{R}^{m+n}$ as

$$h(L) := M(L)\left[M(L)^t M(L)\right]^{-1} M(L)^t, \text{ where } M(L) := \begin{pmatrix} \mathrm{id}_m \\ L \end{pmatrix}.$$

We refer the reader to [DDKT21, Subsection 6.1], [DRT22, Page 470], and [HT21, Subsection A.6] for a more expository introduction to these objects.

Proof. Extending this proof from the interior case to boundary points setting is identical to the argument presented in [DRT22, Corollary 4.4], but now relying on proposition 9.0.2. ☐

Proof of proposition 9.0.2. First of all, by standard arguments, cf. [DRT22, Page 465], we can assume without loss of generality that $\mathcal{F}$ is an autonomous functional, i.e., it does not depend on the variable in $\mathbb{R}^{m+n}$. Hence we will denote $\mathcal{B}_{\mathcal{F}}(\pi) \equiv \mathcal{B}_{\mathcal{F}}(x, \pi)$ and $K_{\mathcal{F}} \equiv K_{\mathcal{F},x}$. We can set the m-manifold of class $C^{1,\alpha}$ given by $\overline{\Gamma} = \Gamma + (N_0\Gamma \cap \pi)$. In particular, by remark 8.1.2, we have a

Whitney decomposition $\overline{\mathcal{W}}$ of $\mathbf{B}_{r_0} \setminus \overline{\Gamma}$. We denote with x_C and $\overline{x}_C$ respectively the center of the cube C and the orthogonal projection of x_C on $\overline{\Gamma}$. We consider a Whithney's partition of unity $\{\overline{\varphi}_C\}_{C\in\mathcal{W}}$, and a C^1 function $\overline{\rho}$ satisfying all the conclusions of lemma 8.1.1.

We choose the following vector field $g \in C^1_c(\mathbf{B}_{2r}, \mathbb{R}^{m+n})$ as a test for the first variation:

$$g(x) := \psi^2(x) \sum_{C\in\mathcal{W}} \overline{\varphi}^2_C(x) g_C(x), \quad \text{where} \quad g_C(x) := \mathcal{B}_{\mathcal{F}^*}(\pi^\perp)(\mathbf{p}_{N_{\overline{x}_C}\overline{\Gamma}}(x - \overline{x}_C)),$$

where $\psi \in C^\infty_c(\mathbf{B}_{2r}, [0,1])$ such that $\psi|_{\mathbf{B}_r} \equiv 1$. It is important to choose g using the Whitney decomposition, since it ensures that $g|_{\overline{\Gamma}} \equiv 0$, in particular $g|_\Gamma \equiv 0$, and then (7.2.2) holds. By direct computations we obtain that

$$Dg = \sum_{C\in\mathcal{W}} \left[2\psi\overline{\varphi}^2_C\,(g_C)\cdot(\nabla\psi)^t + \psi^2\overline{\varphi}^2_C Dg_C + 2\psi^2\overline{\varphi}_C g_C \cdot (\nabla\overline{\varphi}_C)^t \right], \tag{9.0.2}$$

$$Dg_C = \mathcal{B}_{\mathcal{F}^*}(\pi^\perp) \circ \mathbf{p}_{N_{\overline{x}_C}\overline{\Gamma}}. \tag{9.0.3}$$

Equation (9.0.2) together with (7.2.2) assures that

$$-\int \langle \mathcal{H}_\mathcal{F}, g\rangle = \int \sum_{C\in\mathcal{W}} \mathcal{B}_\mathcal{F} : \left[2\psi\overline{\varphi}^2_C\,(g_C)\cdot(\nabla\psi)^t + \psi^2\overline{\varphi}^2_C Dg_C + 2\psi^2\overline{\varphi}_C g_C \cdot (\nabla\overline{\varphi}_C)^t \right]. \tag{9.0.4}$$

We set the following notation

$$R_1 := \int \sum_{C\in\mathcal{W}} \psi^2(x)\overline{\varphi}^2_C(x) \langle \mathcal{H}_\mathcal{F}(x), g_C(x)\rangle \, \mathrm{d}\|\mathbf{V}\|(x),$$

$$R_2 := \int \sum_{C\in\mathcal{W}} 2\psi(x)\overline{\varphi}^2_C(x)\mathcal{B}_\mathcal{F}(T_xM) : \left(g_C(x)\cdot\nabla\psi(x)^t\right) \mathrm{d}\|\mathbf{V}\|(x),$$

$$R_3 := \int \sum_{C\in\mathcal{W}} 2\psi^2(x)\overline{\varphi}_C(x)\mathcal{B}_\mathcal{F}(T_xM) : \left(g_C(x)\cdot\nabla\overline{\varphi}_C(x)^t\right) \mathrm{d}\|\mathbf{V}\|(x),$$

$$L_1 := -\int \sum_{C\in\mathcal{W}} \psi^2(x)\overline{\varphi}^2_C(x)\mathcal{B}_\mathcal{F}(T_xM) : \mathcal{B}_{\mathcal{F}^*}(\pi^\perp)\circ\mathbf{p}_{N_{\overline{x}_C}\overline{\Gamma}} \mathrm{d}\|\mathbf{V}\|(x).$$

By (9.0.4) and (9.0.3) we obtain that

$$L_1 = R_1 + R_2 + R_3. \tag{9.0.5}$$

We estimate $|L_1|$ from below. By the definition of L_1 and the uniformly scalar atomic condition, definition 7.3.1, recalling that $\psi|_{\mathbf{B}_r} \equiv 1$,we get

$$\begin{aligned} |L_1| &\geq K_\mathcal{F} \int \sum_{C\in\mathcal{W}} \psi^2(x)\overline{\varphi}^2_C(x)\|T_xM - \pi\|^2 \mathrm{d}\|\mathbf{V}\|(x) \\ &\overset{(8.1.2)}{\geq} K_\mathcal{F} C \int_{\mathbf{B}_r} \|T_xM - \pi\|^2 \mathrm{d}\|\mathbf{V}\|(x) = K_\mathcal{F} C r^m \mathbf{E}(\pi, 0, r), \end{aligned} \tag{9.0.6}$$

where here and in the rest of this proof $C = C(m, n, r_0) > 0$ is defined in (8.1.2). The right-hand side of (9.0.6) is exactly the desired left hand side in the Caccioppoli-type inequality (9.0.1), up to the factor r^m. Therefore, it remains to bound $|R_1| + |R_2| + |R_3|$ from above with the right hand side in (9.0.1) (again up to the factor r^m) plus a term that can be reabsorbed in the left hand side of (9.0.1). With this aim in mind, let us estimate the term R_3. We have that

$$R_3 = \int \sum_{C\in\mathcal{W}} 2\psi^2(x)\overline{\varphi}_C(x)\mathcal{B}_{\mathcal{F}}(T_xM) : \left(g_C(x)\cdot\nabla\overline{\varphi}_C(x)^t\right) \mathrm{d}\|\mathbf{V}\|(x).$$

By straightforward linear algebra computations, we have that

$$\mathcal{B}_{\mathcal{F}}(\pi)^t \cdot \mathcal{B}_{\mathcal{F}^*}(\pi^\perp) = 0 \tag{9.0.7}$$

which in turn implies

$$R_3 = \int \sum_{C\in\mathcal{W}} 2\psi^2(x)\overline{\varphi}_C(x)\left(\mathcal{B}_{\mathcal{F}}(T_xM) - \mathcal{B}_{\mathcal{F}}(\pi)\right) : \left(g_C(x)\cdot\nabla\overline{\varphi}_C(x)^t\right) \mathrm{d}\|\mathbf{V}\|(x).$$

We apply Young's inequality to obtain

$$\begin{aligned}|R_3| \le \frac{K_{\mathcal{F}}C}{4}\int \psi^4(x)\|T_xM-\pi\|^2\mathrm{d}\|\mathbf{V}\|(x) \\ + c(m,n,K_{\mathcal{F}})\int\left|\sum_{C\in\mathcal{W}}\overline{\varphi}_C(x)g_C(x)\cdot(\nabla\overline{\varphi}_C(x))^t\right|^2\mathrm{d}\|\mathbf{V}\|(x).\end{aligned} \tag{9.0.8}$$

To bound the second summand on the right hand side of the last inequality, we proceed as follows

$$\begin{aligned}\sum_{C\in\mathcal{W}}\mathbf{p}_{N_{\overline{x}_C}\Gamma}(x-\overline{x}_C)\cdot(\nabla\overline{\varphi}_C(x))^t &= \sum_{C\in\mathcal{W}}\left(\mathbf{p}_{N_{\overline{x}_C}\Gamma}(x-\overline{x}_C) - \mathbf{p}_{N_{\overline{x}_C}\Gamma}(x-\overline{x})\right)\cdot(\nabla\overline{\varphi}_C(x))^t \\ &= \sum_{C\in\mathcal{W}}\mathbf{p}_{N_{\overline{x}_C}\Gamma}(\overline{x}-\overline{x}_C)\cdot(\nabla\overline{\varphi}_C(x))^t,\end{aligned}$$

where in the first equality we have used that $\sum_{C\in\mathcal{W}}\nabla\overline{\varphi}_C \equiv 0$. Plugging the equality above in (9.0.8), we get that

$$\begin{aligned}|R_3| &\le \frac{K_{\mathcal{F}}C}{4}r^m\mathbf{E}(\pi,0,r) + c\int\sum_{C\in\mathcal{W}}\overline{\varphi}_C(x)|\mathbf{p}_{N_{\overline{x}_C}\Gamma}(\overline{x}-\overline{x}_C)|^2|\nabla\overline{\varphi}_C(x)|^2\mathrm{d}\|\mathbf{V}\|(x) \\ &\overset{(8.1.1)}{\le} \frac{K_{\mathcal{F}}C}{4}r^m\mathbf{E}(\pi,0,r) + c_1\int\sum_{C\in\mathcal{W}}\overline{\varphi}_C(x)\frac{|\mathbf{p}_{N_{\overline{x}_C}\Gamma}(\overline{x}-\overline{x}_C)|^2}{\operatorname{diam}^2(C)}\mathrm{d}\|\mathbf{V}\|(x) \\ &\overset{(8.0.1)}{\le} \frac{K_{\mathcal{F}}C}{4}r^m\mathbf{E}(\pi,0,r) + \kappa^2c_1\int\sum_{C\in\mathcal{W}}\overline{\varphi}_C(x)\frac{|\overline{x}-\overline{x}_C|^{2+2\alpha}}{\operatorname{diam}^2(C)}\mathrm{d}\|\mathbf{V}\|(x) \\ &\le \frac{K_{\mathcal{F}}C}{4}r^m\mathbf{E}(\pi,0,r) + \kappa^2c_1\int\sum_{C\in\mathcal{W}}\overline{\varphi}_C(x)|\overline{x}-\overline{x}_C|^{2\alpha}\mathrm{d}\|\mathbf{V}\|(x) \\ &\le \frac{K_{\mathcal{F}}C}{4}r^m\mathbf{E}(\pi,0,r) + \kappa^2r^{m+2\alpha}c_1,\end{aligned} \tag{9.0.9}$$

where $c_1 = c_1(m,n,\|\mathcal{F}\|_{C^2},\mathcal{W}) > 0$.

Turning our attention to R_1, thanks to the hypothesis that $\mathcal{H}_{\mathcal{F}}$ belongs to L^2, we apply Young's inequality and Jensen inequality to get

$$
\begin{aligned}
|R_1| &= \left| \int \psi^2(x) \left\langle \mathcal{H}_{\mathcal{F}}(x), \sum_{C\in\mathcal{W}} \overline{\varphi}_C^2(x) g_C(x) \right\rangle \mathrm{d}\|\mathbf{V}\|(x) \right| \\
&\leq r^2 \|\mathcal{H}_{\mathcal{F}}\|^2_{\mathrm{L}^2(\mathbf{B}_{2r})} + \frac{C_1(m,n)}{r^2} \int \sum_{C\in\mathcal{W}} \psi^4(x) \overline{\varphi}_C^4(x) |g_C(x)|^2 \mathrm{d}\|\mathbf{V}\| \qquad (9.0.10)\\
&\leq r^2 \|\mathcal{H}_{\mathcal{F}}\|^2_{\mathrm{L}^2(\mathbf{B}_{2r})} + \frac{C_1(m,n)}{r^2} \int \sum_{C\in\mathcal{W}} \overline{\varphi}_C(x) |g_C(x)|^2 \mathrm{d}\|\mathbf{V}\|.
\end{aligned}
$$

We now use (9.0.7) to estimate the summand R_2 as follows

$$
\begin{aligned}
|R_2| &\leq \left| \int \sum_{C\in\mathcal{W}} 2\psi(x)\overline{\varphi}_C^2(x) \left(\mathcal{B}_{\mathcal{F}}(T_xM) - \mathcal{B}_{\mathcal{F}}(\pi)\right) : \left(g_C(x)\cdot \nabla\psi(x)^t\right) \mathrm{d}\|\mathbf{V}\|(x) \right| \\
&\leq 2 \int \|\psi\| \|\nabla\psi\| \|\mathcal{B}_{\mathcal{F}}(T_xM) - \mathcal{B}_{\mathcal{F}}(\pi)\| \sum_{C\in\mathcal{W}} \overline{\varphi}_C^2(x) |g_C(x)| \mathrm{d}\|\mathbf{V}\|(x) \\
&\leq C_2(m,n,\|\mathcal{F}\|_{C^2}) \int \|\psi\| \|T_xM - \pi\| \sum_{C\in\mathcal{W}} \overline{\varphi}_C^2(x) |g_C(x)| \mathrm{d}\|\mathbf{V}\|(x) \\
&\leq \frac{r^2 K_{\mathcal{F}} C}{4} \int_{\mathbf{B}_{2r}} \|\psi\|^2 \|T_xM - \pi\|^2 + \frac{C_2(m,n,\|\mathcal{F}\|_{C^2})}{r^2} \sum_{C\in\mathcal{W}} \overline{\varphi}_C^2(x) |g_C(x)|^2 \mathrm{d}\|\mathbf{V}\|(x),
\end{aligned}
$$

where in the third inequality we have used that $\mathcal{F}$ is C^2 and that the Grassmannian is compact, and in the fourth inequality we have used again Young's inequality. Since the last chain of inequalities is true for any ψ choosen as above, we can take a sequence $\{\psi_i\}_{i\in\mathbb{N}} \subset C_c^\infty(\mathbf{B}_{2r}[0,1])$ such that ψ_i converges to the indicator functions of $\mathbf{B}_r$. Therefore we obtain that

$$
\begin{aligned}
|R_2| &\leq \frac{K_{\mathcal{F}} C}{4} \int_{\mathbf{B}_r} \|T_xM - \pi\|^2 + \frac{C_2}{r^2} \int_{\mathbf{B}_{2r}} \sum_{C\in\mathcal{W}} \overline{\varphi}_C^4(x) |g_C(x)|^2 \\
&= \frac{K_{\mathcal{F}} C}{4} r^m \mathbf{E}(\pi,0,r) + \frac{C_2}{r^2} \int_{\mathbf{B}_{2r}} \sum_{C\in\mathcal{W}} \overline{\varphi}_C(x) |g_C(x)|^2,
\end{aligned} \qquad (9.0.11)
$$

where $C_2 = C_2(m,n,\|\mathcal{F}\|_{C^2}) > 0$. We finally use (9.0.9), (9.0.10), and (9.0.11) to estimate

$$
\begin{aligned}
|R_1| + |R_2| + |R_3| \leq{}& \frac{K_{\mathcal{F}} C}{2} r^m \mathbf{E}(\pi,0,r) + r^2 \|\mathcal{H}_{\mathcal{F}}\|^2_{\mathrm{L}^2(\mathbf{B}_{2r})} + \kappa^2 r^{m+2\alpha} c_1 \\
&+ \frac{C_2}{r^2} \int_{\mathbf{B}_{2r}} \sum_{C\in\mathcal{W}} \overline{\varphi}_C(x) |\mathcal{B}_{\mathcal{F}^*}(\pi^\perp)(\mathbf{p}_{N_{\overline{x}_C}\Gamma}(x - \overline{x}_C))|^2 \mathrm{d}\|\mathbf{V}\|,
\end{aligned} \qquad (9.0.12)
$$

where $c_1 = c_1(m,n,\|\mathcal{F}\|_{C^2},\mathcal{W}) > 0$, and $C_2 = C_2(m,n,\|\mathcal{F}\|_{C^2}) > 0$. It only remains to bound the last summand of the previous inequality. We firstly recall the equality in [DRT22, Equation 3.5] which states that $\mathcal{B}_{\mathcal{F}}(\pi^\perp) = \mathcal{F}(\pi)\pi^\perp - \pi D\mathcal{F}(\pi)\pi^\perp$ and thus we obtain the following

$$\begin{aligned}
|\mathcal{B}_{\mathcal{F}^*}(\pi^\perp)(\mathbf{p}_{N_{\overline{x}_C}\overline{\Gamma}}(x-\overline{x}_C))| &\leq \|\mathcal{F}(\pi)\pi^\perp - \pi D\mathcal{F}(\pi)\pi^\perp\| |\mathbf{p}_{N_{\overline{x}_C}\overline{\Gamma}}(x-\overline{x}_C)| \\
&\leq \|\mathcal{F}\|_{C^2} |\mathbf{p}_{N_{\overline{x}_C}\overline{\Gamma}}(x) - \mathbf{p}_{N_{\overline{x}_C}\overline{\Gamma}}(\overline{x}_C)| \\
&\leq \|\mathcal{F}\|_{C^2} \left(|\mathbf{p}_{N_{\overline{x}_C}\overline{\Gamma}}(x)| + |\mathbf{p}_{N_{\overline{x}_C}\overline{\Gamma}}(\overline{x}_C)| \right) \\
&\overset{(8.0.1)}{\leq} \|\mathcal{F}\|_{C^2} \left(|\mathbf{p}_{N_{\overline{x}_C}\overline{\Gamma}}(x)| + \kappa |\overline{x}_C|^{1+\alpha} \right) \\
&\leq \|\mathcal{F}\|_{C^2} \left(|(\mathbf{p}_{N_{\overline{x}_C}\overline{\Gamma}} - \mathbf{p}_{\pi^\perp})(x)| + |\mathbf{p}_{\pi^\perp}(x)| + \kappa r^{1+\alpha} \right) \\
&\leq \|\mathcal{F}\|_{C^2} \left(|(\mathbf{p}_{N_{\overline{x}_C}\overline{\Gamma}} - \mathbf{p}_{\pi^\perp})(x)| + \mathrm{d}(x,\pi) + \kappa r^{1+\alpha} \right) \\
&\overset{(8.0.1)}{\leq} \|\mathcal{F}\|_{C^2} \left(\kappa |\overline{x}_C|^\alpha |x| + \mathrm{d}(x,\pi) + \kappa r^{1+\alpha} \right) \\
&\leq 4\|\mathcal{F}\|_{C^2} \left(\mathrm{d}(x,\pi) + \kappa r^{1+\alpha} \right).
\end{aligned}$$

The chain of inequalities above with (9.0.12) provides the following estimate

$$\begin{aligned}
|R_1| + |R_2| + |R_3| \leq{}& \frac{K_{\mathcal{F}} C}{2} r^m \mathbf{E}(\pi, 0, r) + r^2 \|\mathcal{H}_{\mathcal{F}}\|^2_{\mathrm{L}^2(\mathbf{B}_{2r})} \\
&+ c_2(m, n, \|\mathcal{F}\|_{C^2}, \mathcal{W}) \left(\kappa^2 r^{m+2\alpha} + \int_{\mathbf{B}_{2r}} \frac{\mathrm{d}^2(x,\pi)}{r^2} \mathrm{d}\|\mathbf{V}\| \right).
\end{aligned}$$

Combining this inequality with (9.0.6), and recalling (9.0.5), we can reabsorb $\frac{K_{\mathcal{F}} C}{2} r^m \mathbf{E}(\pi, 0, r)$ on the left hand side and conclude the proof of (9.0.1). □

Chapter 10

Excess decay at boundary points

In this section, we will work under the following assumption 3. It is clear that assumption 3 is more restrictive than assumption 2.

Assumption 3. We assume assumption 1. Additionally, $\mathbf{V} = \mathbf{V}[u]$ and $\mathcal{H}_{\mathcal{F}} = \mathcal{H}_{\mathcal{F}}[u]$, where $u : \Omega \subset \mathrm{B}_{r_0} \subset \mathbb{R}^m \to \mathbb{R}^n$ is a Lipschitz function with $\mathcal{H}_{\mathcal{F}}[u] \in \mathrm{L}^p(\Omega)$ for some $p > m$. We also set $\Gamma \cap \mathbf{B}_{4r_0} = \partial(\mathrm{graph}(u)) \cap \mathbf{B}_{4r_0}$ and $\partial\Omega = \mathbf{p}(\Gamma)$ splits B_{4r_0} into two disjoint open sets, namely Ω and $\mathrm{B}_{4r_0} \setminus \Omega$. Moreover, there exists $\sigma \in (0,1)$ such that for every $r \in (0, 4r_0)$, we have

$$\frac{\|\mathbf{V}[u]\|(\mathbf{B}_r)}{\omega_m r^m} \leq \frac{1}{2} + \sigma. \tag{10.0.1}$$

We recall a lemma that relates the stationarity of the function u with the stationarity of the varifold $\mathbf{V}[u]$ induced by u. This lemma is proved in [DDKT21] for the case of interior points. Let us set the notation to state it:

$$\mathcal{A}(L) := \sqrt{M(L)^t M(L)}, \quad \mathcal{I}_{\mathcal{F}}(L) \ := \mathcal{A}(L)\mathcal{F}\left(h(L)\right) \qquad \forall L \in \mathbb{R}^m \otimes \mathbb{R}^n. \tag{10.0.2}$$

where h and $M(L)$ are defined in remark 9.0.4.

Lemma 10.0.1. *Assume assumption 3. If for some positive constants C and $q \geq 1$ it holds*

$$|\delta_{\mathcal{F}}\mathbf{V}[u](g)| \leq C\|g\|_{\mathrm{L}^q(\mathrm{B}_{4r_0}\times\mathbb{R}^m)}, \quad \forall g \in C^1_c\left(\mathrm{B}_{4r_0} \times \mathbb{R}^m, \mathbb{R}^{m+n}\right) \ \textit{with } g|_{\Gamma} \equiv 0,$$

then there exists $C' = C'(C, m, p, q) > 0$ such that

$$\left|\int_{\Omega} \langle D\mathcal{I}_{\mathcal{F}}(Du), D\zeta\rangle\right| \mathrm{d}\mathcal{H}^m \leq C'\|\zeta\mathcal{A}^{\frac{1}{q}}(Du)\|_{\mathrm{L}^q(\mathrm{B}_{4r_0})}, \quad \forall \zeta \in C^1_c\left(\mathrm{B}_{4r_0}, \mathbb{R}^n\right) \ \textit{with } \zeta|_{\mathbf{p}(\Gamma)} \equiv 0. \tag{10.0.3}$$

Moreover, if $C = 0$, thus $C' = 0$.

The proof of lemma 10.0.1 is a straightforward extension of [DDKT21, Proposition 6.8] to our boundary setting. Furthermore, [DDKT21, Proposition 6.8] can be adapted to give the equivalence between the two properties. However, we choose to state only the exact statement we will use.

We now use the mass ratio bound (10.0.1) in assumption 3 to prove the following technical lemma, that will allow us to apply the Caccioppoli inequality (proposition 9.0.2).

Lemma 10.0.2. *Under assumption 3, there exists $C_d = C_d(m,n,\alpha) > 0$, $c_0 = c_0(m,n,\alpha) > 0$ and $L_u \in \mathbb{R}^m \otimes \mathbb{R}^n$ with $T_0\Gamma \subset \mathrm{im}(h(L_u))$ such that $\|L_u - (Du)_r\| \le C_d r^\alpha + C_d\sigma$ for any $r \in (0, c_0)$.*

Proof. Without loss of generality, we can assume that $\Gamma = \mathbf{B}_{4r_0} \cap \mathbb{R}^{m-1} \times \{0\}$ by a standard procedure of straightening the boundary (for instance, using [DLNS23, Lemma 3.1]). By the Taylor expansion of the mass (c.f. [DS11]), we obtain that

$$C_0 r^{m+\alpha} + 2\left(\|\mathbf{V}[u]\|(\mathbf{B}_r) - \frac{\omega_m r^m}{2}\right) \ge \int_{\{x_m \ge 0\}\cap \mathrm{B}_r} \|Du\|^2.$$

Thus the control over the mass ratio enables us to straightforwardly derive that

$$\|(Du)_r\| \le C_0 r^\alpha + 2\sigma.$$

We choose $L_u := \lim_{s\to 0}(Du)_s$ which, by the last inequality, satisfies the desired inequality. It is easy to see that $T_0\Gamma = \mathbb{R}^{m-1}\times\{0\} \subset \mathrm{im}(h(L_u))$, since $Du(x) = x_m v_0$ for any $x = (x', x_m) \in \mathbb{R}^{m-1}\times\{0\}$ and a fixed $v_0 \in \mathbb{R}^n$. □

One of the crucial parts of the regularity theory is to prove an excess decay with a precise rate of decay. Let us fix the following shorthand notation for the excess of the function u:

$$\begin{gathered} E(x,r,L) := \fint_{\mathrm{B}(x,r)\cap\Omega} \|Du(z) - L\|^2 \mathrm{d}z, \\ E(x,r) := E(x,r,(Du)_r), \text{ and } E(r) := E(0,r). \end{gathered} \tag{10.0.4}$$

We now prove the excess decay at boundary points for the function u, i.e., we prove that the derivative Du of u becomes closer in L^2-norm to a linear map as we decrease the radius of balls centered at the origin. The proof follows a similar argument as the one for [DRT22, Proposition 4.5].

Proposition 10.0.3 (Excess decay)**.** *Under assumption 3, there exists a positive constant $C_e = C_e(m,n,\|\mathcal{F}\|_{C^2}, K_{\mathcal{F}}, \|u\|_{\mathrm{Lip}}, \Gamma) > 0$ with the following property. For every $\varepsilon \in (0, 4^{-1}C_u^{-1})$, there exist $\delta = \delta(\varepsilon) > 0$ such that*

$$r^{\min\{\alpha, 1-\frac{m}{p}\}}\|\mathcal{H}_{\mathcal{F}}[u]\|_{\mathrm{L^p}(\Omega\times\mathbb{R}^n)} \le E(r) \le \delta \text{ and } \sigma \le \delta \tag{10.0.5}$$

imply

$$E(\varepsilon r, L_u) \le C_e \varepsilon^{2\alpha} E(r). \tag{10.0.6}$$

Proof. As in the proof of proposition 9.0.2, we can again assume without loss of generality that $\mathcal{F}$ is an autonomous functional.

We prove our statement by a contradiction argument. Assume that for every $C_e > 0$ there exist $\varepsilon \in (0, 4^{-1}C_u^{-1})$ such that, for any $\delta, \sigma > 0$ satisfying

$$r^{\min\{\alpha, 1-\frac{m}{p}\}}\|\mathcal{H}_{\mathcal{F}}[u]\|_{\mathrm{L^p}(\Omega\times\mathbb{R}^n)} \le E(r) \le \delta \text{ and } \sigma \le \delta, \tag{10.0.7}$$

for some r, (10.0.6) does not hold, i.e.,

$$E(\varepsilon r, L_u) > C_e \varepsilon^{2\alpha} E(r). \tag{10.0.8}$$

We divide our proof into three steps. In Step 1, we prove that a certain blowup sequence for u converges in $\mathrm{W}^{1,2}(\mathrm{B}_r \cap \Omega)$ to a limit function u_0. After that, we show in Step 2 that the function u_0 is a weak solution of an elliptic system of PDEs, subsequently we use regularity theory for elliptic PDEs to obtain an estimate for the second derivative of u_0. We close our argument in Step 3, where we apply the Caccioppoli inequality, corollary 9.0.3, together with the elliptic estimates from Step 2, to get a contradiction with (10.0.8).

Step 1: We choose $\sigma_j = \delta_j \varepsilon^\alpha$ and $\delta_j^2 := E(r_j)$ where $r_j \to 0$ satisfying both (10.0.7) and (10.0.8). For j large enough such that $r_j \leq \min\{c_0, \sigma^{1/\alpha}\}$, we pick L given by lemma 10.0.2. We set $\Omega_j := r_j^{-1}\Omega$, $\Gamma_j := r_j^{-1}\Gamma$, and the blowup sequence as follows

$$\begin{aligned} u_j \colon \Omega_j &\to \mathbb{R}^n \\ z &\mapsto \frac{u(r_j z) - (u)_{r_j} - r_j (Du)_{r_j} z}{\delta_j r_j}. \end{aligned}$$

We assume that $\delta_j > 0$, otherwise there is nothing to prove. It is easy to see that $\Omega_j \to \{x \in \mathbb{R}^m : x_m \geq 0\}$ and $\Gamma_j \to \mathbb{R}^{m-1}$ as j goes to $+\infty$. Furthermore, we list some properties of the sequence u_j that will be used in this proof. They are:

(a) $Du_j(z) = \delta_j^{-1}\left(Du(r_j z) - (Du)_{r_j}\right)$, which is a trivial computation;

(b) $(Du_j)_1 = 0$, which is a straightforward consequence of item (a);

(c) $\fint_{\mathrm{B}_1 \cap \Omega_j} \|Du_j\|^2 = 1$, which follows changing variables and using item (a);

(d) $\int_{\mathrm{B}_1 \cap \Omega_j} \|u_j - (u_j)_1\|^2$ is uniformly bounded. This follows from the Poincarè-Witinger inequality and item (c);

(e) $E(\varepsilon r_j, L_u) > C_e \varepsilon^{2\alpha} E_{u_j}(1)$, where we set $E_{u_j}(r)$ to be the excess, as defined above in (10.0.4), for the function u_j, i.e., $E_{u_j}(r) := \fint_{\mathrm{B}_r \cap \Omega_j} \|Du_j(z) - (Du_j)_r\|^2 \mathrm{d}z$. This item follows from (10.0.8), item (b), and the definition of u_j.

Denote the halfball $\mathrm{B}_s^+ := \mathrm{B}_s \cap \{(x', x_m) \in \mathbb{R}^{m-1} \times \mathbb{R} : x_m > 0\}, \forall s > 0$. As a consequence of item (c) and item (d), we obtain that (u_j) is bounded in $\mathrm{W}^{1,2}(\mathrm{B}_1^+)$. Since $\mathrm{W}^{1,2}(\mathrm{B}_1^+)$ is reflexive, we can assume that

$$u_j \rightharpoonup u_0 \text{ in } \mathrm{W}^{1,2}(\mathrm{B}_1^+) \text{ and } u_j \to u_0 \text{ in } \mathrm{L}^2(\mathrm{B}_1^+).$$

By classical trace theory, c.f. [Eva10, Section 5.5], we have the following convergence

$$u_j \to u_0 \text{ in } \mathrm{L}^2(\mathbb{R}^{m-1} \cap \mathrm{B}_1^+) \Rightarrow u_0|_{\mathbb{R}^{m-1} \cap \mathrm{B}_1} \equiv 0.$$

Moreover, we also have that there exists a matrix $(Du)_0$ such that $(Du)_{r_j} \to (Du)_0$ thanks to the fact that $\{(Du)_{r_j}\}_{j \in \mathbb{N}}$ is equibounded.

Step 2: We start defining, for all $A \in \mathbb{R}^n \otimes \mathbb{R}^m$, the following sequence of operators

$$\mathcal{I}_j(A) := \frac{1}{\delta_j^2}\left[\mathcal{I}_{\mathcal{F}}(\delta_j A + (Du)_{r_j}) - \mathcal{I}_{\mathcal{F}}((Du)_{r_j}) - \delta_j \langle D\mathcal{I}_{\mathcal{F}}((Du)_{r_j}), A\rangle\right], \tag{10.0.9}$$

where $\mathcal{I}_{\mathcal{F}}$ is defined above in eq. (10.0.2). One can check that $\mathcal{I}_j(A) \to D^2\mathcal{I}_{\mathcal{F}}((Du)_0)[A,A]$ in the C^2-topology. We now claim that u_0 is a weak solution of an elliptic system of PDEs, namely,

$$\int_{\mathrm{B}_1^+} D^2\mathcal{I}_{\mathcal{F}}((Du)_0)[Du_0, D\zeta]\mathrm{d}\mathcal{H}^m = 0 \text{ for all } \zeta \in C_c^\infty(\mathrm{B}_1^+,\mathbb{R}^n), \quad \zeta|_{\mathbf{p}(\Gamma)} \equiv 0. \tag{10.0.10}$$

For the fluency of the text, we let the proof of this claim to the end. Since u_0 is a weak solution of the elliptic PDE in (10.0.10), we have that

$$\begin{aligned}\sup_{\mathrm{B}_{1/2}^+}\|D^2u_0\|^2 \leq \|u_0\|_{C^{2,\alpha}(\mathrm{B}_{1/2}^+)} &\overset{\text{Schauder Est.}}{\leq} C_0\|u_0\|_{C^{0,\alpha}(\mathrm{B}_{1/2}^+)} \\ \overset{\text{DG-N-M}}{\leq} C_0\|u_0\|_{\mathrm{L}^2(\mathrm{B}_1^+)} &\overset{\text{Poincaré Ineq.}}{\leq} C_0\|Du_0\|_{\mathrm{L}^2(\mathrm{B}_1^+)} \overset{\text{item (c)}}{\leq} C_0,\end{aligned} \tag{10.0.11}$$

where DG-N-M stands for the De Giorgi-Nash-Moser inequality and we put item (c) into account to use the Poincaré inequality.

Step 3: We now apply the Caccioppoli inequality, i.e., corollary 9.0.3, for L chosen at the beginning of the proof and $r = \varepsilon r_j$ to obtain

$$\begin{aligned}C_c⨍_{\mathrm{B}_{\varepsilon r_j}\cap\Omega}\|Du(y)-L_u\|^2\mathrm{d}y \leq \frac{1}{(\varepsilon r_j)^2}⨍_{\mathrm{B}_{C_u\varepsilon r_j}\cap\Omega}|u(y)-(u)_{C_u\varepsilon r_j}-L_u(y)|^2\mathrm{d}y \\ + (\varepsilon r_j)^2⨍_{\mathrm{B}_{C_u\varepsilon r_j}\cap\Omega}|\mathcal{H}[u](y)|^2\mathrm{d}y + \kappa^2(\varepsilon r_j)^{2\alpha}.\end{aligned}$$

Using Hölder inequality, we guarantee that

$$\begin{aligned}C_c⨍_{\mathrm{B}_{\varepsilon r_j}\cap\Omega}\|Du(y)-L_u\|^2\mathrm{d}y \leq \frac{1}{(\varepsilon r_j)^2}⨍_{\mathrm{B}_{C_u\varepsilon r_j}\cap\Omega}|u(y)-(u)_{C_u\varepsilon r_j}-L_u(y)|^2\mathrm{d}y \\ + C_0(\varepsilon r_j)^{2-\frac{2m}{p}}\|\mathcal{H}_{\mathcal{F}}[u]\|^2_{\mathrm{L}^p} + \kappa^2(\varepsilon r_j)^{2\alpha}.\end{aligned} \tag{10.0.12}$$

We work on the integral in the right-hand side of (10.0.12) as follows

$$\begin{aligned}&\frac{1}{(C_u\varepsilon r_j)^2}⨍_{\mathrm{B}_{C_u\varepsilon r_j}\cap\Omega}|u(y)-(u)_{C_u\varepsilon r_j}-L_u(y)|^2\mathrm{d}y \\ &\overset{\text{Poincaré Ineq.}}{\leq} ⨍_{\mathrm{B}_{C_u\varepsilon r_j}\cap\Omega}|Du(y)-L_u|^2\mathrm{d}y + (\|L_u\|C_u\varepsilon r_j)^2 \\ &\leq (\|L_u\|C_u\varepsilon r_j)^2 + \|L_u-(Du)_{C_u\varepsilon r_j}\|^2 + ⨍_{\mathrm{B}_{C_u\varepsilon r_j}\cap\Omega}|Du(y)-(Du)_{C_u\varepsilon r_j}|^2\mathrm{d}y.\end{aligned}$$

Using this computations, (10.0.12) turns into

$$\begin{aligned}C_c⨍_{\mathrm{B}_{\varepsilon r_j}\cap\Omega}\|Du(y)-L_u\|^2\mathrm{d}y \leq (\|L_u\|C_u\varepsilon r_j)^2 + \|L_u-(Du)_{C_u\varepsilon r_j}\|^2 + C_0(\varepsilon r_j)^{2-\frac{2m}{p}}\|\mathcal{H}_{\mathcal{F}}[u]\|^2_{\mathrm{L}^p} \\ + \kappa^2(\varepsilon r_j)^{2\alpha} + \frac{1}{C_u^2}⨍_{\mathrm{B}_{C_u\varepsilon r_j}\cap\Omega}|Du(y)-(Du)_{C_u\varepsilon r_j}|^2\mathrm{d}y.\end{aligned}$$

Rewriting the inequality above in terms of u_j (changing variables, dividing by δ_j^2, using the definition of the u_j's, and item (a)) and inserting (10.0.7), we derive

$$\begin{aligned} C_c \fint_{\mathrm{B}_\varepsilon \cap \Omega_j} \|Du_j(y) - \ell_j\|^2 \mathrm{d}y \leq \left(\frac{\|L_u\| C_u \varepsilon r_j}{\delta_j} \right)^2 + \varepsilon^{2\alpha} \delta_j^2 + C_0 \|\mathcal{H}_{\mathcal{F}}[u]\|_{\mathrm{L}^p}^2 \frac{(\varepsilon r_j)^{2-\frac{2m}{p}}}{\delta_j^2} \\ + \kappa^2 \frac{(\varepsilon r_j)^{2\alpha}}{\delta_j^2} + \fint_{\mathrm{B}_{C_u\varepsilon} \cap \Omega_j} |Du_j(y) - (Du_j)_{C_u\varepsilon}|^2 \mathrm{d}y, \end{aligned} \tag{10.0.13}$$

where $\ell_j := \delta_j^{-1}(L_u - (Du)_{r_j})$. We now focus on bounding the limits of the terms appearing in (10.0.13). It is well known that

$$\begin{aligned} C_e \varepsilon^{2\alpha} \overset{\text{items (b) and (c)}}{=} C_e \varepsilon^{2\alpha} E_{u_j}(1) \overset{\text{item (e)}}{<} \fint_{\mathrm{B}_{\varepsilon r_j} \cap \Omega} \|Du(y) - L_u\|^2 \mathrm{d}y \\ = \fint_{\mathrm{B}_\varepsilon \cap \Omega_j} \|Du_j(y) - \ell_j\|^2 \mathrm{d}y. \end{aligned} \tag{10.0.14}$$

As a consequence of (10.0.7), it holds

$$\lim_{j \to +\infty} \left(\frac{\|L_u\|^2 C_u^2 \varepsilon^2 r_j^2}{\delta_j^2} + C_0 \frac{(\varepsilon r_j)^{2-\frac{2m}{p}} \|\mathcal{H}_{\mathcal{F}}[u]\|_{\mathrm{L}^p}^2}{\delta_j^2} + \kappa^2 \frac{(\varepsilon r_j)^{2\alpha}}{\delta_j^2} \right) \leq C_b \kappa^2 \varepsilon^{2\alpha}. \tag{10.0.15}$$

Combining (10.0.13), (10.0.14), and (10.0.15), we attain

$$\begin{aligned} C_e C_c \varepsilon^{2\alpha} &\leq C_b \kappa^2 \varepsilon^{2\alpha} + \limsup_{j \to \infty} \fint_{\mathrm{B}_{C_u\varepsilon} \cap \Omega_j} |Du_j(y) - (Du_j)_{C_u\varepsilon}|^2 \mathrm{d}y \\ &= C_b \kappa^2 \varepsilon^{2\alpha} + \fint_{\mathrm{B}^+_{C_u\varepsilon}} |Du_0(y) - (Du_0)_{C_u\varepsilon}|^2 \mathrm{d}y \\ &\overset{\text{Poincaré Ineq.}}{\leq} C_b \kappa^2 \varepsilon^{2\alpha} + C_u^2 \varepsilon^2 \fint_{\mathrm{B}^+_{C_u\varepsilon}} |D^2 u_0(y)|^2 \mathrm{d}y \overset{(10.0.11)}{\leq} C_b \kappa^2 \varepsilon^{2\alpha}. \end{aligned}$$

Adjusting the constants in the last inequality, we finally find the desired contradiction, as well we finish the proof of this proposition.

Proof of the claim (10.0.10)**:** Without loss of generality we can assume that $\mathbf{p}(\Gamma) = \mathrm{B}_{r_0} \cap \mathbb{R}^{m-1}$. Indeed, it is a standard procedure of straightening/flattening out the boundary, see [Eva10, Subsection 3.2.3]. If the boundary is not flat, i.e., $\mathbf{p}(\Gamma) \neq \mathrm{B}_{r_0} \cap \mathbb{R}^{m-1}$, we take a smooth function Φ such that $\Phi(0) = 0, D\Phi(0) = 0$, and $\Phi(\mathbf{p}(\Gamma)) = \mathrm{B}_{r_0} \cap \mathbb{R}^{m-1}$. So, $u_0 \circ \Phi$ satisfies (10.0.10), which assures that u_0 satisfies a similar elliptic PDE. For more details on this standard argument, we refer the reader to [Eva10, Subsection 3.2.3].

Fix a flattened boundary $\mathbf{p}(\Gamma) = \mathrm{B}_{r_0} \cap \mathbb{R}^{m-1}$, denote $\mathrm{B}_s^+ := \mathrm{B}_s \cap \{(x', x_m) \in \mathbb{R}^{m-1} \times \mathbb{R} : x_m > 0\}$ for every $s > 0$, and $q \in \mathbb{R}$ the conjugate exponent of p, i.e., such that $p^{-1} + q^{-1} = 1$. Let $\zeta \in C_c^\infty(\mathrm{B}_1^+, \mathbb{R}^n)$ a test vector field with $\zeta(z', 0) = 0$ for every $z' \in \mathbb{R}^{m-1}$. Then we define the sequence $\zeta_j(z) := \zeta(\frac{z}{r_j})$ which for each j also satisfies $\zeta_j(z', 0) = 0$ for every $z' \in \mathbb{R}^{m-1}$.

Our aim now is to apply lemma 10.0.1. To this aim, we estimate the left-hand side of (10.0.3) in lemma 10.0.1 as follows

$$\begin{aligned}\int_{\mathrm{B}_1^+} \langle D\mathcal{I}_{\mathcal{F}}(Du(z)), D\zeta_j(z)\rangle \mathrm{d}z &= r_j^{-1}\int_{\mathrm{B}_1^+} \langle D\mathcal{I}_{\mathcal{F}}(Du(z)), D\zeta\left(\frac{z}{r_j}\right)\rangle \mathrm{d}z \\ &= r_j^{m-1}\int_{\mathrm{B}_1^+} \langle D\mathcal{I}_{\mathcal{F}}(Du(r_j z)), D\zeta\,(z)\rangle \mathrm{d}z.\end{aligned} \tag{10.0.16}$$

Notice that the domain of integration does not change under the change of variables since ζ has compact support. Thus, by (10.0.16), we obtain that

$$\begin{aligned}\int_{\mathrm{B}_1^+} \langle D\mathcal{I}_{\mathcal{F}}(Du(z)), D\zeta_j(z)\rangle \mathrm{d}z &= r_j^{m-1}\int_{\mathrm{B}_1^+} \langle D\mathcal{I}_{\mathcal{F}}(Du(r_j z)) - D\mathcal{I}_{\mathcal{F}}((Du)_{r_j}), D\zeta\,(z)\rangle \mathrm{d}z \\ &\overset{((a))}{=} r_j^{m-1}\int_{\mathrm{B}_1^+} \langle D\mathcal{I}_{\mathcal{F}}((Du)_{r_j} + \delta_j Du_j(z)) - D\mathcal{I}_{\mathcal{F}}((Du)_{r_j}), D\zeta\,(z)\rangle \mathrm{d}z \\ &\overset{(10.0.9)}{=} \delta_j r_j^{m-1}\int_{\mathrm{B}_1^+} \langle D\mathcal{I}_j(Du_j(z), D\zeta\,(z)\rangle \mathrm{d}z,\end{aligned} \tag{10.0.17}$$

where we used the compactness of the support of ζ and the divergence theorem for the first equality. We now focus on the right-hand side of (10.0.3). Recalling the definition of $\mathcal{A}$ and that u is Lipschitz, we have that $\|\zeta_j \mathcal{A}^{1/q}(Du)\|_{\mathrm{L}^q(\mathrm{B}_r\cap\Omega)} \leq C_0\|\zeta_j\|_{\mathrm{L}^q(\mathrm{B}_r\cap\Omega)}$, where we change variables to get that

$$\|\zeta_j \mathcal{A}^{1/q}(Du)\|_{\mathrm{L}^q(\mathrm{B}_r\cap\Omega)} \leq C_0 r_j^{\frac{m}{q}}\|\zeta\|_{\mathrm{L}^q(\mathrm{B}_r\cap\Omega)}. \tag{10.0.18}$$

We now use lemma 10.0.1, (10.0.17), and (10.0.18), to derive that

$$\begin{aligned}\delta_j r_j^{m-1}\int_{\mathrm{B}_1^+} \langle D\mathcal{I}_j(Du_j(z)), D\zeta\,(z)\rangle \mathrm{d}z &= \delta_j r_j^{m-1}\int_{\mathrm{B}_1^+} \langle D\mathcal{I}_{\mathcal{F}}(Du(z)), D\zeta_j(z)\rangle \mathrm{d}z \\ &\leq C'\|\zeta_j \mathcal{A}^{1/q}(Du)\|_{\mathrm{L}^q(\mathrm{B}_r\cap\Omega)} \\ &\leq C_0' r_j^{\frac{m}{q}}\|\zeta\|_{\mathrm{L}^q(\mathrm{B}_r\cap\Omega)}.\end{aligned} \tag{10.0.19}$$

Recalling (10.0.7) and the choice of q, we easily obtain that

$$C_0'\frac{r_j^{\frac{m}{q}-m+1}}{\delta_j} = C_0'\frac{r_j^{1-\frac{m}{p}}}{\delta_j} \leq C_0'\frac{\delta_j}{\|\mathcal{H}_{\mathcal{F}}[u]\|_{\mathrm{L}^p(\Omega\times\mathbb{R}^n)}},$$

which in turn, together with (10.0.19), implies that

$$\lim_{j\to+\infty}\int_{\mathrm{B}_1^+} \langle D\mathcal{I}_j(Du_j(z)), D\zeta\,(z)\rangle \mathrm{d}z = 0.$$

By the very same argument of [DRT22, Proposition 4.5], we conclude from the previous equation that

$$\int_{\mathrm{B}_1^+} D^2\mathcal{I}_{\mathcal{F}}((Du)_0)[Du_0, D\zeta] = 0,$$

for all $\zeta \in C_c^\infty(\mathrm{B}_1^+, \mathbb{R}^n)$ with $\zeta(x', 0) = 0, \forall x' \in \mathrm{B}_1 \cap \mathbb{R}^{m-1}$, as claimed in (10.0.10). □

Chapter 11

Boundary regularity

We now define the auxiliary excess for $\varepsilon > 0$, which encompasses the mean curvature rather than only the excess E, as follows

$$e(x,s,L) := E(x,s,L) + \frac{8s^{\min\{\alpha,1-\frac{m}{p}\}}}{\varepsilon^m}\|\mathcal{H}[u]\|_p,$$
$$e(x,s) := e(x,s,(Du)_r), \text{ and } e(s) := e(0,s), \forall s > 0.$$

By proposition 10.0.3, there exists $C_e = C_e(m,n,\|\mathcal{F}\|_{C^2},K_{\mathcal{F}},\|u\|_{\text{Lip}},\Gamma) > 0$ with the following property: setting $\gamma := \min\{\alpha, 1-m/p\}$ and

$$\varepsilon < \min\left\{(12C_e)^{-\frac{1}{2\alpha}}, C_e^{-\frac{1}{2\alpha-\gamma}}, 2^{-\frac{2}{\gamma}}, 8^{-\frac{1}{\gamma}}, \frac{1}{4C_u}\right\}, \tag{11.0.1}$$

then there exist $\delta = \delta(\varepsilon) > 0$ such that (10.0.5) implies (10.0.6), i.e.,

$$\begin{cases} r^\gamma\|\mathcal{H}[u]\|_p \le E(r) \le \delta \\ \sigma \le \delta \end{cases} \implies E(\varepsilon r, L_u) \le C_e\varepsilon^{2\alpha}E(r). \tag{11.0.2}$$

We prove a decay for e in the next corollary, which is a consequence of the excess decay, proposition 10.0.3. We lastly choose

$$r_1 := \min\left\{\frac{\delta^{1/2\alpha}}{6C_d}, \left(\frac{\delta}{2\|\mathcal{H}[u]\|_p}\right)^{\frac{1}{\gamma}}\right\} \text{ and } \sigma < \min\left\{\delta, \frac{1}{6C_d}\right\}. \tag{11.0.3}$$

Corollary 11.0.1. *Assume assumption 3, (11.0.1), and (11.0.3). Then we have*

$$e(\varepsilon^j r, L_u) \le 2^{-2j}e(r) \quad \textit{for every } j\in\mathbb{N} \textit{ and } r \in (0,r_1).$$

Moreover, there exists $\eta \in (0,1)$ and $c_e = c_e(r,\varepsilon) > 0$ such that

$$e(s,L_u) \le c_e s^{2\eta} \quad \textit{for all } s \in (0,r).$$

Proof. Performing the same computations of lemma 10.0.2, the excess $E(r)$ can be taken small enough, up to choose r_1 and σ small enough. Hence, from (11.0.3), we can fix $r > 0$ such that

$$r^\gamma \|\mathcal{H}[u]\|_p + E(r) \leq \delta. \tag{11.0.4}$$

We wish to prove that

$$e(\varepsilon r, L_u) \leq 2^{-2} e(r). \tag{11.0.5}$$

To this end we consider two cases.

Case 1: if $r^\gamma \|\mathcal{H}[u]\|_p \leq E(r)$, we can apply (11.0.2) to deduce that

$$\begin{aligned} e(\varepsilon r, L_u) &\overset{(11.0.2)}{\leq} C_e \varepsilon^{2\alpha} E(r) + \frac{8(\varepsilon r)^\gamma}{\varepsilon^m} \|\mathcal{H}[u]\|_p \leq (C_e \varepsilon^{2\alpha-\gamma}) \varepsilon^\gamma E(r) + \frac{8(\varepsilon r)^\gamma}{\varepsilon^m} \|\mathcal{H}[u]\|_p \\ &\overset{(*)}{\leq} \varepsilon^\gamma e(r) \leq 2^{-2} e(r), \end{aligned}$$

which is precisely (11.0.5). Here $(*)$ follows from the fact that $\varepsilon < \min\{C_e^{-\frac{1}{2\alpha-\gamma}}, 2^{-2/\gamma}\}$, as assumed in (11.0.1).

Case 2: if $r^\gamma \|\mathcal{H}[u]\|_p \geq E(r)$, we proceed as follows:

$$\begin{aligned} e(\varepsilon r, L_u) &\leq \varepsilon^{-m} E(r, L_u) + 8\varepsilon^{-m+\gamma} r^\gamma \|\mathcal{H}[u]\|_p \leq \left(\varepsilon^{-m} r^\gamma + 8\varepsilon^{-m+\gamma} r^\gamma\right) \|\mathcal{H}[u]\|_p \\ &= \left(\frac{1}{8} + \varepsilon^\gamma\right) \frac{8r^\gamma}{\varepsilon^m} \|\mathcal{H}[u]\|_p \overset{(**)}{\leq} \frac{1}{4} \frac{8r^\gamma}{\varepsilon^m} \|\mathcal{H}[u]\|_p \leq 2^{-2} e(r), \end{aligned}$$

which is exactly (11.0.5). In $(**)$ we used that $\varepsilon < 8^{-1/\gamma}$, as we have assumed in (11.0.1). We conclude that (11.0.4) implies (11.0.5), i.e.,

$$r^{1-\frac{m}{p}} \|\mathcal{H}[u]\|_p + E(r) \leq \delta \quad \Longrightarrow \quad e(\varepsilon r, L_u) \leq 2^{-2} e(r). \tag{11.0.6}$$

We observe that (11.0.4) holds also with $\varepsilon^j r$ in place of r for every $j \in \mathbb{N}$. In fact, we have that

$$E(\varepsilon^j r, L_u) \leq 2E(\varepsilon^j r, L_u) + 2\|L_u - (Du)_{\varepsilon^j r}\|^2 \overset{(10.0.6)}{\leq} 2C_e \varepsilon^{2j\alpha} E(r) + C_0(\sigma^2 + r^{2\alpha}),$$

which, thanks to the smallness of r, σ and ε assumed in (11.0.1) and (11.0.3), ensures

$$(\varepsilon^j r)^{1-m/p} \|\mathcal{H}[u]\|_p + E(\varepsilon^j r) \leq \delta/2.$$

Then, applying (11.0.6), we obtain $e(\varepsilon^j r, L_u) \leq 2^{-2j} e(r)$ for any $j \in \mathbb{N}$. The latter surely implies the moreover part of the lemma by standard techniques, see for instance [HL11, Theorem 3.1]. □

We finally have all the tools to state and prove our main theorem. We will rewrite all the assumptions made up to now as part of the hypothesis of the theorem for the reader's convenience.

Theorem 11.0.2 (Boundary regularity theorem)**.** *Let $m, n \geq 2$, $\mathcal{F}$ be an integrand of class C^2 on the m-Grasmannian bundle $\mathbf{Gr}(\mathbf{B}(x, 4r_0))$ satisfying USAC, Γ be an $(m-1)$-submanifold of class $C^{1,\alpha}$ in $\mathbf{B}(x, 4r_0)$ with reach $\kappa \leq (2r_0^\alpha)^{-1}$ and such that $x \in \Gamma$. Let Ω be an open subset of $\mathbf{B}(x, 4r_0) \cap (\mathbb{R}^m \times \{0\})$ and $\mathbf{V} = \mathbf{V}[u] \in \mathbf{V}_m(\mathbf{B}(x, 4r_0))$ be an m-varifold induced by the graph of $u \in Lip(\Omega, \mathbb{R}^n)$, and $\partial\mathrm{graph}(u) = \Gamma$. Assume that the anisotropic first variation $\delta_\mathcal{F}\mathbf{V}$ is a Radon measure on $\mathbf{B}(x, 4r_0) \setminus \Gamma$ and the anisotropic mean curvature $\mathcal{H}_\mathcal{F} \in \mathrm{L}^p(\mathbf{B}(x, 4r_0)), p > m$. Then* *there exists* $\delta = \delta(m, n, p, \|\mathcal{F}\|_{C^2}, K_{\mathcal{F}}, \|u\|_{\mathrm{Lip}}, \Gamma) > 0$ *satisfying the following property. If* $\sigma \in (0, \delta)$ *is such that*

$$\|\mathbf{V}\|(\mathbf{B}(x, r)) \leq \left(\frac{1}{2} + \sigma\right) \omega_m r^m, \quad \textit{for any } r \in (0, 4r_0),$$

then there exist two constants $r_2 > 0$ *and* $\eta \in (0, 1)$ *depending only on* $m, n, p, \|\mathcal{F}\|_{C^2}, K_{\mathcal{F}}, \|u\|_{\mathrm{Lip}}, \Gamma$, *such that* $u \in C^{1,\eta}(\mathrm{B}\,(x, r_2))$.

Proof. Without loss of generality, we can assume $x = 0$. Denote $\gamma := \min\{\alpha, 1 - m/p\}$. The hypothesis of this theorem matches exatcly with assumption 3. We can choose $\varepsilon, \delta > 0$ satisfying (11.0.1) and (11.0.3). We now recall that the excess $E(\cdot, r)$ is continuous with respect to the variable in $\mathbb{R}^{m+n}$. Hence, as in the proof of (11.0.4), there exists $r_2 > 0$ such that

$$r^\gamma \|\mathcal{H}[u]\|_p + E(y, r) \leq \delta, \quad \forall y \in \mathbf{B}_{r_2}, \quad \forall r \in (0, r_1).$$

We apply corollary 11.0.1 to obtain the existence of $\eta \in (0, 1)$ such that

$$e(y, r, L_u) \leq c_e r^{2\eta}, \quad \forall y \in \mathbf{B}_{r_2}, \quad \forall r \in (0, r_1),$$

In particular, since $(Du)_r$ is optimal for $E(p, r, \cdot)$, it is easy to see that

$$E(y, r) \leq E(y, r, L_u) \leq e(y, r, L_u) \leq c_e r^{2\eta}, \quad \forall y \in \mathbf{B}_{r_2}, \forall r \in (0, r_1).$$

This shows that Du restricted to B_{r_2} belongs to a Campanato space, hence it is a Hölder continuous function for some $\eta \in (0, 1)$ which concludes the proof of the theorem. □

Part C

Lipschitz approximation for general almost minimizing currents

Chapter 12

Introduction

The regularity theory is a widely spread theme in mathematics. It splits into various branches, for example, the regularity theory of PDEs, the regularity theory of minimal surfaces, etc. One of the first and most famous problems is the existence and regularity of minimal surfaces, in other words, the existence and regularity of objects that minimizes the area functional over some classes of admissible competing surfaces. The classes of admissible objects where the area functional may be defined can be genuinely different, giving rise to quite different approaches to geometric variational problems. For example, the classical one considers the functional area defined in the class of smooth submanifolds of a fixed ambient Riemannian manifold.

We quickly realize that in this smooth context, the minimization problem for the area functional manifests a lack of compactness that naturally leads to considering and introducing objects that play the role of generalized smooth surfaces in the same guise of what is done when weak solutions are introduced to work around the weaknesses of the space of classical solutions of PDEs. Over the years, many generalizations have been proposed, for instance, the theory of Caccioppoli sets, varifolds, currents, flat chains, etc.

In this note, we will focus on the theory of currents that minimize (in a relaxed sense) the area functional. According to the definition given by I. Tamanini, [Tam84, Eq. 1.2], we work with the relaxed minimality condition, which we call the ω-almost minimality condition (see Definition 13.0.4). This condition is natural since it arises from practical problems as F. Maggi noticed in his book [Mag12, Chapter III], where for the codimension 1 setting, he considers an almost minimality condition that is a particular case of the ω-almost minimality.

A natural question to ask is: do these generalized surfaces have good regularity properties provided they minimize area?. Aiming at answering this question, in arbitrary dimension and codimension, in the setting of integral currents, F. Almgren Jr. has introduced his long and intricate, but still rich and beautiful, program in [Alm00] to prove regularity results for interior points of area minimizing currents. He stated that the singular set has Hausdorff dimension at most $m-2$. His theory was revisited by C. De Lellis and E. Spadaro, in a series of works (see [DS14]) where they furnished a different approach using new techniques of geometric analysis which give a much shorter proof and they also strengthened the main result. More recently, in [Sko21], A. Skorobogatova improved Almgren's estimate, she proves that the upper Minkowski dimension of the interior singular set is at most $m-2$.

In this article, we aim to perform the first part of the regularity program used in the aforementioned works, which is (almost-) monotonicity results, the strong Lipschitz approximation, and the strong excess decay at interior points. We aim at proving these first part of the framework for ω-almost area minimizing currents. Moreover, in Proposition 13.0.5, we prove that the setting of [DLSS18] is a particular case of the general ω-minimality. We also give a nice example (Example 13.0.6) of a current that satisfies the ω-almost minimality condition and it is not covered by any of the definitions considered in the works mentioned before.

Chapter 13

Preliminaries

The goal of this section is to set standard notations on currents theory that will be used throughout this paper.

We use $\mathbf{B}(p,r) \subset \mathbb{R}^{m+n}$ for the open balls centered at $p \in \mathbb{R}^{m+n}$ and of radius $r \in]0,+\infty[$ of the ambient space $\mathbb{R}^{m+n}$, and we fix $\pi_0 := \mathbb{R}^m \times \{0\} \subset \mathbb{R}^{m+n}$. For any linear subspace $\pi \subset \mathbb{R}^{m+n}$, $\pi^\perp$ is its orthogonal complement in $\mathbb{R}^{m+n}$, $\mathbf{p}_\pi$ is the orthogonal projection onto π, and $\mathbf{p} := \mathbf{p}_{\pi_0}$.

We define the **tilted disk** $\mathrm{B}_r(p,\pi) := \mathbf{B}(p,r) \cap (p+\pi)$ and $\mathbf{C}(p,r,\pi)$ the **tilted cylinder** as the set $\left\{(x+y) : x \in \mathrm{B}_r(p,\pi), y \in \pi^\perp\right\}$. We also set $\mathbf{C}(p,r) := \mathbf{C}(p,r,\pi_0)$ and $\mathrm{B}_r(p) := \mathrm{B}_r(p,\pi_0)$. Moreover, $\mathbf{C}_r := \mathbf{C}(0,r) = \mathbf{C}(0,r,\pi_0)$.

We also assume that each linear subspace π of $\mathbb{R}^{m+n}$ is oriented by a k-vector $\vec{\pi} := v_1 \wedge \cdots \wedge v_k$, where $(v_i)_{i\in\{1,\dots,k\}}$ is an orthonormal base of π and, with an abuse of notation, we write $|\pi_2 - \pi_1|$ standing for $|\vec{\pi}_2 - \vec{\pi}_1|$, where $|\cdot|$ is the norm associated to the canonical inner product of k-vectors.

For any $s \in [0,+\infty[$ we also set $\mathcal{H}^s$ as the s-dimensional Hausdorff measure in $\mathbb{R}^{m+n}$. We recall the definition of density of a given $T \in \mathcal{D}_m(U)$, where $U \subseteq \mathbb{R}^{m+n}$ is an open set and $\mathcal{D}_m(U)$ is the set of m-dimensional current in U at a given point $p \in \mathbb{R}^{m+n}$. We say that $\Theta^m(T,p) \in [0,+\infty]$ is the **m-dimensional density of T at** p, if

$$\Theta^m(T,p) = \lim_{r\to 0} \frac{\|T\|(\mathbf{B}(p,r))}{\mathcal{H}^m\left(\mathrm{B}_r(0)\right)},$$

whenever the limit exists.

For standard notations and classical results on the theory of currents which will be used in this note, we refer the reader to the classical treatise of [Fed69]. For the theory of multi-valued maps, we refer the reader to [DS15].

Definition 13.0.1 (Excess and height)**.** Given an integer rectifiable m-dimensional current T in $\mathbb{R}^{m+n}$ with finite mass and compact support, i.e., $T \in \mathbf{I}_m(\mathbb{R}^{m+n})$ and m-planes π, and π', we define **the excesses of T in balls and cylinders** as

$$\mathbf{E}\left(T, \mathbf{B}(p,r), \pi\right) := \frac{1}{2\omega_m r^m} \int_{\mathbf{B}(p,r)} |\vec{T} - \vec{\pi}|^2 d\|T\|,$$

$$\mathbf{E}\left(T, \mathbf{C}(p,r,\pi), \pi'\right) := \frac{1}{2\omega_m r^m} \int_{\mathbf{C}(p,r,\pi)} \left|\vec{T} - \vec{\pi}'\right|^2 d\|T\|,$$

we will use the shorthand notation $\mathbf{E}\left(T, \mathbf{C}(p,r,\pi)\right)$ for $\mathbf{E}\left(T, \mathbf{C}(p,r,\pi), \pi\right)$. We define also **the height function in a set** $A \subset \mathbb{R}^{m+m}$ **with respect to the** m**-plane** π as

$$\mathbf{h}(T, A, \pi) := \sup_{x,y \in \operatorname{spt}(T)\cap A} |\mathbf{p}_{\pi^\perp}(x) - \mathbf{p}_{\pi^\perp}(y)| \,.$$

Definition 13.0.2 (Optimal planes for the excess)**.** We say that an m-dimensional plane π **optimizes the excess of** T **in a ball** $\mathbf{B}(p,r)$ if

$$\mathbf{E}\left(T, \mathbf{B}(p,r)\right) := \min_{\pi'} \mathbf{E}\left(T, \mathbf{B}(p,r), \pi'\right) = \mathbf{E}\left(T, \mathbf{B}(p,r), \pi\right). \tag{13.0.1}$$

Observe that in general the plane optimizing the excess is not unique and $\mathbf{h}\left(T, \mathbf{B}(p,r), \pi\right)$ might depend on the optimizer π.

Definition 13.0.3 (Optimal planes)**.** A m-plane π is called an **optimal plane**, if it optimizes the height function among the planes that are optimal for the excess, i.e.,

$$\mathbf{h}\left(T, \mathbf{B}(p,r), \pi\right) = \min\left\{\mathbf{h}\left(T, \mathbf{B}(p,r), \pi'\right) : \pi' \textit{ satisfies } (13.0.1)\ \right\} =: \mathbf{h}\left(T, \mathbf{B}(p,r)\right).$$

Henceforth, $\mathbf{h}\left(T, \mathbf{C}(p,r,\pi)\right)$ will stand for $\mathbf{h}\left(T, \mathbf{C}(p,r,\pi), \pi\right)$

Lastly, we recall the definition of ω-almost minimality where we also briefly discuss the high level of generality that this condition represents.

Definition 13.0.4 (Almost minimality condition)**.** Let T be an m-dimensional integer rectifiable current in $\mathbb{R}^{m+n}$, i.e., $T \in \mathbf{I}_m(\mathbb{R}^{m+n})$. We say that T is ω**-almost area minimizing**, if there exist $\mathbf{s} > 0$ and an absolutely continuous function $\omega : (0, \mathbf{s}) \to (0, +\infty)$ such that $\omega(s) = o(1)$ when $s \to 0^+$, and for every $p \in \operatorname{spt}(T) \setminus \operatorname{spt}(\partial T)$,

$$\|T\|(\mathbf{B}(p,s)) \leq (1 + \omega(s))\, \|T + \partial S\|(\mathbf{B}(p,s)), \quad \forall s \in (0,\mathbf{s}), \quad \forall S \in \mathbf{I}_{m+1}(\mathbf{B}(p,s)).$$

In the special case that $\omega(s) = \mathbf{A}s^{\boldsymbol{\alpha}}$ for some $\mathbf{A} \geq 0$ and $\boldsymbol{\alpha} \in (0,1]$, we say that T is $(\mathbf{A}, \mathbf{s}, \boldsymbol{\alpha})$**-almost area minimizing** if, for every $p \in \operatorname{spt}(T) \setminus \operatorname{spt}(\partial T)$, it holds that

$$\|T\|(\mathbf{B}(p,s)) \leq (1 + \mathbf{A}s^{\boldsymbol{\alpha}})\, \|T + \partial S\|(\mathbf{B}(p,s)), \quad \forall s \in (0,\mathbf{s}), \quad \forall S \in \mathbf{I}_{m+1}(\mathbf{B}(p,s)).$$

If $\omega \equiv 0$, we say that T is **area minimizing**.

It is easily seen that all the previous cases treated in the literature are a particular case of the definition above, as follows.

Proposition 13.0.5. *Let* $\boldsymbol{\Omega} \geq 0$ *and* $T \in \mathbf{I}_m(\mathbb{R}^{m+n})$*, if* T *is a* $\boldsymbol{\Omega}$*-minimal current as in* [DLSS18, Definition 1.1]*, i.e.,*

$$\mathbb{M}(T) \leq \mathbb{M}(T + \partial S) + \boldsymbol{\Omega}\mathbb{M}(S), \quad \forall S \in \mathbf{I}_{m+1}(\mathbb{R}^{m+n}), \tag{13.0.2}$$

then T *is* $(\mathbf{A}, \infty, 1)$*-almost area minimizing with* $\mathbf{A} = c\boldsymbol{\Omega}$ *for some positive constant* $c = c(m,Q,T) > 0$*.*

Proof. For any $r > 0, p \in \mathbb{R}^{m+n}$ and $S \in \mathbf{I}_{m+1}(\mathbf{B}(p,r))$, we have that

$$\|T\|(\mathbb{R}^{m+n} \setminus \mathbf{B}(p,r)) = \|T + \partial S\|(\mathbb{R}^{m+n} \setminus \mathbf{B}(p,r)), \tag{13.0.3}$$

since the support of S and ∂S is contained in the ball $\mathbf{B}(p,r)$. Using classical density estimates, c.f. [Fed69, Section 4.1.28(5)], the latter equation leads to

$$\begin{aligned}\|T\|(\mathbf{B}(p,r)) &\overset{(13.0.2),(13.0.3)}{\leq} \|T+\partial S\|(\mathbf{B}(p,r)) + \mathbf{\Omega}\|S\|(\mathbf{B}(p,r)) \\ &\leq \|T+\partial S\|(\mathbf{B}(p,r)) + \mathbf{\Omega} c_1 r^{m+1} \\ &\leq \|T+\partial S\|(\mathbf{B}(p,r)) + \mathbf{\Omega} c_2 r \|T+\partial S\|(\mathbf{B}(p,r)) \\ &= (1+\mathbf{\Omega} c_2 r)\|T+\partial S\|(\mathbf{B}(p,r)).\end{aligned}$$

We set $\mathbf{A} = c_2\mathbf{\Omega}$ and $\boldsymbol{\alpha} = 1$ and, since the inequality holds for any p and $r > 0$, we obtain $\mathbf{s} = +\infty$. □

Following I. Tamanini's ideas, [Tam84], we give two examples to justify the different definitions of almost minimality that we mentioned. Notice that the ω-almost minimality encompasses the other definitions, then we will work in this generality.

Example 13.0.6. We let T to be the reduced boundary of the Caccioppoli set $E \subset \mathbb{R}^{m+1}$ which is a minimizer of the following variational problem:

$$\inf\left\{\mathcal{P}(F,\mathbf{B}(p,s)) + \int_F H(q)\,\mathrm{d}q : F \text{ is a Caccioppoli set and } F\Delta E \subset\subset \mathbf{B}(p,s)\right\}, \tag{13.0.4}$$

where H is a prescribed mean curvature function that belongs to $\mathrm{L}^p(\mathbb{R}^{m+1})$ with the crucial condition that $p > m$ and $\mathcal{P}$ denotes the perimeter measure of Caccioppoli sets, see [Mag12]. Given any F as in (13.0.4), thanks to $p > m$ we must apply the Holder inequality to derive

$$\begin{aligned}\mathcal{P}(E,\mathbf{B}(p,s)) - \mathcal{P}(F,\mathbf{B}(p,s)) &\leq \int_{E\Delta F} |H(x)|\,\mathrm{d}x \\ &\leq \|H\|_{\mathrm{L}^p(\mathbf{B}(p,s))}\mathcal{H}^m(\mathbf{B}(p,s))^{1-\frac{1}{p}} \\ &\leq \omega_m^{1-\frac{1}{p}}\|H\|_{\mathrm{L}^p(\mathbf{B}(p,s))} s^{m-\frac{m}{p}}.\end{aligned}$$

Again using the argument with the density estimates, we easily see that T is a

$$\left(c_0(m)\omega^{1-\frac{1}{p}}\|H\|_{\mathrm{L}^p(\mathbb{R}^{m+1})}, \infty, 1-\frac{m}{p}\right)\text{-almost minimizer in } \mathbb{R}^{m+1}.$$

We now provide an example of a ω-almost minimizer which does not belong to the class of $(\mathbf{A},\mathbf{s},\boldsymbol{\alpha})$-almost minimizer.

Example 13.0.7. We consider the function

$$\begin{aligned} f: \quad (0,1)\subset\mathbb{R} &\to (0,1)\subset\mathbb{R} \\ t &\mapsto \int_0^t \left(\ln(\tfrac{e}{s})\right)^{-1}\mathrm{d}s.\end{aligned}$$

We have that $f(0) = 0$ and we set $f(t) = f(-t), \forall t \in (-1,0)$. So, if we consider the Caccioppoli set given by $E := \mathrm{epi}(u)$ and $Q_t := (-t,t)^2 \subset \mathbb{R}^2$, we have that

$$\mathcal{P}(E,Q_t) - t = 2\int_0^t \left(\sqrt{1+(\ln(\frac{e}{s}))^{-2}} - 1\right) ds \approx t(\ln(\frac{e}{t}))^{-2},$$

So, we have that the 1-current induced by E is ω-almost minimizer with $\omega(t) = (\ln(\frac{e}{t}))^{-2}$.

Chapter 14

The ω-almost monotonicity formulas

Let us now state the ω-almost monotonicity formula with an additive error term which is the analogous of [DSS17a, Proposition 2.1] for our setting of ω-almost minimizers. Henceforth, we will denote by $(z-p)^\perp$ the projection of the vector $z-p$ onto the orthogonal complement of the approximate tangent to T at z.

Proposition 14.0.1 (ω-almost monotonicity formula)**.** *Let $T \in \mathbf{I}_m(\mathbb{R}^{m+n})$ be an ω-almost minimizer and $p \in \operatorname{spt}(T)\setminus \operatorname{spt}(\partial T)$. There are dimensional constants $C_0, r_0 > 0$ such that*

$$\int_{\mathbf{B}(p,r)\setminus\mathbf{B}(p,s)} \frac{\left|(z-p)^\perp\right|^2}{|z-p|^{m+2}} d\|T\|(z) \leq C_0 \left(\frac{\|T\|\left(\mathbf{B}(p,r)\right)}{\omega_m r^m} - \frac{\|T\|\left(\mathbf{B}(p,s)\right)}{\omega_m s^m} + \int_s^r \frac{\omega(\rho)}{\rho}\,\mathrm{d}\rho \right), \quad (14.0.1)$$

for all $0 < s < r < r_0 < \mathbf{s}$. Moreover, the function $r \mapsto \frac{\|T\|(\mathbf{B}(p,r))}{\omega_k r^m} + \int_s^r \frac{\omega(\rho)}{\rho}\,\mathrm{d}\rho$ is nondecreasing. Furthermore, when ω satisfies a Dini condition of the following type, $\int_0^r \frac{\omega(\rho)}{\rho}\,\mathrm{d}\rho < +\infty$ then the function $r \mapsto \frac{\|T\|(\mathbf{B}(p,r))}{\omega_k r^m} + \int_0^r \frac{\omega(\rho)}{\rho}\,\mathrm{d}\rho$ is nondecreasing.

Proof. We define the integral current

$$W := 0 \times\!\!\!\!\times\, \partial\left(T \llcorner \mathbf{B}(p,r)\right),$$

and test the ω-almost minimality for it to obtain

$$\|T\|\left(\mathbf{B}(p,r)\right) \leq (1+\omega(r))\|W\|\left(\mathbf{B}(p,r)\right) \leq \frac{r}{m}\|\partial T\|\left(\mathbf{B}(p,r)\right) + c_1(m,T)\omega(r)r^m, \quad (14.0.2)$$

where we use classical density estimates in the second inequality. We now set the mass function $\mathfrak{m}(r) := \|T\|\left(\mathbf{B}(p,r)\right)$ and observe that $\mathfrak{m}$ is a nondecreasing function and thus it is a function of bounded variation. We can decompose its distributional derivative $D\mathfrak{m}$, which is a nonnegative measure, as $D\mathfrak{m} = \mathfrak{m}'\mathcal{H}^1 + \mu_s$ and μ_s is the singular part of $D\mathfrak{m}$. In (14.0.2), we multiply mr^{-m-1} and add $\frac{\mathfrak{m}'(r)+\mu_s}{r^m}$ to obtain

$$\frac{\mathfrak{m}'(r)+\mu_s}{r^m} - \frac{1}{r^m}\|\partial T\|\left(\mathbf{B}(p,r)\right) \leq -\frac{m\mathfrak{m}(r)}{r^{m+1}} + mc_1(m,T)\frac{\omega(r)}{r} + \frac{D\mathfrak{m}}{r^m},\ \forall r \in (0,\mathbf{s}).$$

We integrate the latter inequality on the interval (s,t), where $\mathbf{s} > t > s$, thus we reach

$$\underbrace{\int_s^t \frac{1}{\rho^m} d\mu_s(\rho)}_{I^s} + \underbrace{\int_s^t \frac{\mathfrak{m}'(\rho) - \|\partial T\|(\mathbf{B}(p,\rho))}{\rho^m} \,\mathrm{d}\mathcal{H}^1(\rho)}_{I^a} \leq \frac{\mathfrak{m}(r)}{r^m} - \frac{\mathfrak{m}(s)}{s^m} + mc_1(m,T)\int_s^t \rho^{-1}\omega(\rho)\,\mathrm{d}\rho,$$

notice that we have used the following equality

$$\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\frac{\mathfrak{m}(\rho)}{\rho^m}\right) = -\frac{m\mathfrak{m}(\rho)}{\rho^{m+1}} + \frac{D\mathfrak{m}}{\rho^m}.$$

In the proof of [DSS17a, Proposition 2.1], it is shown that $I := I^s + I^a$ bounds, up to a dimensional constant, the left-hand side of (14.0.1) without the use of any minimality condition. So, it finishes the proof of our result. □

We now state a second ω-almost monotonicity formula for the ω-almost minimality condition which now comes with multiplicative error terms, these result is the analogous for interior points of [HM19, Proposition 2.3] which is stated to boundary points and only for the special case $\omega(r) = C(m,n)r^\alpha$.

Proposition 14.0.2 (ω-almost monotonicity formula)**.** *Let $T \in \mathbf{I}_m(\mathbb{R}^{m+n})$ be an ω-almost minimizer and $p \in \mathrm{spt}(T)\setminus\mathrm{spt}(\partial T)$, then there exists a dimensional constant $r_1 = r_1(m,n,\omega) > 0$, such that*

$$\int_{\mathbf{B}(p,t)\setminus\mathbf{B}(p,s)} e^{m\omega(|z-p|)} \frac{\left|(z-p)^\perp\right|^2}{2|z-p|^{m+2}} d\|T\|(z) \leq e^{m\omega(t)}\frac{\|T\|\left(\mathbf{B}(p,t)\right)}{t^m} - e^{m\omega(s)}\frac{\|T\|\left(\mathbf{B}(p,s)\right)}{s^m}, \quad (14.0.3)$$

for every $0 < s < t < r_1 < \mathbf{s}$.

Proof. We start defining

$$S := T \llcorner \mathbf{B}(p,r) \quad \text{and} \quad W := 0 \mathbin{⨳} \partial S.$$

By the ω-almost minimizing property of T, we deduce for $r < \mathbf{s}$ that

$$\begin{aligned} \|T\|\left(\mathbf{B}(p,r)\right) &\leq (1+\omega(r))\|W\|(\mathbf{B}(p,r)) \\ &= (1+\omega(r))\frac{r}{m}\|\partial S\|(\mathbf{B}(p,r)). \end{aligned} \quad (14.0.4)$$

For a.e. $\rho \leq r_1 < \mathbf{s}$, we conclude that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\rho}\left(\frac{\|T\|\left(\mathbf{B}(p,\rho)\right)}{\rho^m}\right) &= \frac{-m\|T\|\left(\mathbf{B}(p,\rho)\right)}{\rho^{m+1}} + \frac{\|T\|'\left(\mathbf{B}(0,t)\right)}{\rho^m} \\ &= \frac{-m\|T\|\left(\mathbf{B}(p,\rho)\right)}{\rho^{m+1}} + \frac{\|T\|'\left(\mathbf{B}(0,t)\right)}{\rho^m} \\ &\quad + \frac{m\|T\|\left(\mathbf{B}(p,\rho)\right)}{\left(1+\omega(\rho)\right)\rho^{m+1}} - \frac{m\|T\|\left(\mathbf{B}(p,\rho)\right)}{\left(1+\omega(\rho)\right)\rho^{m+1}} \\ &= \frac{m\|T\|(\mathbf{B}(p,\rho)}{\rho^{m+1}}\left(\frac{1}{1+\omega(\rho)} - 1\right) \\ &\quad + \underbrace{\frac{\|T\|'\left(\mathbf{B}(0,t)\right)}{\rho^m} - \frac{m\|T\|\left(\mathbf{B}(p,\rho)\right)}{\left(1+\omega(\rho)\right)\rho^{m+1}}}_{I_1}. \end{aligned} \quad (14.0.5)$$

We only use (14.0.4) to bound I_1 from below as follows

$$I_1 \geq \frac{1}{\rho^m}\left(\|T\|'(\mathbf{B}(p,\rho) - \|\partial S\|(\mathbf{B}(p,\rho)\right). \tag{14.0.6}$$

Therefore, by (14.0.5) and (14.0.6), we obtain that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\frac{\|T\|\left(\mathbf{B}(p,\rho)\right)}{\rho^m}\right) &\geq \frac{m\|T\|(\mathbf{B}(p,\rho)}{\rho^{m+1}}\left(\frac{1}{1+\omega(\rho)}-1\right)+\frac{1}{\rho^m}\left(\|T\|'(\mathbf{B}(p,\rho) - \|\partial S\|(\mathbf{B}(p,\rho)\right)\\ &\geq -\frac{m\omega(\rho)\|T\|(\mathbf{B}(p,\rho))}{\rho^{m+1}}+\frac{1}{\rho^m}\left(\|T\|'(\mathbf{B}(p,\rho) - \|\partial S\|(\mathbf{B}(p,\rho)\right).\end{aligned}$$

Since ω is absolutely continuous, by the Lebesgue differentiation theorem, we have that ω is differentiable almost everywhere on $(0,r_1)$, hence the latter equation provides, for a.e. $\rho\in(0,r_1)$,

$$\frac{\mathrm{d}}{\mathrm{d}\rho}\left(e^{m\omega(\rho)}\frac{\|T\|\left(\mathbf{B}(p,\rho)\right)}{\rho^m}\right) \geq e^{m\omega(\rho)}\frac{1}{\rho^m}\left(\|T\|'(\mathbf{B}(p,\rho) - \|\partial S\|(\mathbf{B}(p,\rho)\right). \tag{14.0.7}$$

We denote by $(\cdot)^\perp$ the projection onto the approximate tangent plane to T at p. Thus, using classical theory of slicing of currents, we have for a.e. $0\leq s<t$ that

$$\int_s^t \|\partial S\|(\mathbf{B}(p,\rho))\,\mathrm{d}\rho = \int_{\mathbf{B}(p,t)\setminus\mathbf{B}(p,s)}\frac{\left|x^\perp\right|}{|x|}d\|T\|.$$

Integrating (14.0.7) on $(s,t)\subset(0,r_1)$ and applying [HM19, Lemma 2.2] (their proof works straightforwardly for f, as in their notation, absolutely continuous, we just need to recall that ω is absolutely continuous), we conclude

$$\begin{aligned}e^{m\omega(t)}\frac{\|T\|\left(\mathbf{B}(p,t)\right)}{t^m} - e^{m\omega(s)}\frac{\|T\|\left(\mathbf{B}(p,s)\right)}{s^m} &\geq \int_{\mathbf{B}(p,t)\setminus\mathbf{B}(p,s)}\frac{e^{m\omega(|z-p|)}}{|z-p|^m}\left(1-\frac{\left|(z-p)^\perp\right|}{|z-p|}\right)d\|T\|(z)\\ &\geq \int_{\mathbf{B}(p,t)\setminus\mathbf{B}(p,s)} e^{m\omega(|z-p|)}\frac{\left|(z-p)^\perp\right|^2}{2|z-p|^{m+2}}d\|T\|(z).\end{aligned}$$

□

A very useful result often implicitly used in our theory is the following. To enunciate and prove it, we fix the notation of the flat distance between two m-dimensional integer rectifiable currents T and S, i.e., $T,S\in\mathscr{I}_m(U)$, U open and $A\Subset U$ as follows:

$$\mathcal{F}_A(T,S)=\inf\left\{\|R\|(A)+\|\tilde{T}\|(A): T-S=R+\partial\tilde{T} \text{ with } R\in\mathbf{I}_m(U) \text{ and } \tilde{T}\in\mathbf{I}_{m+1}(U)\right\}.$$

Lemma 14.0.3 (Sequences of ω_k-almost minimizers)**.** *For each $k\in\mathbb{N}$, let $U\subset\mathbb{R}^{m+n}$ be an open set, we assume that*

(a) $T_k\in\mathbf{I}_m(U)$ *is ω_k-almost area minimizing currents in U,*

(b) $\partial T_k=0$ *for each $k\in\mathbb{N}$,*

(c) $\limsup_{k\to+\infty}\|T_k\|(U)<+\infty$,

(d) $r_{1k}=r_1(m,n,\omega_k)$ *from Proposition 14.0.2 satisfies* $r_0:=\liminf_{k\to+\infty}r_{1k}>0$,

(e) $\omega := \limsup_{k\to+\infty} \omega_k$ *satisfies the assumptions of Definition 13.0.4, i.e., its domain contains* $(0, r_0)$, *it is absolutely continuous and* $\omega = o(1)$.

Then we have that there exists $T \in \mathbf{I}_m(U)$ *such that*

(i) $T_k \rightharpoonup T$,

(ii) T *is* ω*-almost minimizing in* U,

(iii) $\|T\| \leq \liminf_{k\to+\infty} \|T_k\| \leq \limsup_{k\to+\infty} \|T_k\| \leq (1 + \sup\omega) \|T\|$,

(iv) $\mathcal{F}(T_k, T) \to 0$ *as* k *goes to* $+\infty$,

(v) $\mathrm{spt}(T_k)$ *converges in the Hausdorff distance sense to* $\mathrm{spt}(T)$.

Proof. Thanks to ((b)) and ((c)), we can use standard compactness results (one can consult [Fed69, Section 4.2]) to ensure the existence of $T \in \mathbf{I}_m(U)$ such that $T_k \rightharpoonup T$, up to a subsequence, in the sense of currents so ((i)) is proved. The equivalence between ((i)) and ((iv)) is given by [Sim14, Theorem 7.2] again using ((b)) and ((c)). Now we want to prove ((ii)) and ((iii)). Let us write $T_k - T = R_k + \partial\tilde{T}_k$ in $\mathbf{B}(p, R+2)$, $p \in U, R \in (0, \mathbf{s})$ with

$$\limsup_{k\to+\infty}\Big(\|R_k\|(\mathbf{B}(p, R+1)) + \|\tilde{T}_k\|(\mathbf{B}(p, R+1))\Big) = 0.$$

Thus, since all the measures involved are Radon measures, for almost every $s \in (R, R+1)$, it follows that

$$\limsup_{k\to+\infty}\|R_k\|(\mathbf{B}(0, s)) = 0 \tag{14.0.8}$$

and

$$\limsup_{k\to+\infty}\mathbb{M}\left(\langle\tilde{T}_k, d, s\rangle\right) = 0. \tag{14.0.9}$$

Note that (14.0.9) follows directly from the formula of the slice and the fact that T_k converges to T in the sense of currents. We use again the slice formula to get

$$T_k \llcorner \mathbf{B}(p, s) = T \llcorner \mathbf{B}(p, s) + R_k \llcorner \mathbf{B}(p, s) - \langle\tilde{T}_k, d, s\rangle + \partial(\tilde{T}_k \llcorner \mathbf{B}(p, s)). \tag{14.0.10}$$

The ω_k-almost minimality condition gives

$$\|T_k\|(\mathbf{B}(p, s)) \leq (1 + \omega_k(s)) \|T_k + \partial\tilde{T}_k\| (\mathbf{B}(p, s)).$$

Putting into account the latter inequality, the triangle inequality and (14.0.10), we obtain that

$$\|T_k\|(\mathbf{B}(p, s)) \leq (1 + \omega_k(s)) \left(\|T\|(\mathbf{B}(p, s)) + \|R_k\|(\mathbf{B}(p, s)) + \mathbb{M}\left(\langle\tilde{T}_k, d, s\rangle\right) + 2\|\partial\tilde{T}_k\|(\mathbf{B}(p, s))\right).$$

Note that, by construction, it follows that $\|\partial\tilde{T}_k\|(\mathbf{B}(p, s)) \to 0$ as $k \to +\infty$. Finally, by the lower semicontinuity of the mass, (14.0.8), (14.0.9) and the last equation passed through $\limsup_{k\to+\infty}$, we conclude that

$$\begin{aligned} \|T\|(\mathbf{B}(p, s)) \leq \liminf_{k\to+\infty} \|T_k\|(\mathbf{B}(p, s)) &\leq \limsup_{k\to+\infty} \|T_k\|(\mathbf{B}(p, s)) \\ &\leq \left(1 + \limsup_{k\to+\infty} \omega_k(s)\right) \|T\|(\mathbf{B}(p, s)), \end{aligned} \tag{14.0.11}$$

for any $p \in U$ and a.e. $s \in (0, R)$, which ensures ((ii)) and ((iii)).

We proceed with the proof of ((v)) using a contradiction argument. We take $K \subset \mathbb{R}^{m+n}$ a compact subset and assume that there exists $\eta_0 > 0$ and $q_k \in K \cap \operatorname{spt}(T_k)$ with $\mathrm{d}(q_k, \operatorname{spt}(T)) > \eta_0 > 0$ for k sufficiently large. Since K is compact, up to a subsequence, we denote by q_0 the limit of $(q_k)_{k\in\mathbb{N}}$. Clearly, we have that $\mathrm{d}_H(q_0, \operatorname{spt}(T)) \geq \eta_0 > 0$. Hence, by convergence of Radon measures, i.e., (14.0.11) we have that

$$0 = \|T\| \left(\mathbf{B}\Big(q_0, \frac{3\eta_0}{4}\Big)\right) \geq \limsup_{k\to+\infty} \|T_k\| \left(\mathbf{B}\Big(q_0, \frac{\eta_0}{2}\Big)\right). \tag{14.0.12}$$

Provided k is sufficiently large, it is easy to see that $\mathbf{B}\big(q_k, \frac{\eta_0}{8}\big) \subset \mathbf{B}\big(q_0, \frac{\eta_0}{4}\big)$ which in turn, by the ω_k-almost monotonicity formula, Proposition 14.0.2, implies that

$$\begin{aligned} \|T_k\| \left(\mathbf{B}\Big(q_0, \frac{\eta_0}{2}\Big)\right) &\overset{(14.0.3)}{\geq} e^{m(\omega_k(\eta_0/4) - \omega_k(\eta_0/2))} 2^m \|T_k\| \left(\mathbf{B}\Big(q_k, \frac{\eta_0}{8}\Big)\right) \\ &\geq C(m,k,\eta_0) e^{m(\omega_k(\eta_0/4) - \omega_k(\eta_0/2))}, \end{aligned} \tag{14.0.13}$$

where $C(m, k, \eta_0) > C > 0$ where $C > 0$ is a positive constant independent of k because $q_k \in \operatorname{spt}(T_k)$ and T_k is an integer current and so the density of T_k in q_k is always a positive integer number greater or equal to 1. Notice that

$$\begin{aligned} C(m,k,\eta_0) e^{m(\omega_k(\eta_0/4) - \omega_k(\eta_0/2))} &\geq \frac{C(m,k,\eta_0)}{e^{m\omega_k(\eta_0/2)}} \geq \frac{C(m,k,\eta_0)}{e^{m\omega_k(\eta_0/2)}} \\ &\geq \frac{C(m,k,\eta_0)}{e^{m \limsup_{k\to+\infty} \max_{s\in[0,r_{1k}]}(\omega_k(s))}} \overset{(*)}{>} 0, \end{aligned} \tag{14.0.14}$$

recall that r_{1k} from the ω_k-almost monotonicity formula depends on m, n and ω_k, which ensures $(*)$ thanks to ((d)) and ((e)). This finishes the proof of the theorem, since (14.0.12), (14.0.13) and (14.0.14) are in contradiction. $\square$

Chapter 15

Almgren's stratification for ω-almost area minimizing currents

The stratification process allows us to estimate the Hausdorff dimension of the set of points at which the current becomes infinitesimally more flat. Although this seems to be a good measurement of regularity, since it mimics the behavior of smooth manifolds, the fact that the density and codimension are arbitrary makes the 'becoming flat' property insufficient to derive regularity. The famous Federer's example $\{(z,w) \in \mathbb{C}^2 : z^2 = w^3\}$ confirms it at the origin.

Let us define regular and singular interior points.

Definition 15.0.1. Let T be an ω-almost area minimizing integral current in $\mathbb{R}^{m+n}$, we define the set of regular points as follows

$$\mathrm{Reg}(T) := \{p \in \mathrm{spt}(T) : \mathrm{spt}(T) \cap \mathbf{B}(p,r) \text{ is a } C^{1,\alpha} \text{ submanifold of } \mathbb{R}^{m+n} \text{ for some } \alpha, r > 0\},$$

as well the set of singular points is defined as $\mathrm{Sing}(T) := \mathrm{spt}(T) \setminus (\mathrm{spt}(\partial T) \cup \mathrm{Reg}(T))$.

Let us prove an important property for the density function and the existence of blowup limits. To that end, fix the notation $\iota_{p,r}(x) := r^{-1}(x-p)$.

Lemma 15.0.2 (Minimal tangent cones and density's upper semi-continuity)**.** *Let T be an ω-almost area minimizing integral current in $\mathbb{R}^{m+n}$, $\partial T = 0$ and $p \in \mathrm{spt}(T)$. Then $\Theta^m(T,p) \geq 1$ and*

(i) *$\Theta^m(T,q)$ exists everywhere and is an upper semi-continuous functions of $q \in \mathbb{R}^{m+n}$;*

(ii) *For each sequence $r_k \to 0$, there is a subsequence $\{r_{k'}\}$ such that $(\iota_{p,r_{k'}})_\# T =: T_{p.r_{k'}} \rightharpoonup C$, where C is an integer* area minimizing cone *in $\mathbb{R}^{m+n}$ with $\Theta^m(T,p) = \Theta^m(C,0)$.*

Remark 15.0.3. The uniqueness of C is a long-standing open problem in the literature, i.e., as we change the sequence $\{r_k\}_{k\in\mathbb{N}}$ we may end up with different limits. At this point, the conical property and the minimality are the takeaways. □

Proof. The existence everywhere of the density is a direct consequence of the ω-almost monotonicity formula (Proposition 14.0.2), as well the upper semi-continuity of the density with respect to the point.

It remains to prove the existence of a limit of the blowup sequence $\{T_{p.r_{k'}}\}_{k'\in\mathbb{N}}$ and the conical property of this limit.

The existence of C follows from Lemma 14.0.3 which also gives the minimality of C. Indeed, since $\omega_{k'}(r) = \omega(r_{k'}r)$ is the minimality error of $T_{p,r_{k'}}$, we have that $\lim_{k'} \omega_{r_{k'}} \equiv 0$ hence C is an integer area minimizing current.

We prove the conical property as follows. Since we are dealing with Radon measures, for almost every $\rho > 0$, we have

$$\|T_{p,r_{k'}}\|(\mathbf{B}(0,\rho)) \to \|C\|(\mathbf{B}(0,\rho)) \Rightarrow$$
$$\Rightarrow \frac{\|C\|(\mathbf{B}(0,\rho))}{\omega_m \rho^m} = \lim_{k'\to\infty} \frac{\|T_{p,r_{k'}}\|(\mathbf{B}(0,\rho))}{\omega_m \rho^m} = \lim_{k'\to\infty} \frac{\|T\|(\mathbf{B}(p,r_{k'}\rho))}{\omega_m \rho^m r_{k'}^m} = \Theta^m(T,p).$$

The last inequality stated that the mass ration of C is equal to a fixed constant, namely $\Theta^m(T,p)$, for almost every ρ. Such fact implies that the left-hand side of the ω-almost monotonicity formula (Proposition 14.0.2) vanishes. Combined with [Sim14, Lemma 2.40], we obtain that C is a cone. ☐

The blowup limits above (C's of Lemma 15.0.2) are called **tangent cones to** T **at** p. Whenever it occurs that $\operatorname{spt}(C)$ is a m-dimensional subspace of $\mathbb{R}^{m+n}$, C will be named a **tangent plane to** T **at** p.

Even though, Almgren's stratification theorem gives an estimate, it is **not** an estimate for the singular set, since existence of flat tangent planes does **not** imply regularity in the arbitrary density and codimension setting.

Theorem 15.0.4 (Almgren's stratification theorem for ω-almost area minimizing currents)**.** *Let T be an ω-almost area minimizing integral current in $\mathbb{R}^{m+n}$ and $\partial T = 0$. Then, for any $\eta > 0$, there exists a* tangent plane *to T at p for $\mathcal{H}^{m-2+\eta}$-a.e. $p \in \operatorname{spt}(T)$.*

Proof. As remarked, the proof is an adaptation of [Sim14, Theorem 3.3, Chapter 7]. Recalling that [Sim14, Theorem 3.3, Chapter 7] is just an application of [Sim14, Theorem A.4], we will as well make a list of definitions to be able to apply [Sim14, Theorem A.4] (since [Sim14, Theorem A.4] does not require any minimality condition).

We now define the set $\mathscr{F} \subset \mathbf{I}_m(\mathbb{R}^{m+n})$ as follows

$$\mathscr{F} := \left\{ S : S = \lim_{i\to+\infty} (\iota_{x_i,\lambda_i})_\# T, \{x_i\}_{i\in\mathbb{N}} \text{ is a converging sequence, and } \lambda_i \to 0 \right\}.$$

By compactness, i.e., using Lemma 14.0.3, we have that any $S \in \mathscr{F}$ is an integer *area minimizing* m-current. It is also straightforward to verify that

$$(\iota_{q,\lambda})_\# \mathscr{F} = \mathscr{F}, \quad \forall q \in \mathbb{R}^{m+n}, \quad \lambda > 0.$$

We define, for any $A \in 2^{\mathbb{R}^{m+n}}$, L an m-dimensional subspace of $\mathbb{R}^{m+n}$, $\lambda > 0$, and $q \in \mathbb{R}^{m+n}$, the height function as follows:

$$h(A,L,\lambda,q) = \sup_{y\in A\cap \mathbf{B}(q,\lambda)} |\mathbf{p}_{L^\perp}(y-q)|.$$

For any $S \in \mathscr{F}$ and $\beta > 0$, we set

$$\mathscr{T}_\beta(S) := \{q \in \operatorname{spt}(S) : h(\operatorname{spt}(S), L, \lambda, q) \leq \beta\lambda \text{ for some } \lambda > 0 \text{ and an } m\text{-plane } L \subset \mathbb{R}^{m+n}\}.$$

It is straightforward to see that

$$(\iota_{q,\lambda})_\# \mathscr{T}_\beta(S) = \mathscr{T}_\beta((\iota_{q,\lambda})_\# S), \quad \forall q \in \mathbb{R}^{m+n}, \quad \lambda > 0.$$

Using the ω-almost monotonicity formula (Proposition 14.0.2), we readily check that

$$\begin{cases} \{S_j\}_{j\in\mathbb{N}} \subset \mathscr{T}, S_j \rightharpoonup S \in \mathscr{F} \\ \{p_j\}_{j\in\mathbb{N}} \subset \operatorname{spt}(S_j), p_j \to p \in \mathscr{T}_\beta(S) \end{cases} \Rightarrow \quad p_j \in \operatorname{spt}(S_j).$$

Denoting $N = \binom{m+n}{n}$ and, for each $S \in \mathscr{F}$, define $\varphi_S^0(q) := \Theta^m(S,q)$ and $\varphi_S^k(q) := \Theta^m(S,q) N_S^k(q)$ for $k \in \{1, \ldots, N\}$ where N_S^k is the k-th component of the orientation $\vec{S}(q)$. We also set

$$\mathscr{F} : \{\varphi_S : S \in \mathscr{T}\} \text{ and } \operatorname{sing}\varphi_S := \operatorname{spt}\Theta^m(S, \cdot) \setminus \mathscr{T}_\beta(S).$$

By [Sim14, Theorem A.4 (‡)] (the notations and definitions above matches the same notations and definition prior to Theorem A.4), we obtain the existence of $d \in \mathbb{N} \cap [0, m-1]$ such that $\dim_H(\operatorname{sing}\varphi_S) \leq d$ for each $S \in \mathscr{F}$. In another words, we obtain for any $\eta > 0$

$$0 = \mathcal{H}^{d+\alpha}(\operatorname{sing}\varphi_S) = \mathcal{H}^{d+\alpha}(\operatorname{spt}(S) \setminus \mathscr{T}_\beta(S)), \quad \forall \beta > 0.$$

Taking a sequence $\beta_j \to 0$, we obtain from the last equality

$$0 = \mathcal{H}^{d+\alpha}(\operatorname{spt}(S) \setminus \cup_{j\in\mathbb{N}} \mathscr{T}_{\beta_j}(S)).$$

Furthermore, it is easy to see, by the definition of h, that it holds

$$q \in \cup_{j\in\mathbb{N}} \mathscr{T}_{\beta_j}(S) \Leftrightarrow T \text{ has a tangent plane at } q.$$

Finally, by the last two displayed equations, it remains to prove that $d \leq m-2$. We argue by contradiction, assume that $d > m-2$ which implies, since d is an integer, $d = m-1$. Also from [Sim14, Theorem A.4 (‡‡)], we also get the existence of $S \in \mathscr{T}$ and an d-dimensional plane L such that $\operatorname{sing}\varphi_S = L$. Since S is a minimizing cone without boundary, it is well known (see for instance [Sim14, Lemma 3.5, Chapter 7]) that it splits into $[\![\mathbb{R}^{m-1}]\!] \times S_0$ where S_0 is an 1-dimensional minimizing cone. Since S_0 has no boundary, it has to be a line, i.e., $S_0 = k\,[\![\ell]\!]$. Such a fact, surely implies that $S = [\![\mathbb{R}^{m-1}]\!] \times S_0 = [\![\mathbb{R}^{m-1}]\!] \times k\,[\![\ell]\!]$ is flat, hence $\operatorname{sing}\varphi_S = \emptyset$ which is a contradiction. Then $d \leq m-2$ and we are done. □

Chapter 16

Strong Lipschitz approximation

The goal of this section is to prove our main theorem regarding the Lipschitz approximation, i.e., Theorem 16.4.1, in which we provide superlinear estimates.

We now define the excess which is one of the most important concepts in the regularity theory of currents, since it measures the deviation of the current with respect to an m-plane.

Definition 16.0.1 (Excess measure)**.** Let $T \in \mathbf{I}_m(\mathbb{R}^{m+n})$ satisfying Assumption 4. We define the **excess measure** as

$$\mathbf{e}_T(A) := \|T\|(A \times \mathbb{R}^n) - Q\mathcal{H}^m(A), \quad \forall A \subset \mathrm{B}_r(\mathbf{p}(p)),$$

and the **cylindrical excess** as

$$\mathbf{E}(T, \mathbf{C}(p,s)) = \frac{\mathbf{e}_T(\mathrm{B}_s(\mathbf{p}(p)))}{\omega_m s^m}, \quad \forall s \in (0,r),$$

where from now on we denote $\omega_m := \mathcal{H}^m(\mathrm{B}_1(0))$.

In what follows, we will work under two assumptions which are the constancy assumption (CA) and the no boundary assumption (NB) described below.

Assumption 4. There exist a point $p \in \mathrm{spt}(T) \setminus \mathrm{spt}(\partial T)$, a radius $r > 0$ and an integer $Q \in \mathbb{N} \setminus \{0\}$ such that

$$(\mathbf{p}_\# T) \llcorner \mathbf{C}(p, 4r) = \Theta^m(\mathbf{p}_\# T, p) [\![\mathrm{B}_{4r}(\mathbf{p}(p))]\!] := Q [\![\mathrm{B}_{4r}(\mathbf{p}(p))]\!], \tag{CA}$$

$$\partial T \llcorner \mathbf{C}(p, 4r) = 0. \tag{NB}$$

The assumption above is not restrictive when the strong Lusin type Lipschitz approximation is applied in the regularity theory, because of the following lemma which is the analogous of [DS16a, Lemma 1.6].

Lemma 16.0.2. *Whenever T is an ω-almost area minimizing current in $\mathbb{R}^{m+n}$, $p \in \mathrm{spt}(T) \setminus$*

spt(∂T), *there exists a geometric constant* $\eta = \eta(m,n,Q) > 0$, *such that, if*

$$\Theta^m(T,p) = Q \tag{16.0.1}$$

$$\frac{\|T\|(\mathbf{B}(p,4r))}{\omega_m(4r)^m} \le Q + \eta, \tag{16.0.2}$$

$$\max\{\omega(\mathbf{s}), \mathbf{E}(T,\mathbf{B}(p,4r)) := \mathbf{E}(T,\mathbf{B}(p,4r), \pi_0)\} < \eta, \tag{16.0.3}$$

then we have that $(\mathbf{p}_{\pi_0})_\# T \llcorner \mathbf{B}(p,4r) = Q [\![\mathrm{B}_{4r}(\mathbf{p}(p))]\!]$.

Proof. Assume by contradiction that we have a sequence T_k of ω_k-almost area minimizing currents with ω_k converging uniformly to $f \equiv 0$ as $k \to \infty$, and a sequence of real numbers $\eta_k \to 0$ as k goes to $+\infty$ both satisfying (16.0.2) and (16.0.3) such that

$$(\mathbf{p}_{\pi_0})_\# T_k \llcorner \mathbf{B}(p,4r) \ne Q [\![\mathrm{B}_{4r}(\mathbf{p}(p))]\!], \forall k \in \mathbb{N}.$$

Since $\|T_k\|(\mathbf{B}(p,4r)) + \|\partial T_k\|(\mathbf{B}(p,4r))$ is uniformly bounded w.r.t. k, we can then apply the Compactness Theorem for integral currents which gives the existence of an integral current $T_\infty \in \mathbf{I}_m(\mathbf{B}(p,4r))$ such that $T_k \rightharpoonup T_\infty$ in the sense of currents. By the ω-almost monotonicity formula, Proposition 14.0.1, we have that the convergence in the sense of currents implies the convergence of the measures and the Hausdorff convergence of the supports, Lemma 14.0.3. Hence, by (16.0.3), we directly obtain that

$$\mathbf{E}(T_\infty, \mathbf{B}(p,4r), \pi_0) = 0,$$

which leads to spt$(T_\infty) \subset \mathbf{B}(p,4r) \cap \pi_0$ and then $T_\infty = Q_\infty [\![\mathrm{B}_{4r}(\mathbf{p}(p))]\!]$ for some integer Q_∞. Using the convergence of $\|T_k\| \rightharpoonup \|T_\infty\|$, (16.0.2) and (16.0.1), we obtain that $Q_\infty = Q$. Now, since T_k has no boundary in $\mathbf{B}(p,r)$, i.e., $\partial T_k \llcorner \mathbf{B}(p,r) = 0$, for k large enough, we obtain by the Constancy Lemma ([Fed69, 4.1.17]) that

$$(\mathbf{p}_{\pi_0})_\# T_k \llcorner \mathbf{B}(p,4r) = Q_k [\![\mathrm{B}_{4r}(\mathbf{p}(p))]\!].$$

Since we have the convergence of T_k to T_∞ in the sense of currents, we obtain that $Q_k = Q$, for k large enough, which gives the contradiction. □

16.1 Weak Lipschitz approximation

The weak Lipschitz approximation does not need any minimality condition to be proven, then we refer the reader directly to [DS14, Proposition 2.2].

Proposition 16.1.1 (Weak Lusin type Lipschitz approximation)**.** *There exist two positive geometric constants* $\varepsilon_{wl} = \varepsilon_{wl}(m,n,Q) > 0$ *and* $C_{wl} = C_{wl}(m,n,Q) > 0$ *such that, if*

(a) T *satisfies Assumption 4,*

(b) $\mathbf{E}(T,\mathbf{C}(p,4r)) < \varepsilon_{wl}$,

(c) $\beta \in (0, \frac{1}{2m}]$,

hold, we have that exist a set $K \subset \mathrm{B}_{\frac{7r}{2}}(\mathbf{p}(p))$ *and a Lipschitz function* $f : \mathrm{B}_{\frac{7r}{2}}(\mathbf{p}(p)) \to \mathcal{A}_Q(\mathbb{R}^n)$ *which satisfies*

$$\mathrm{Lip}(f) \leq C_{wl}\mathbf{E}(T, \mathbf{C}(p, 4r))^{\beta}, \tag{16.1.1}$$

$$\mathbf{G}_f \llcorner (K \times \mathbb{R}^n) = T \llcorner (K \times \mathbb{R}^n), \tag{16.1.2}$$

$$\|T - \mathbf{G}_f\| \left(\mathrm{B}_{\frac{7r}{2}}(\mathbf{p}(p)) \setminus K\right) \leq C_{wl} r^m \mathbf{E}(T, \mathbf{C}(p, 4r))^{1-2\beta},$$

$$\mathcal{H}^m \left(\mathrm{B}_{\frac{7r}{2}}(\mathbf{p}(p)) \setminus K\right) \leq C_{wl} r^m \mathbf{E}(T, \mathbf{C}(p, 4r))^{1-2\beta}.$$

Let us now enunciate the analogous of [DLSS18, Lemma 2.2] with our more general almost minimality condition.

Lemma 16.1.2 (Homotopy Lemma)**.** *Let* T *be* ω*-almost area minimizing which satisfies Assumption 4 and* $3r < \mathbf{s}$*. There are positive geometric constants* $\varepsilon_h = \varepsilon_h(m, n, Q) > 0$ *and* $C_h = C_h(m, n, Q) > 0$ *such that, whenever* $\mathbf{E}(T, \mathbf{C}(p, 4r)) < \varepsilon_h < \varepsilon_{wl}$*, it holds*

$$\|T\|(\mathbf{C}(p, 3r)) \leq \|R\|(\mathbf{C}(p, 3r)) + C_h \omega(r) r^m \mathbf{E}(T, \mathbf{C}(p, 4r))^{\frac{1}{2}},$$

for every $R \in \mathbf{I}_m(\mathbf{C}(p, 3r))$ *with* $\partial R = \partial (T \llcorner \mathbf{C}(p, 3r))$.

Proof. In the proof of [DLSS18, Lemma 2.2], the authors do not use any minimality property to show the existence of an $(m+1)$-current $S'' \in \mathbf{I}_{m+1}(\mathbf{C}(p, 3r))$ (notice that S'' is the same notation that the authors use for such current) such that

$$\partial S'' = (T - R) \llcorner \mathbf{C}(p, 3r), \tag{16.1.3}$$

$$\|S''\|(\mathbf{C}(p, 3r)) \leq C_0 r^{m+1} \mathbf{E}(T, \mathbf{C}(p, 3r))^{\frac{1}{2}}. \tag{16.1.4}$$

Then, applying our ω-minimality condition on T, we obtain that

$$\begin{aligned}
\|T\|(\mathbf{C}(p, 3r)) &\leq (1 + \omega(r)) \|T - \partial S''\|(\mathbf{C}(p, 3r)) \\
&\overset{(16.1.3)}{\leq} \|R\|(\mathbf{C}(p, 3r)) + \omega(r)\|T - \partial S''\|(\mathbf{C}(p, 3r)) \\
&\overset{(*)}{\leq} \|R\|(\mathbf{C}(p, 3r)) + C_1 \frac{\omega(r)}{r} r^{m+1} \\
&\overset{(*)}{\leq} \|R\|(\mathbf{C}(p, 3r)) + C_2 \frac{\omega(r)}{r} \|S''\|(\mathbf{C}(p, 3r)) \\
&\overset{(16.1.4)}{\leq} \|R\|(\mathbf{C}(p, 3r)) + C_3 \omega(r) r^m \mathbf{E}(T, \mathbf{C}(p, 3r))^{\frac{1}{2}},
\end{aligned}$$

where in $(*)$ we use standard density estimates, see for instance [Fed69, Section 4.1.28]. □

Definition 16.1.3 (Barycenter of a Q-tuple)**.** We define $\boldsymbol{\eta}(P) = \frac{1}{Q}\sum_{i=1}^{Q} P_i$ for every $P = \sum_{i=1}^{Q} [\![P_i]\!] \in \mathcal{A}_Q(\mathbb{R}^n)$.

Proposition 16.1.4 (Approximation by minimizers of the Dirichlet energy)**.** *Given two positive real numbers* $\eta > 0$ *and* $\beta \in (0, \frac{1}{2m})$*, there exists a positive geometric constant* $\varepsilon_{ha} = \varepsilon_{ha}(m, n, Q, \eta, \beta) > 0$ *such that, if*

(a) T *is* ω*-almost area minimizing and satisfies Assumption 4,*

(b) $\mathbf{E}(T, \mathbf{C}(p, 4r)) < \varepsilon_{ha} < \varepsilon_h$,

(c) $\omega(r) \leq \varepsilon_{ha}\mathbf{E}(T, \mathbf{C}(p, 4r))^{\frac{1}{2}}$,

hold, then there exists $w : \mathrm{B}_{\frac{7r}{2}}(\mathbf{p}(p)) \to \mathcal{A}_Q(\mathbb{R}^n)$ *which minimizes the Dirichlet energy and satisfies*

$$\int_{\mathrm{B}_{2r}(\mathbf{p}(p))} |Df|^2 \, d\mathcal{H}^m \leq \eta \mathbf{e}_T(\mathrm{B}_{4r}(\mathbf{p}(p))),$$

$$\frac{1}{r^2}\int_{\mathrm{B}_{2r}(\mathbf{p}(p))} \mathcal{G}(f, g)^2 \, d\mathcal{H}^m + \int_{\mathrm{B}_{2r}(\mathbf{p}(p))} (|Df| - |Dw|)^2 \, d\mathcal{H}^m \leq \eta \mathbf{e}_T(\mathrm{B}_{4r}(\mathbf{p}(p))),$$

$$\int_{\mathrm{B}_{2r}(\mathbf{p}(p))} |D(\boldsymbol{\eta} \circ f) - D(\boldsymbol{\eta} \circ w)|^2 \, d\mathcal{H}^m \leq \eta \mathbf{e}_T(\mathrm{B}_{4r}(\mathbf{p}(p))).$$

where f *is the weak Lusin type Lipschtiz approximation given by Proposition 16.1.1.*

Proof. Note that condition ((a)) and ((b)) put us in position to apply the Homotopy Lemma (Lemma 16.1.2) and then the proof of [DS14, Theorem 3.1] follows straightforwardly. □

Remark 16.1.5. Note that the approximation above at interior points is weaker than the one we can get at boundary points. Indeed, for m-currents that minimize the area in $\mathbb{R}^{m+n}$ and take the boundary with arbitrary boundary multiplicity, in [NR22, Theorem 4.12], we proved that w is indeed multi-copies of a classical harmonic function. However, at interior points, we can prove that w is a minimizer of the Dirichlet energy but not necessarily Q copies of an harmonic function. □

Lemma 16.1.6 (Weak excess decay)**.** *For every* $\eta > 0$*, there exist a positive geometric constants* $\varepsilon_{we} = \varepsilon_{we}(m, n, Q, \eta) > 0$ *and a positive constant* $C_{we} = C_{we}(m, n, Q, \eta) > 0$*, if*

(a) *T is ω-almost area minimizing under Assumption 4,*

(b) $\mathbf{E}(T, \mathbf{C}(p, 4r)) < \varepsilon_{we} < \varepsilon_{ha}$,

(c) $A \subset \mathrm{B}_r(\mathbf{p}(p))$ *Borel set with* $\mathcal{H}^m(A) \leq \varepsilon_{we}\omega_m r^m$,

hold, then we have that

$$\mathbf{e}_T(A) \leq \eta \mathbf{e}_T(\mathrm{B}_{4r}(\mathbf{p}(p)))r^m + C_{we}\omega(r)^2 r^m. \tag{16.1.5}$$

Proof. We first assume that $\omega(\mathbf{s}) \geq \varepsilon_{ha}\mathbf{E}(T, \mathbf{C}(p, 4r))^{\frac{1}{2}}$ which leads to

$$\mathbf{e}_T(A) = \mathbf{E}(T, \mathbf{C}(p, 4r))\omega_m(4r)^m \leq C_0\mathbf{E}(T, \mathbf{C}(p, 4r)) \leq \varepsilon_{ha}^{-2}\omega^2(\mathbf{s}),$$

which gives (16.1.5). On the other hand, if we have $\omega(\mathbf{s}) \leq \varepsilon_{ha}\mathbf{E}(T, \mathbf{C}(p, 4r))^{\frac{1}{2}}$ which is ((c)) of Proposition 16.1.4, then we can apply it and the proof goes as in [DLSS18, Proposition 3.2]. □

We fix the notation for the **density of the excess measure** as follows

$$\mathbf{d}_T(q) := \limsup_{s \to 0} \frac{\mathbf{e}_T(\mathrm{B}_s(\mathbf{p}(q)))}{\omega_m s^m}. \tag{16.1.6}$$

16.2 Higher integrability of the excess measure's density

One crucial point in the theory which will allow us to improve our weak excess decay, Lemma 16.1.6, is the higher integrability of the function $\mathbf{d}_T$, it means that $\mathbf{d}_T \in \mathrm{L}^p(\mathrm{B}_{2r}(\mathbf{p}(p)))$ for some $p > 1$. By the Taylor expansion, we can easily compare $|Df|$ (from the harmonic approximation, Proposition 16.1.4) and $\mathbf{d}_T$, hence we can reduce the problem of studying the higher integrability of minimizers of the Dirichlet energy to study it for $\mathbf{d}_T$. It will be used to prove the strong excess decay, Theorem 16.3.1. For a more detailed discussion about this topic we refer the reader to [Spa12], [DS14, Section 6] and [Spa10].

Proposition 16.2.1 (Higher integrability of the density of the excess measure)**.** *There exist positive geometric constants $a = a(m, n, Q) > 1, \varepsilon_a = \varepsilon_a(m, n, Q) > 0$ and $C_a = C_a(m, n, Q) > 0$ such that, if*

(a) *T is ω-almost area minimizing under Assumption 4,*

(b) *$\mathbf{E}(T, \mathbf{C}(p, 4r)) < \varepsilon_a < \varepsilon_{we}$,*

hold, then

$$\int_{\{\mathbf{d}_T \leq 1\}\cap \mathrm{B}_{2r}(\mathbf{p}(p))} \mathbf{d}_T^a \, \mathrm{d}\mathcal{H}^m \leq C_a \left[\mathbf{E}(T, \mathbf{C}(p, 4r))^a + \omega(r)^2 \mathbf{E}(T, \mathbf{C}(p, 4r))^{a-1}\right] r^m. \tag{16.2.1}$$

Proof. The inequality is trivial if we consider that $\mathbf{E}(T, \mathbf{C}(p, 4r)) = 0$, we then assume w.l.o.g. that $\mathbf{E} := \mathbf{E}(T, \mathbf{C}(p, 4r)) > 0$. We claim that

- There exist constants $\gamma = \gamma(m, \varepsilon_{we}) \geq 2^m$ and $\theta = \theta(m, n, \varepsilon_{we}) > 0$ such that, for every $c \in \left[1, \frac{1}{\gamma \mathbf{E}}\right]$ and $s \in \left[2r, 4r(1 - c^{-\frac{1}{m}})\right]$, we have

$$\begin{aligned}\int_{\{c\gamma\mathbf{E} \leq \mathbf{d}_T \leq 1\}\cap \mathrm{B}_s(\mathbf{p}(p))} \mathbf{d}_T \, \mathrm{d}\mathcal{H}^m \leq \gamma^{-\theta} r^m &\int_{\{\frac{c\mathbf{E}}{\gamma} \leq \mathbf{d}_T \leq 1\}\cap \mathrm{B}_{s+rc^{-\frac{1}{m}}}(\mathbf{p}(p))} \mathbf{d}_T \, \mathrm{d}\mathcal{H}^m \\ &+ C_{we} c^{-1} \omega(r)^2 r^m,\end{aligned} \tag{16.2.2}$$

We now show how to obtain the statement of the theorem, i.e., equation (16.2.1). We want to apply the claim for $c = \gamma^{2k}$, to that end, we need

$$\gamma^{2k} \leq \frac{1}{\gamma \mathbf{E}} \Rightarrow k \leq \frac{1}{2}\left(\log_\gamma \mathbf{E}^{-1} - 1\right).$$

Henceforth we denote by k_0 the biggest number in $\mathbb{N}$ that satisfies the latter inequality, and we define $s_1 := 2r$ and $s_k = s_{k-1} + r\gamma^{-\frac{2k}{m}}, \forall k \leq k_0$. Recall that s_k is increasing, thus, we may apply the claim for $c = \gamma^{2k}$ and $s = s_k$,

$$\begin{aligned}\int_{\{\gamma^{2k+1}\mathbf{E} \leq \mathbf{d}_T \leq 1\}\cap \mathrm{B}_{s_k}(\mathbf{p}(p))} \mathbf{d}_T \, \mathrm{d}\mathcal{H}^m \overset{(16.2.2)}{\leq} \gamma^{-\theta} r^m &\int_{\{\gamma^{2k-1}\mathbf{E} \leq \mathbf{d}_T \leq 1\}\cap \mathrm{B}_{s_{k+1}}(\mathbf{p}(p))} \mathbf{d}_T \, \mathrm{d}\mathcal{H}^m \\ &+ C_{we} \gamma^{-2k} \omega(r)^2 r^m, \quad \forall k \in \{1, \dots, k_0\}.\end{aligned}$$

So, if we iterate the last equation, it is then immediate to see that

$$\begin{aligned}\int_{\{\gamma^{2k+1}\mathbf{E}\leq \mathbf{d}_T\leq 1\}\cap \mathrm{B}_{2r}(\mathbf{p}(p))} \mathbf{d}_T \, d\mathcal{H}^m &\overset{(16.2.2)}{\leq} \gamma^{-(k-1)\theta} r^m \int_{\{\gamma^{2k_0-1}\mathbf{E}\leq \mathbf{d}_T\leq 1\}\cap \mathrm{B}_{s_{k_0}}(\mathbf{p}(p))} \mathbf{d}_T \, d\mathcal{H}^m \\ &\quad + C_{we}\omega(r)^2 r^m \sum_{i=0}^{k-2} \gamma^{-2(k-i)+i\theta}, \quad \forall k \in \{1,\dots,k_0\}.\end{aligned} \tag{16.2.3}$$

We thus fix any number $a = a(m,n,Q) \in (1, 1+\theta/2)$, if necessary, we reduce θ in order to have $\theta < 2/m$ and define the sets $A_0 := \{\mathbf{d}_T < \gamma\mathbf{E}\} \cap \mathrm{B}_{2r}(\mathbf{p}(p))$, $A_k := \{\gamma^{2k-1}\mathbf{E} \leq \mathbf{d}_T \leq \gamma^{2k+1}\mathbf{E}\} \cap \mathrm{B}_{2r}(\mathbf{p}(p))$, $\forall k \in \{1,\dots,k_0\}$, and $A_{k_0+1} := \{\gamma^{2k_0+1}\mathbf{E} \leq \mathbf{d}_T \leq 1\} \cap \mathrm{B}_{2r}(\mathbf{p}(p))$. Therefore, we obtain that

$$\begin{aligned}\int_{\{\mathbf{d}_T\leq 1\}\cap \mathrm{B}_{2r}(\mathbf{p}(p))} \mathbf{d}_T^a \, d\mathcal{H}^m &\leq \sum_{k=0}^{k_0+1}\int_{A_k} \mathbf{d}_T^a \, d\mathcal{H}^m \leq \sum_{k=0}^{k_0+1} \gamma^{(2k+1)(a-1)}\mathbf{E}^{a-1}\int_{A_k} \mathbf{d}_T \, d\mathcal{H}^m \\ &\overset{(16.2.3)}{\leq} C_1 \sum_{k=0}^{k_0+1}\left[\gamma^{k(2(a-1)-\theta)} r^m \mathbf{E}^a + C_2\mathbf{E}^{a-1}\omega(r)^2 r^m \sum_{i=0}^{k-2}\gamma^{k(2(a-1)-\frac{2}{m})-i(\frac{2}{m}-\theta)}\right] \\ &\leq C_3 r^m \mathbf{E}^a + C_3\mathbf{E}^{a-1}\omega(r)^2 r^m \sum_{k=0}^{k_0+1}\gamma^{k(2(a-1)-\theta)},\end{aligned}$$

which concludes the proof of the theorem.

Let us prove the initial claim (16.2.2). Let us fix the constant $\eta = 2^{-2m-N}$ and the corresponding dimensional constant $M = M(m,n) > 0$ which is the constant given by the Besicovich's covering theorem, c.f. [EG92, Section 1.5.2], the natural number $N \in \mathbb{N}$ such that $M < 2^{N-1}$ and

$$\gamma = \gamma(m, \varepsilon_{we}) := \max\left\{2^m, \varepsilon_{we}^{-1}\right\} \tag{16.2.4}$$

So, as in the claim, take arbitrary numbers $c \in \left[1, \frac{1}{\gamma\mathbf{E}}\right]$ and $s \in \left[2r, 4r(1-c^{-\frac{1}{m}})\right]$. We obtain the following inequality by classical rescaling of the excess

$$\mathbf{E}(T, \mathbf{C}(q,t)) \leq \left(\frac{4r}{t}\right)^m \mathbf{E}(T, \mathbf{C}(q,4r)) \leq c\mathbf{E}, \quad \forall t \geq 4rc^{-\frac{1}{m}}. \tag{16.2.5}$$

Note that, by (16.2.5), we can define

$$4r(q) := \min\left\{t \in \left[0, 4rc^{-\frac{1}{m}}\right] : \mathbf{E}(T, \mathbf{C}(q,t)) \leq c\mathbf{E}\right\},$$

it remains to show that $r(q) > 0$ and it follows for almost every $q \in \{c\gamma\mathbf{E} \leq \mathbf{d}_T \leq 1\} \cap \mathbf{C}(q,s)$, since

$$\lim_{s\to 0}\mathbf{E}(T, \mathbf{C}(q,s)) \overset{(16.1.6)}{=} \mathbf{d}_T(q) \geq c\gamma\mathbf{E} \overset{(16.2.4)}{\geq} 2^m c\mathbf{E}.$$

By the definition of $r(q)$ and the fact that it is a positive constant, we obviously have the inequalities

$$\mathbf{E}(T, \mathbf{C}(q, 4r(q))) \leq c\mathbf{E}, \tag{16.2.6}$$

$$\mathbf{E}(T, \mathbf{C}(q,t)) \geq c\mathbf{E}, \quad \forall t \in (0, 4r(q)). \tag{16.2.7}$$

Now, we aim to apply the weak excess decay, Lemma 16.1.6, to the current $T \llcorner \mathbf{C}(q, 4r(q))$ and the set $A = \{c\gamma\mathbf{E} \leq \mathbf{d}_T\} \cap \mathrm{B}_{4r(q)}(\mathbf{p}(q))$. It is clear that we are under hypothesis ((a)) of Lemma 16.1.6, let us check ((b)):

$$\mathbf{E}(T, \mathbf{C}(q, 4r(q))) \overset{(16.2.6)}{\leq} c\mathbf{E} \leq \frac{E}{\gamma E} = \gamma^{-1} \overset{(16.2.4)}{<} \varepsilon_{we},$$

and ((c)) follows from:

$$\begin{aligned}\mathcal{H}^m(A) \leq \frac{1}{c\gamma\mathbf{E}} \int_A \mathbf{d}_T \, \mathrm{d}\mathcal{H}^m(q) &\overset{\text{Fatou's Lemma}}{\leq} \frac{1}{c\gamma\mathbf{E}} \lim_{s\to 0} \int_A \mathbf{E}(T, \mathbf{C}(q, s)) \, \mathrm{d}\mathcal{H}^m(q) \\ &\overset{(16.2.6)}{\leq} \frac{\mathcal{H}^m(\mathrm{B}_{4r(q)}(\mathbf{p}(q)))}{\gamma} \\ &\overset{(16.2.4)}{<} \varepsilon_{we}\omega_m(4r(q))^m.\end{aligned}$$

Now, recalling that $\eta = 2^{-2m-N}$, we are able to apply the weak excess decay, Lemma 16.1.6, and Lebesgue's differentiation theorem to derive

$$\begin{aligned}\int_A \mathbf{d}_T \, \mathrm{d}\mathcal{H}^m \leq \mathbf{e}_T(A) &\overset{(16.1.5)}{\leq} 2^{-2m-N}\mathbf{e}_T(\mathrm{B}_{4r(q)}(\mathbf{p}(q)))r(q)^m + C_{we}\omega(r(q))^2 r(q)^m \\ &= 2^{-N}\mathbf{E}(T, \mathbf{C}(q, 4r(q))) + C_{we}\omega(r(q))^2 r(q)^m \\ &\overset{(16.2.6)}{\leq} 2^{-N}c\mathbf{E} + C_{we}\omega(r(q))^2 r(q)^m \\ &\overset{(16.2.7)}{\leq} 2^{-N}\mathbf{E}(T, \mathbf{C}(q, r(q))) + C_{we}\omega(r(q))^2 r(q)^m \\ &= 2^{-N}\mathbf{e}_T(\mathrm{B}_{r(q)}(\mathbf{p}(q)))r(q)^m + C_{we}\omega(r(q))^2 r(q)^m.\end{aligned} \tag{16.2.8}$$

From the latter chain of inequalities (16.2.8) and the fact that $\{\mathbf{d}_T > 1\} \cap \mathrm{B}_{r(q)}(\mathbf{p}(q)) \subset A$, we obtain

$$\int_{\{\mathbf{d}_T > 1\}\cap \mathrm{B}_{r(q)}(\mathbf{p}(q))} \mathbf{d}_T \, \mathrm{d}\mathcal{H}^m \leq \int_A \mathbf{d}_T \, \mathrm{d}\mathcal{H}^m \leq 2^{-N}\mathbf{e}_T(\mathrm{B}_{r(q)}(\mathbf{p}(q)))r(q)^m + C_{we}\omega(r(q))^2 r(q)^m. \tag{16.2.9}$$

It is easy to see that

$$\int_{\{\mathbf{d}_T < \frac{c\mathbf{E}}{\gamma}\}\cap \mathrm{B}_{r(q)}(\mathbf{p}(q))} \mathbf{d}_T \, \mathrm{d}\mathcal{H}^m \leq \frac{c\mathbf{E}}{\gamma}\omega_m r(q)^m \overset{(16.2.7)}{\leq} \gamma^{-1}\mathbf{e}_T(\mathrm{B}_{r(q)}(\mathbf{p}(q))), \tag{16.2.10}$$

Putting the Lebesgue differentiation theorem, (16.2.9) and (16.2.10) into account, we infer that

$$\begin{aligned}\mathbf{e}_T(\mathbf{B}(\mathbf{p}(q), r(q))) &= \int_{\{\mathbf{d}_T < \frac{c\mathbf{E}}{\gamma}\}\cap \mathrm{B}_{r(q)}(\mathbf{p}(q))} \mathbf{d}_T \, \mathrm{d}\mathcal{H}^m + \int_{\{\frac{c\mathbf{E}}{\gamma} \leq \mathbf{d}_T \leq 1\}\cap \mathrm{B}_{r(q)}(\mathbf{p}(q))} \mathbf{d}_T \, \mathrm{d}\mathcal{H}^m \\ &\quad + \int_{\{\mathbf{d}_T > 1\}\cap \mathrm{B}_{r(q)}(\mathbf{p}(q))} \mathbf{d}_T \, \mathrm{d}\mathcal{H}^m \\ &\overset{(16.2.9),(16.2.10)}{\leq} \left(2^{-N}r(q)^m + \gamma^{-1}\right)\mathbf{e}_T(\mathrm{B}_{r(q)}(\mathbf{p}(q))) + C_{we}\omega(r(q))^2 r(q)^m \\ &\quad + \int_{\{\frac{c\mathbf{E}}{\gamma} \leq \mathbf{d}_T \leq 1\}\cap \mathrm{B}_{r(q)}(\mathbf{p}(q))} \mathbf{d}_T \, \mathrm{d}\mathcal{H}^m,\end{aligned}$$

which surely implies

$$\left[1-2^{-N}-\gamma^{-1}\right]\mathbf{e}_T(\mathbf{B}(\mathbf{p}(q),r(q)))\leq\int_{\{\frac{c\mathbf{E}}{\gamma}\leq\mathbf{d}_T\leq 1\}\cap \mathrm{B}_{r(q)}(\mathbf{p}(q))}\mathbf{d}_T\,\mathrm{d}\mathcal{H}^m+C_{we}\omega(r(q))^2r(q)^m. \tag{16.2.11}$$

Since $\{c\gamma\mathbf{E}\leq\mathbf{d}_T\leq 1\}\cap \mathrm{B}_{r(q)}(\mathbf{p}(q))\subset A$, it is guaranteed that

$$\begin{aligned}\int_{\{c\gamma\mathbf{E}\leq\mathbf{d}_T\leq 1\}\cap \mathrm{B}_{r(q)}(\mathbf{p}(q))}\mathbf{d}_T\,\mathrm{d}\mathcal{H}^m &\leq\int_A\mathbf{d}_T\,\mathrm{d}\mathcal{H}^m\\ &\overset{(16.2.8)}{\leq} 2^{-N}\mathbf{e}_T(\mathrm{B}_{r(q)}(\mathbf{p}(q)))r(q)^m+C_{we}\omega(r(q))^2r(q)^m\\ &\overset{(16.2.11)}{\leq}\frac{2^{-N}}{1-2^{-N}-\gamma^{-1}}r(q)^m\int_{\{\frac{c\mathbf{E}}{\gamma}\leq\mathbf{d}_T\leq 1\}\cap \mathrm{B}_{r(q)}(\mathbf{p}(q))}\mathbf{d}_T\,\mathrm{d}\mathcal{H}^m\\ &\quad+C_{we}\omega(r(q))^2r(q)^m,\end{aligned} \tag{16.2.12}$$

using that $\gamma\geq 2^m$, we can bound $\frac{2^{-N}}{1-2^{-N}-\gamma^{-1}}$ by 2^{-N+1}. Recalling that M is given by the Besicovich's covering theorem, [EG92, Section 1.5.2], we choose M families $\{\mathcal{A}_1,\dots,\mathcal{A}_M\}$ of closed disjoint balls $\mathrm{B}_{r(q)}(\mathbf{p}(q))$ with center in $\mathrm{B}_s(\mathbf{p}(p))$ such that their union covers $\{c\gamma\mathbf{E}\leq\mathbf{d}_T\leq 1\}\cap\mathrm{B}_s(\mathbf{p}(p))$, then by (16.2.12) we have that

$$\begin{aligned}\int_{\{c\gamma\mathbf{E}\leq\mathbf{d}_T\leq 1\}\cap \mathrm{B}_s(\mathbf{p}(p))}\mathbf{d}_T\,\mathrm{d}\mathcal{H}^m &\leq\sum_{i=1}^M\sum_{q\in\mathcal{A}_i}\int_{\{c\gamma\mathbf{E}\leq\mathbf{d}_T\leq 1\}\cap \mathrm{B}_{r(q)}(\mathbf{p}(q))}\mathbf{d}_T\,\mathrm{d}\mathcal{H}^m\\ &\overset{(16.2.12)}{\leq} 2^{-N+1}\sum_{i=1}^M\sum_{q\in\mathcal{A}_i}\left[\int_{\{\frac{c\mathbf{E}}{\gamma}\leq\mathbf{d}_T\leq 1\}\cap \mathrm{B}_{r(q)}(\mathbf{p}(q))}\mathbf{d}_T\,\mathrm{d}\mathcal{H}^m\right.\\ &\quad\left.+C_{we}\omega(r(q))^2r(q)^m\right]r(q)^m\\ &\overset{(*)}{\leq} 2^{-N+1}Mc^{-1}r^m\int_{\{\frac{c\mathbf{E}}{\gamma}\leq\mathbf{d}_T\leq 1\}\cap \mathrm{B}_{s+rc^{-\frac{1}{m}}}(\mathbf{p}(p))}\mathbf{d}_T\,\mathrm{d}\mathcal{H}^m\\ &\quad+C_{we}c^{-1}r^m\sum_{i=1}^M\sum_{q\in\mathcal{A}_i}\omega(r(q))^2\\ &\leq 2^{-N+1}Mc^{-1}r^m\int_{\{\frac{c\mathbf{E}}{\gamma}\leq\mathbf{d}_T\leq 1\}\cap \mathrm{B}_{s+rc^{-\frac{1}{m}}}(\mathbf{p}(p))}\mathbf{d}_T\,\mathrm{d}\mathcal{H}^m\\ &\quad+C_{we}c^{-1}\omega(r)^2r^m,\end{aligned}$$

where in $(*)$ we use that $r(q)\leq rc^{-\frac{1}{m}}, c^{-1}\leq 1$ and $M<2^{N-1}$. Now, we choose

$$\theta=\theta(m,n,\varepsilon_{we}):=-\log_\gamma\left(\frac{M}{2^{N-1}}\right),$$

with this number settled and recalling that $c^{-1}\leq 1$, we finish the proof of the claim. □

16.3 Strong excess decay

We now enunciate the strong excess decay which is a straightened version of the weak statement, Lemma 16.1.6, in which we cut off assumption ((c)) and improve (16.1.5) to (16.3.1). This stronger decay will allow us to improve all our previous approximation to then give the proof of the main theorem, Theorem 16.4.1.

Theorem 16.3.1 (Strong excess decay)**.** *There exist positive geometric constants $\varepsilon_{se} = \varepsilon_{se}(m, n, Q) > 0$, $C_{se} = C_{se}(m, n, Q) > 0$, and $\gamma_{se} = \gamma_{se}(m, n, Q) > 0$ such that, if*

(a) *T is ω-almost area minimizing under Assumption 4,*

(b) $\mathbf{E}(T, \mathbf{C}(p, 4r)) < \varepsilon_{se} < \varepsilon_a$,

hold, thus, for every Borel set $A \subset \mathrm{B}_{\frac{5r}{4}}(\mathbf{p}(p))$, we have that

$$\mathbf{e}_T(A) \leq C_{se}\left(\mathbf{E}(T, \mathbf{C}(p, 4r))^{\gamma_{se}} + \mathcal{H}^m(A)^{\gamma_{se}}\right)\left(\mathbf{E}(T, \mathbf{C}(p, 4r)) + \omega(r)^2\right) r^m. \tag{16.3.1}$$

Proof. The proof of this result is similar to the one given in [DS14, Theorem 7.1] where the authors consider area minimizing currents with support contained in an ambient manifold. We set $\mathbf{E} := \mathbf{E}(T, \mathbf{C}(p, 4r))$. Given $\beta \in (0, \frac{1}{2m})$, as it is done in [DS14, Proposition 7.3], using a regularization by convolution technique, we can build up a set $B \subset [r, 2r]$ with $\mathcal{H}^1(B) > r/2$ such that, for every $z \in B$, there exists a Lipschitz function $g : \mathrm{B}_z(\mathbf{p}(p)) \to \mathcal{A}_Q(\mathbb{R}^n)$ satisfying

$$\begin{aligned} \mathrm{Lip}(g) &\leq C_1\mathbf{E}^{\beta}, \\ g|_{\partial \mathrm{B}_z(\mathbf{p}(p))} &= f|_{\partial \mathrm{B}_z(\mathbf{p}(p))}, \\ \int_{\mathrm{B}_z(\mathbf{p}(p))} |Dg|^2 \,\mathrm{d}\mathcal{H}^m &\leq \int_{\mathrm{B}_z(\mathbf{p}(p))\cap K} |Df|^2 \,\mathrm{d}\mathcal{H}^m + C_1\mathbf{E}^{\gamma_0}\left(\mathbf{E} + \omega(r)^2\right) r^m, \end{aligned} \tag{16.3.2}$$

where f and K are given by the weak approximation from Proposition 16.1.1, and $\gamma_0 = \gamma_0(m, n, Q) > 0$ and $C_1 = C_1(m, n, Q) > 0$ are positive geometric constants. There is a radius $z \in B$ and a current $P \in \mathbf{I}_m(\mathbb{R}^{m+n})$ with

$$\partial P = \partial\left(T \llcorner \mathbf{C}(p, z) - \mathbf{G}_f \llcorner \mathbf{C}(p, z)\right) \tag{16.3.3}$$

$$\|P\|(\mathbf{C}(p, z)) \leq C_2\mathbf{E}^{1+\gamma_1} r^m. \tag{16.3.4}$$

We now take γ to be $\min\{\gamma_0, \gamma_1\}$. Thanks to (16.3.3) and Lemma 16.1.2, we are apt to apply [DLSS18, Equation 2.2] in our context which, together with (16.3.4), provides us the following

$$\|T\|(\mathbf{C}(p, z)) \leq \|\mathbf{G}_g\|(\mathbf{C}(p, z)) + C_3\mathbf{E}^{1+\gamma} r^m + C_3\omega(r)^2 r^m \mathbf{E}^{\frac{3}{4}} + C_3\int_{\mathrm{B}_z(\mathbf{p}(p))} \mathcal{G}(g, f)\,\mathrm{d}\mathcal{H}^m. \tag{16.3.5}$$

We proceed with a simple algebraic argument, notice that for any nonzero real numbers a and b, we have that $0 \leq (ba)^2 - 2ab + 1$ which leads to $2a \leq ba^2 + 1/b$. Therefore, taking $a = \omega(r)^2 r^m \mathbf{E}^{3/4}$ and $b = \omega(r)^{-2} r^{-m}\mathbf{E}^{\gamma - 3/2}$, we have that

$$2\omega(r)^2 r^m \mathbf{E}^{\frac{3}{4}} \leq \omega(r)^2 r^m \mathbf{E}^{\gamma} + \omega(r)^2 r^m \mathbf{E}^{\frac{3}{2}-\gamma} \leq \omega(r)^2 r^m \mathbf{E}^{\gamma} + \omega(r)^2 r^m \mathbf{E}^{1+\gamma},$$

where in the last inequality we used that $\gamma < 1/4$. By the latter inequality and (16.3.5), we get that

$$\|T\|(\mathbf{C}(p,z)) \le \|\mathbf{G}_g\|(\mathbf{C}(p,z)) + C_3\mathbf{E}^\gamma\left(\omega(r)^2 + \mathbf{E}\right) r^m + C_3\int_{\mathrm{B}_z(\mathbf{p}(p))} \mathcal{G}(g,f)\,\mathrm{d}\mathcal{H}^m. \tag{16.3.6}$$

Now, we apply the Taylor expansion, [DS15, Corollary 3.3], for $\mathbf{G}_g$ together with (16.3.2) to get that

$$\|\mathbf{G}_g\|(\mathbf{C}(p,z)) \le Q\mathcal{H}^m(\mathrm{B}_z(\mathbf{p}(p))) + \int_{\mathrm{B}_z(\mathbf{p}(p))\cap K} \frac{|Df|^2}{2} + C_1\mathbf{E}^\gamma\left(\mathbf{E} + \omega(r)^2\right) r^m, \tag{16.3.7}$$

and for $\mathbf{G}_f$ to obtain that

$$\|\mathbf{G}_f\|(\mathbf{C}(p,z)\cap K) \ge Q\mathcal{H}^m(\mathrm{B}_z(\mathbf{p}(p))\cap K) + \int_{\mathrm{B}_z(\mathbf{p}(p))\cap K} \frac{|Df|^2}{2} - C_4\mathbf{E}^\gamma\left(\mathbf{E} + \omega(r)^2 r^m\right). \tag{16.3.8}$$

So, in order to estimate the excess measure of the bad set, we put the past inequalities into account as follows:

$$\begin{aligned}
\mathbf{e}_T(\mathrm{B}_z(\mathbf{p}(p))\setminus K) &= \|T\|(\mathbf{C}(p,z)\setminus K) - Q\mathcal{H}^m(\mathrm{B}_z(\mathbf{p}(p))\setminus K)\\
&= \|T\|(\mathbf{C}(p,z)) - \|T\|(\mathbf{C}(p,z)\cap K) - Q\mathcal{H}^m(\mathrm{B}_z(\mathbf{p}(p))\setminus K)\\
&\overset{(16.3.6),(16.3.7)}{\le} Q\mathcal{H}^m(\mathrm{B}_z(\mathbf{p}(p))) - Q\mathcal{H}^m(\mathrm{B}_z(\mathbf{p}(p))\setminus K)\\
&\quad + \int_{\mathrm{B}_z(\mathbf{p}(p))\cap K} \frac{|Df|^2}{2} + C_1\mathbf{E}^\gamma\left(\mathbf{E} + \omega(r)^2\right) r^m\\
&\quad + C_3\mathbf{A}\int_{\mathrm{B}_z(\mathbf{p}(p))} \mathcal{G}(g,f)\,\mathrm{d}\mathcal{H}^m - \|T\|(\mathbf{C}(p,z)\cap K)\\
&\overset{(16.3.8)}{\le} C_5\mathbf{E}^\gamma\left(\mathbf{E} + \omega(r)^2\right) r^m + C_3\mathbf{A}\int_{\mathrm{B}_z(\mathbf{p}(p))} \mathcal{G}(g,f)\,\mathrm{d}\mathcal{H}^m\\
&\quad + \|\mathbf{G}_f\|(\mathbf{C}(p,z)\cap K) - \|T\|(\mathbf{C}(p,z)\cap K)\\
&= C_5\mathbf{E}^\gamma\left(\mathbf{E} + \omega(r)^2\right) r^m + C_3\mathbf{A}\int_{\mathrm{B}_z(\mathbf{p}(p))} \mathcal{G}(g,f)\,\mathrm{d}\mathcal{H}^m,
\end{aligned}$$

where in the last inequality we use that in the good set K the current $\mathbf{G}_f$ induced by the weak approximation f of the current T coincides with the current T, i.e. (16.1.2). Now we notice that

$$\int_{\mathrm{B}_z(\mathbf{p}(p))} \mathcal{G}(g,f)\,\mathrm{d}\mathcal{H}^m \le C_6\mathbf{E}^\gamma\left(\mathbf{E} + \omega(r)^2\right) r^m,$$

the proof of this fact is given in the proof of [DS14, Theorem 4.1], the argument given by the authors works line by line in our setting. The latter inequality together with the long chain of inequalities above furnish

$$\mathbf{e}_T(\mathrm{B}_z(\mathbf{p}(p))\setminus K) \le C_7\mathbf{E}^\gamma\left(\mathbf{E} + \omega(r)^2\right) r^m. \tag{16.3.9}$$

Let us now handle the term $\mathbf{e}_T(A)$. To that end, we recall that $|Df|^2$ is a L^1 function, so, almost every point of K is a Lebesgue point of $|Df|^2$ which together with the Taylor expansion ensure that

$$
\begin{aligned}
|Df|^2(x) &= \lim_{t\to 0}\frac{1}{\omega_m t^m}\int_{\mathrm{B}_t(x)\cap K}|Df|^2\,\mathrm{d}\mathcal{H}^m \le C_8\lim_{t\to 0}\frac{\mathbf{e}_{\mathbf{G}_f}(\mathrm{B}_t(x)\cap K)}{\omega_m t^m}\\
&\overset{(16.1.2)}{=} C_8\lim_{t\to 0}\frac{\mathbf{e}_T(\mathrm{B}_t(x)\cap K)}{\omega_m t^m}\le C_8\limsup_{t\to 0}\frac{\mathbf{e}_T(\mathrm{B}_t(x))}{\omega_m t^m} = C_8\mathbf{d}_T(x).
\end{aligned}
\tag{16.3.10}
$$

We now are in position to apply Proposition 16.2.1 and we also recall that$\mathbf{d}_T \le C_9\mathbf{E}^{1+\gamma} < 1$ in K. Therefore, given a Borel set $A \subset \mathrm{B}_z(\mathbf{p}(p))$, we proceed as follows:

$$
\begin{aligned}
\mathbf{e}_T(A\cap K) &\overset{\text{Taylor}}{\le} \int_{A\cap K}\frac{|Df|^2}{2}\,\mathrm{d}\mathcal{H}^m + C_{10}\mathbf{E}^{1+\gamma}\\
&\overset{\text{Holder ineq.}}{\le} \mathcal{H}^m(A\cap K)^{\frac{a-1}{a}}\left(\int_{A\cap K}\frac{|Df|^{2a}}{2}\,\mathrm{d}\mathcal{H}^m\right)^{\frac{1}{a}} + C_{10}\mathbf{E}^{1+\gamma}\\
&\overset{(16.3.10)}{\le} \frac{C_8}{2^{\frac{1}{a}}}\mathcal{H}^m(A)^{\frac{a-1}{a}}\left(\int_{A\cap K}\mathbf{d}_T^a\,\mathrm{d}\mathcal{H}^m\right)^{\frac{1}{a}} + C_{10}\mathbf{E}^{1+\gamma}\\
&\le C_8\mathcal{H}^m(A)^{\frac{a-1}{a}}\left(\int_{\{\mathbf{d}_T\le 1\}\cap \mathrm{B}_z(\mathbf{p}(p))}\mathbf{d}_T^a\,\mathrm{d}\mathcal{H}^m\right)^{\frac{1}{a}} + C_{10}\mathbf{E}^{1+\gamma}\\
&\overset{(16.2.1)}{\le} C_8C_a\mathcal{H}^m(A)^{\frac{a-1}{a}}\left[\mathbf{E}^a r^m + \omega(r)^2 r^m\mathbf{E}^{a-1}\right]^{\frac{1}{a}} + C_{10}\mathbf{E}^{1+\gamma}\\
&\le C_8C_a\mathcal{H}^m(A)^{\frac{a-1}{a}}\left[\mathbf{E}+\omega(r)^2\right]r^m + C_{10}\mathbf{E}^{1+\gamma},
\end{aligned}
\tag{16.3.11}
$$

where in the last inequality we used (c) to obtain that $\mathbf{E}\left[1+\omega(r)^2\mathbf{E}^{-1}\right]^{1/a}$ is close to $\mathbf{E}$ and consequently smaller than $\mathbf{E}+\omega(r)^2$. Finally, recalling that $z > r$ and $A \subset \mathrm{B}_z(\mathbf{p}(p))$, possibly choosing γ smaller depending on a, we put (16.3.11), the triangle inequality and (16.3.9) together to easily conclude the proof of the theorem. Lastly, we remark that $z \in B$ and $\mathcal{H}^1(B) > \frac{r}{2}$, then we surely can take $z \in \left[\frac{5}{4}r, 2r\right]$. □

16.4 Superlinear Lipschitz approximation

We now state our main strong approximation theorem which is the analogous of [DLSS18, Theorem 1.4] to our general almost minimality condition, and we also provide some improvements.

Theorem 16.4.1 (Strong Lusin type Lipschitz approximation)**.** *There exist positive geometric constants $\varepsilon_{la} = \varepsilon_{la}(m,n,Q) > 0$, $C = C(m,n,Q) > 0$, and $\gamma_{la} = \gamma_{la}(m,n,Q) \in (0, 1/(2m))$ such that, if*

(a) *T is ω-almost area minimizing under Assumption 4,*

(b) $\mathbf{E}(T, \mathbf{C}(p, 4r)) < \varepsilon_{la}$*,* *hold, then there exist a Lipschitz map* $f : \mathrm{B}_r(\mathbf{p}(p)) \to \mathcal{A}_Q(\mathbb{R}^n)$ *and* $K \subset \mathrm{B}_r(\mathbf{p}(p))$ *such that*

$$\mathrm{Lip}(f) \leq C\mathbf{E}(T, \mathbf{C}(p, 4r))^{\gamma}, \tag{16.4.1}$$

$$\mathbf{G}_f \llcorner (K \times \mathbb{R}^n) = T \llcorner (K \times \mathbb{R}^n), \tag{16.4.2}$$

$$\mathcal{H}^m(\mathrm{B}_r(\mathbf{p}(p)) \setminus K) \leq C\mathbf{E}(T, \mathbf{C}(p, 4r))^{\gamma} \left[\mathbf{E}(T, \mathbf{C}(p, 4r)) + \omega^2(r)\right] r^m, \tag{16.4.3}$$

$$\mathbf{e}_T(A \setminus K) \leq C\mathbf{E}(T, \mathbf{C}(p, 4r))^{\gamma} \left[\mathbf{E}(T, \mathbf{C}(p, 4r)) + \omega^2(r)\right] r^m, \tag{16.4.4}$$

$$\int_{A \setminus K} \frac{|Df|^2}{2} \, \mathrm{d}\mathcal{H}^m \leq C\mathbf{E}(T, \mathbf{C}(p, 4r))^{3\gamma} \left[\mathbf{E}(T, \mathbf{C}(p, 4r)) + \omega^2(r)\right] r^m, \tag{16.4.5}$$

$$\left|\mathbf{e}_T(A) - \int_A \frac{|Df|^2}{2} \, \mathrm{d}\mathcal{H}^m\right| \leq C\mathbf{E}(T, \mathbf{C}(p, 4r))^{\gamma} \left[\mathbf{E}(T, \mathbf{C}(p, 4r)) + \omega^2(r)\right] r^m, \tag{16.4.6}$$

where the last inequality holds for every Borel set $A \subset \mathrm{B}_r(\mathbf{p}(p))$.

We will usually refer to the set K as the **good set** and $\mathrm{B}_r(\mathbf{p}(p)) \setminus K$ as the **bad set**.

Proof of Theorem 16.4.1. Fix $\varepsilon_{la} =: \varepsilon$ and $\gamma_{la} =: \gamma$. We begin noticing that ε is supposed to be smaller than ε_{se}, in particular, it is smaller than ε_{ha} and thus condition ((b)) is satisfied with ε_{ha}. We start taking $\gamma < 1/2m$ and thus we take K and f to be the weak approximation given by Proposition 16.1.1. So, it is clear that (16.4.1) and (16.4.2) follows respectively from (16.1.1) and (16.1.2). In order to prove (16.4.3), we define

$$A := \{\mathbf{me}_T > 2^{-m}\mathbf{E}(T, \mathbf{C}(p, 4r))^{2\gamma}\} \cap \mathrm{B}_{9r/8}(\mathbf{p}(p)),$$

notice that [DS14, Proposition 3.2] gives that $\mathcal{H}^m(A) \leq C_1\mathbf{E}(T, \mathbf{C}(p, 4r))^{1-2\gamma}$. Therefore, possibly taking $\gamma < \gamma_{se}$, we use the strong excess decay to obtain

$$\begin{aligned}\mathcal{H}^m(\mathrm{B}_r(\mathbf{p}(p)) \setminus K) &\leq C_2\mathbf{E}(T, \mathbf{C}(p, 4r))^{-2\gamma}\mathbf{e}_T(A) \\ &\overset{(16.3.1)}{\leq} C_2\mathbf{E}(T, \mathbf{C}(p, 4r))^{2\gamma_{se} - 2\gamma(1+\gamma_{se})} \left(\mathbf{E}(T, \mathbf{C}(p, 4r)) + \omega(r)^2\right) r^m,\end{aligned}$$

which, possibly choosing $\gamma < \frac{\gamma_{se}}{2(1+\gamma_{se})}$, furnishes (16.4.3). We fix $\mathbf{E}(T, \mathbf{C}(p, 4r)) = \mathbf{E}$. Now, we firstly prove (16.4.6) for $A = \mathrm{B}_r(\mathbf{p}(p))$ as follows:

$$\begin{aligned}\left|\mathbf{e}_T(\mathrm{B}_r(\mathbf{p}(p))) - \int_{\mathrm{B}_r(\mathbf{p}(p))} \frac{|Df|^2}{2} \, \mathrm{d}\mathcal{H}^m\right| = \Big|&\mathbf{e}_T(\mathrm{B}_r(\mathbf{p}(p))) + \mathbf{e}_{\mathbf{G}_f}(\mathrm{B}_r(\mathbf{p}(p))) \\ &- \mathbf{e}_{\mathbf{G}_f}(\mathrm{B}_r(\mathbf{p}(p))) - \int_{\mathrm{B}_r(\mathbf{p}(p))} \frac{|Df|^2}{2} \, \mathrm{d}\mathcal{H}^m\Big|\end{aligned}$$

Using that $\mathbf{G}_f$ is equal to T on the good set, we obtain that

$$\begin{aligned}\left|\mathbf{e}_T(\mathrm{B}_r(\mathbf{p}(p))) - \int_{\mathrm{B}_r(\mathbf{p}(p))} \frac{|Df|^2}{2}\,\mathrm{d}\mathcal{H}^m\right| &\le \mathbf{e}_T(\mathrm{B}_r(\mathbf{p}(p))\setminus K) + \mathbf{e}_{\mathbf{G}_f}(\mathrm{B}_r(\mathbf{p}(p))\setminus K)\\ &\quad + \left|\mathbf{e}_{\mathbf{G}_f}(\mathrm{B}_r(\mathbf{p}(p))) - \int_{\mathrm{B}_r(\mathbf{p}(p))} \frac{|Df|^2}{2}\,\mathrm{d}\mathcal{H}^m\right|\\ &\overset{(16.3.1),(16.4.3)}{\le} C_2\mathbf{E}^\gamma\left[\mathbf{E}+\omega(r)^2\right]r^m\\ &\quad + \left|\mathbf{e}_{\mathbf{G}_f}(\mathrm{B}_r(\mathbf{p}(p))) - \int_{\mathrm{B}_r(\mathbf{p}(p))} \frac{|Df|^2}{2}\,\mathrm{d}\mathcal{H}^m\right|\\ &\overset{\text{Taylor}}{\le} C_3\mathbf{E}^\gamma\left[\mathbf{E}+\omega(r)^2\right]r^m.\end{aligned}$$

For every Borel set $A\subset \mathrm{B}_r(\mathbf{p}(p))$, notice that (16.4.1) and (16.4.3) give

$$\int_{F\setminus K}|Df|^2 \overset{(16.4.1)}{\le} C_4\mathbf{E}^{2\gamma}\mathcal{H}^m(\mathrm{B}_r(\mathbf{p}(p))\setminus K) \overset{(16.4.3)}{\le} C_5\mathbf{E}^{3\gamma}\left[\mathbf{E}+\omega(r)^2\right]r^m,$$

hence we achieve (16.4.5). The Taylor expansion of the area functional, [DS15, Corollary 3.3] gives

$$\left|\mathbf{e}_{\mathbf{G}_f}(A) - \frac{1}{2}\int_A |Df|^2\right| \le C_5\mathbf{E}^\gamma\left[\mathbf{E}+\omega(r)^2\right]r^m.$$

Therefore, we obtain for every Borel set $A\subset \mathrm{B}_r(\mathbf{p}(p))$ that

$$\begin{aligned}\mathbf{e}_T(A\setminus K) &= \mathbf{e}_T(A) - \mathbf{e}_{\mathbf{G}_f}(A\cap K)\\ &\overset{\text{triangle ineq.}}{\le} \left|\mathbf{e}_T(A) - \frac{1}{2}\int_A|Df|^2\right| + \left|\frac{1}{2}\int_{A\cap K}|Df|^2 - \mathbf{e}_{\mathbf{G}_f}(A\cap K)\right| + \int_{A\setminus K}|Df|^2\\ &\overset{(16.4.6),(16.4.5)}{\le} C_6\mathbf{E}^\gamma\left[\mathbf{E}+\omega(r)^2\right]r^m + \left|\frac{1}{2}\int_{A\cap K}|Df|^2 - \mathbf{e}_{\mathbf{G}_f}(A\cap K)\right|\\ &\overset{\text{Taylor}}{\le} C_7\mathbf{E}^\gamma\left[\mathbf{E}+\omega(r)^2\right]r^m,\end{aligned}$$

which is (16.4.4). With the aim at proving (16.4.6), which we have proved above for the special case $A=\mathrm{B}_r(\mathbf{p}(p))$, we proceed as follows:

$$\begin{aligned}\left|\mathbf{e}_T(A) - \frac{1}{2}\int_A|Df|^2\right| &\le \left|\mathbf{e}_T(A\cap K) - \frac{1}{2}\int_{A\cap K}|Df|^2\right| + \mathbf{e}_T(A\setminus K) + \frac{1}{2}\int_{A\setminus K}|Df|^2\\ &\overset{(16.4.6),(16.4.4),(16.4.5)}{\le} C_8\mathbf{E}^\gamma\left[\mathbf{E}+\omega(r)^2\right]r^m.\end{aligned}$$

□

Part D

Density of the boundary regular set of $2d$ area minimizing currents with arbitrary codimension and multiplicity

Chapter 17

Statement of the main theorem and outline of its proof

Definition 17.0.1 (Regular and singular one-sided boundary points, Definition 0.1 of [DLNS21])**.** Let T be a 2-dimensional integer rectifiable current (i.e. $T \in \mathscr{R}_2^{loc}(\mathbb{R}^{2+n})$) with $\partial T = Q^\star [\![\Gamma]\!], Q^\star \in \mathbb{N} \setminus \{0\}, p \in \Gamma$ and $\Theta^2(T, p) = \frac{Q^\star}{2}$. Then p is called a **regular one-sided boundary point** if T consists, in a neighborhood U of p, of the union of finitely many surfaces with boundary Γ, counted with multiplicities, which meet at Γ transversally. More precisely, if there are:

(i): a finite number $\Sigma_1, \ldots, \Sigma_J$ of oriented embedded surfaces in U,

(ii): and a finite number of positive integers $k_1, \ldots, k_J$ such that:

- (a) $\partial \Sigma_j \cap U = \Gamma \cap U = \Gamma_i \cap U$ (in the sense of differential topology) for every j,
- (b) $\Sigma_j \cap \Sigma_l = \Gamma \cap U$ for every $j \neq l$,
- (c) for all $j \neq l$ and at each $q \in \Gamma$ the tangent cones to Λ_j and Λ_l are distinct,
- (d) $T \llcorner U = \sum_j k_j [\![\Sigma_j]\!]$ and $\sum_j k_j = Q^\star$.

The set $\mathrm{Reg}_b^1(T)$ of **regular boundary one-sided points** is a relatively open subset of Γ.

Definition 17.0.2 (Regular and singular two-sided boundary points, Definition 1.1 of [DDHM18])**.** Let T be a m-dimensional integer rectifiable current (i.e. $T \in \mathscr{R}_2^{loc}(\mathbb{R}^{2+n})$) with $\partial T = Q^\star [\![\Gamma]\!], Q^\star \in \mathbb{N} \setminus \{0\}, p \in \Gamma$ and $\Theta^2(T, p) > \frac{Q^\star}{2}$.

(i): We say that p is a **regular boundary two-sided point for** T if there exist a neighborhood $U \ni p$ and a surface $\Sigma \subset U \cap \Sigma$ such that $\mathrm{spt}(T) \cap U \subset \Sigma$. The set of such points will be denoted by $\mathrm{Reg}_{\mathrm{b}}^2(T)$,

(ii): We also denote $\mathrm{Reg}_b(T) := \mathrm{Reg}_{\mathrm{b}}^1(T) \mathring{\cup} \mathrm{Reg}_{\mathrm{b}}^2(T)$, $\mathrm{Sing}_b(T) := \Gamma \setminus (\mathrm{Reg}_{\mathrm{b}}^1(T) \mathring{\cup} \mathrm{Reg}_{\mathrm{b}}^2(T))$

(iii): We will say that $p \in \mathrm{Sing}_{\mathrm{b}}(T)$ is of **crossing type** if there is a neighborhood U of p and two currents T_1 and T_2 in U with the properties that:

- (a) $T_1 + T_2 = T$ and $\partial T_1 = 0$,

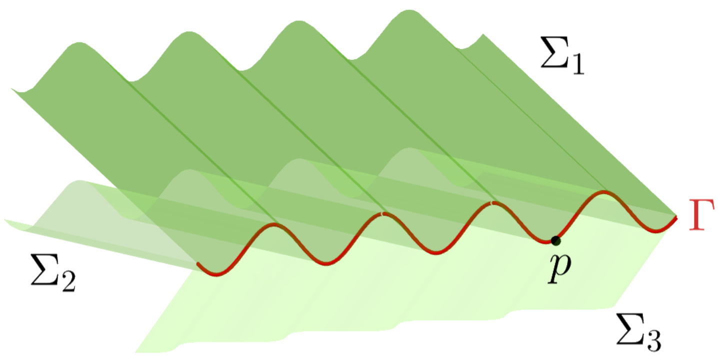


Figure 17.1: *Here $J = 3$ and the current is given by $T = \sum_{j=1}^{3} k_j [\![\Sigma_j]\!]$, then p is a regular one-sided boundary point of T. Note that, each surface Σ_j is taken with an integer multiplicity k_j and the boundary ∂T has multiplicity $Q^\star = k_1 + k_2 + k_3$.*

(b) $p \in \mathrm{Reg}_b(T_2)$.

(iv): If $p \in \mathrm{Sing}_b(T)$ is not of crossing type, we will then say that p is a **genuine boundary singularity point of** T.

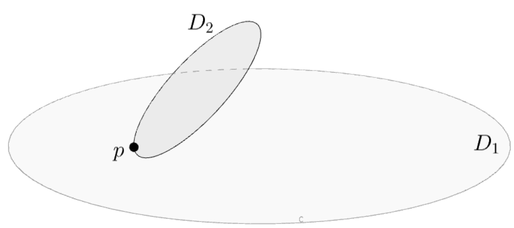


Figure 17.2: *Let $T = [\![D_1]\!] + [\![D_2]\!]$ and $p \in \partial D_2 \cap \mathrm{int}(D_1)$. It is easy to see that p is a crossing type singularity to the 2d current T.*

Our main theorem will be proved under the following assumptions.

Assumption 5. Let $\alpha \in (0, 1]$ and an integer $Q^\star \geq 1$. Consider $\Gamma \subset \Sigma$ a $C^{3,\alpha}$ oriented curve without boundary. Let T be an integral 2-dimensional area minimizing current in $\mathbf{B}(0, 2) \subset \mathbb{R}^{2+n}$ with boundary $\partial T \llcorner \mathbf{B}(0, 2) = Q^\star [\![\Gamma \cap \mathbf{B}(0, 2)]\!]$.

We can now state the main theorem which gives the density of the regular set $\mathrm{Reg}_b(T)$ in Γ, where the regular set allows the existence of both one-sided and two-sided points.

Theorem 17.0.3. *Let T and Γ be as in Assumptions 5. Then $\mathrm{Reg}_b(T)$ is an open dense set in Γ.*

Remark 17.0.4. We mention that we cite some results directly from [DDHM18] where the multiplicity $Q^\star = 1$, even though, we are working with higher multiplicity, the cited results used follows line by line, otherwise we write the proof here in details.

17.1 Outline of the proof of the main theorem

Our main issue is the existence of two kind of points which behave completely different, these points are defined as one-sided points and two-sided points, it means that the current is contained in one half space of the ambient space in the one-sided case and, in the case of two-sided points, the current is not contained in any half-space of the ambient space. In [DDHM18] the authors consider $Q^\star = 1$ and thus $Q - \frac{1}{2}$ is the density of the two-sided regular points in the boundary, they also allow the existence of one-sided and two-sided points, however, their notion of regularity coincides with the one given by Allard (in the case that $Q = 1$) to the context of one-sided points, in fact, Allard has considered what is called **convex barrier**. This equivalence of definitions strongly depends on the fact that the multiplicity at the boundary is 1. If $Q^\star > 1$, as we have defined, the notions of regularity for one-sided and two-sided points given in Definitions 17.0.1 and 17.0.2 does not coincide precisely because in the higher multiplicity setting at one-sided points we have currents with many sheets which are regular and this situation is not covered by Definition 17.0.2 when $Q = Q^\star$. So, similarly to the way that the authors in [DDHM18] rely on Allard's regularity result to reduce the analysis to two-sided points, we will use [DLNS21] to reduce our analysis to two-sided points. In [DLNS21], the authors have proved the analogous of Allard's theorem for higher multiplicity and currents of dimension 2, i.e., assuming that Γ belongs to a **convex barrier**, $2d$ area minimizing currents with $T = Q^\star [\![\Gamma]\!]$ are completely regular at the boundary. This reduction is allowed by Theorem 19.3.6 where we state, for 2-dimensional (C_0, r_0, α_0)-almost area minimizing currents, that, if every two-sided collapsed point (Definition 19.3.4) is regular, then $\mathrm{Reg}_b(T)$ is dense, we reiterate that $\mathrm{Reg}_b(T)$ contains both one-sided and two-sided points.

Now the main goal becomes to prove that two-sided collapsed points are regular, Theorem 19.3.8, and, to this end, we follow the framework given in [DDHM18]. Moreover, this prove is done for area minimizing currents with arbitrary dimension m, codimension n and arbitrary multiplicity $Q^\star$. Firstly, we construct a linear theory for (f^+, f^-) where we define this pair as $(Q - \frac{Q^\star}{2})$-valued functions (Definition 20.1.1), and we study the regularity that we can extract once we assume that it minimizes the Dirichlet energy. If we take $\Omega \subset \mathbb{R}^m$ an open set and γ a $(m-1)$-submanifold, called **interface**, which splits Ω into Ω^+ and Ω^-, we define $(Q - \frac{Q^\star}{2})$-valued functions as a Q-valued function f^+ defined in Ω^+ and a $(Q - Q^\star)$-valued function f^- defined in Ω^- in the sense of Almgren, when these functions glue at γ we say that (f^+, f^-) collapses at the interface as in Definition 20.1.1. So, if we assume that (f^+, f^-) is a $(Q - \frac{Q^\star}{2})$-Dir minimizer which collapses at the interface, then it is given by Q copies of κ in Ω^+ and $(Q - Q^\star)$ copies of κ in Ω^- where κ is a classical harmonic function, Theorem 20.1.4.

The next step is to approximate the current T by Lipschitz maps, indeed, we can do it without assuming any minimizing property of the current and the approximations, Theorem 20.2.3. Furthermore, when we add the condition that T is area minimizing, the approximation becomes minimizers of the Dirichlet energy and thus, by Theorem 20.1.4, we obtain an harmonic approximation of the current T, Theorem 20.2.4. This harmonic approximation is a key point to prove a milder excess decay for the area minimizing current T, Lemma 21.2.1, after that we do a simple argument to get a superlinear decay on the excess of T, Theorem 21.2.2. The superlinear decay is an important tool to prove the uniqueness of tangent cones at two-sided collapsed points of T (Theorem 21.3.1) and also to improve our Lipschitz approximation in the sense of getting better estimates, i.e. superlinear, on the errors of the approximation, Theorem 21.4.1. These constructions allow us to build the center manifolds.

We first construct the Whitney decomposition with the suitable stopping condition to then prove the existence of the so-called $C^{3,\kappa}$ center manifolds $\mathcal{M}^+$ and $\mathcal{M}^-$, see Theorem 22.5.1. After that we build in Theorem 22.6.5 the $\mathcal{M}$-normal approximation which is multivalued Lipschitz maps $\mathcal{N}^+$ and $\mathcal{N}^-$ defined on the center manifold $\mathcal{M}^+$ and $\mathcal{M}^-$, respectively, and taking values in the normal bundle of $\mathcal{M}$ which approximate the m-current T in the desired fashion. After that, we provide a blowup argument in Subsection 23.2 which ensures that $\mathcal{N}^\pm \equiv 0$ and hence the m-dimensional area minimizing current T has to coincide with $\mathcal{M}^+$ in the right portion and with $\mathcal{M}^-$ in the left portion. This gives that any two-sided collapsed point is a two-sided regular point, i.e., Theorem 19.3.8, which finishes the proof of our main theorem as aforementioned, i.e., Theorem 17.0.3.

Chapter 18

Fundamental concepts and results

For basic definitions and standard notations, we refer the reader to the textbooks [Fed69] and [Sim14]. Let us set up some notation that will be used in this work.

Definition 18.0.1. We define for $x \in \mathbb{R}^{m+n}$ and $r > 0$ the function $\iota_{x,r}(y) := \frac{y-x}{r}$. For any current $T \in \mathcal{D}'_m(\mathbb{R}^{m+n})$ let us define the **rescaled current T at x at scale r** as $\iota_{x,r\sharp}T =: T_{x,r}$ and $T_r =: \iota_{0,r\sharp}T$. We call a current T_x a **blowup of T at x**, if there exists a sequence of radii $r_j \to 0$ such that $T_{x,r_j} \to T_x$ in the weak topology.

We also fix the notation of the flat distance between two m-dimensional integer rectifiable currents T and S, i.e., $T, S \in \mathscr{R}_m^{loc}(U)$, U open and $A \Subset U$ as follows:

$$\mathbf{d}_A(T,S) = \inf\left\{\|R\|(A) + \|\tilde{T}\|(A) : T - S = R + \partial\tilde{T} \text{ with } R \in \mathbf{I}_m(U) \text{ and } \tilde{T} \in \mathbf{I}_{m+1}(U)\right\}.$$

Definition 18.0.2. Given three real numbers $C_0 \geq 0, r_0, \alpha_0 > 0$, we say that an m-dimensional integer rectifiable current T (i.e. $T \in \mathscr{R}_m^{loc}(\mathbb{R}^{m+n})$) with $\partial T = Q^\star [\![\Gamma]\!], Q^\star \in \mathbb{N} \setminus \{0\}$, is (C_0, r_0, α_0)-**almost area minimizing at** $x \in \operatorname{spt}(T)$, if we have

$$\|T\|(\mathbf{B}(x,r)) \leq (1 + C_0 r^{\alpha_0})\, \|T + \partial\tilde{T}\|(\mathbf{B}(x,r)), \tag{18.0.1}$$

for all $0 < r < r_0$ and all integral $(m+1)$-dimensional currents $\tilde{T}$ supported in $\mathbf{B}(x,r)$, i.e., for all $\tilde{T} \in \mathbf{I}_{m+1}(\mathbf{B}(x,r))$. The current is called (C_0, r_0, α_0)-**almost area minimizing in** $\mathbb{R}^{m+n}$, if the current T is (C_0, r_0, α_0)-almost area minimizing at each $x \in \operatorname{spt}(T)$ with the same constants $C_0, r_0, \alpha_0 > 0$ independently of the point x. If T is $(0, r_0, \alpha_0)$-almost area minimizing, we say that T is **area minimizing**.

Assumption 6. Let $\alpha \in]0,1]$ and integers $m \geq 2$, $Q^\star \geq 1$. Consider $\Gamma \subset \Sigma$ a $C^{3,\alpha}$ oriented $(m-1)$-submanifold without boundary. Let T be an integral m-dimensional (C_0, r_0, α_0)-almost area minimizing current in $\mathbf{B}(0,2)$ with boundary $\partial T \llcorner \mathbf{B}(0,2) = Q^\star [\![\Gamma \cap \mathbf{B}(0,2)]\!]$.

Let us enunciate the almost monotonicity formula which will often used in what follows. The proof of this formula can be found in [HM19, Lemma 2.1]. Although, it is done for boundary multiplicity one currents, the proof can be readily adapted to the higher multiplicity case as it is done in [DLNS23].

Proposition 18.0.3 (Almost monotonicity formula for boundary points, Proposition 3.3 of [DLNS23])**.** *Let T and Γ be as in Assumption 6, and $x \in \operatorname{spt} T \cap \Gamma$. Set $\alpha_1 := \min\{\alpha_0, \alpha\}$, $0 < r_1 < \min\{r_0, r'\}$.*

Then there is a constant $C_1 = C_1(m, n, C_0, r_0, r', \alpha_0, \alpha, \theta, \|\Gamma\|_{1,\alpha}) > 0$*, such that*

$$e^{C_1 r^{\alpha_1}} \frac{\|T\|(\mathbf{B}(x,r))}{r^m} - e^{C_1 s^{\alpha_1}} \frac{\|T\|(\mathbf{B}(x,s))}{s^m} \geq \int_{\mathbf{B}(x,r)\setminus\mathbf{B}(x,s)} e^{C_1|z-x|^{\alpha_1}} \frac{\left|(z-x)^{\perp}\right|^2}{2|z-x|^{m+2}} d\|T\|(z), \quad (18.0.2)$$

for every $0 < s < r < r_1$*.*

The upper semicontinuity of the density function is well known when restricted to either the interior or the boundary of an area minimizing current, we give a short proof of the validity of this fact at the boundary of almost area minimizing currents. We would like to remark that the upper semicontinuity holds for the restriction of the density function to boundary or interior points, however, it does not hold when it is considered defined on the whole $\operatorname{spt}(T)$, therefore, to get around that we state (iii) as B. White does in the context of area minimizing currents, c.f. [Whi97].

Proposition 18.0.4 (Upper semicontinuity of the density function)**.** *Let* Γ *and* T *be as in Assumption 6. Then*

(*i*) *The function* $x \mapsto \Theta^m(T, x)$ *is upper semicontinuous in* $\operatorname{spt}(T) \cap \Gamma$*,*

(*ii*) *The function* $x \mapsto \Theta^m(T, x)$ *is upper semicontinuous in* $\operatorname{spt}(T) \setminus \Gamma$*,*

(*iii*) *The function* $x \mapsto \begin{cases} \Theta^m(T,x), & x \notin \Gamma \\ 2\Theta^m(T,x), & x \in \Gamma \end{cases}$ *is upper semicontinuous.*

Proof. We define $f(x,r) = e^{C_1 r^{\alpha_1}} r^{-m} \|T\|(\mathbf{B}(x,r))$ and take $x_i \to x, x_i \in \operatorname{spt}(T) \cap \Gamma$, and assume $\mathbf{B}(x_i, r) \subset \mathbf{B}(x, r+\varepsilon), \forall i \in \mathbb{N}, \varepsilon > 0$. Applying the almost monotonocity formula (Proposition 18.0.3), we already know that $f(x_i, \cdot)$ is monotone nondecreasing and so we obtain

$$f(x_i, t) \leq f(x_i, r) \leq e^{C_1 r^{\alpha_1}} r^{-m} \|T\|(\mathbf{B}(x, r+\varepsilon)) = f(x, r+\varepsilon) \overbrace{\left[e^{-C_1 \varepsilon^{\alpha_1}} \left(1 + \frac{\varepsilon}{r}\right)^m\right]}^{I},$$

for any $0 < t < r$, $i \geq i_\varepsilon$ and $\varepsilon > 0$. First let $t \to 0$ we get that for every fixed positive $\varepsilon > 0$ it holds

$$f(x_i, 0^+) = \Theta^m(T, x_i) \leq f(x, r+\varepsilon) \overbrace{\left[e^{-C_1 \varepsilon^{\alpha_1}} \left(1 + \frac{\varepsilon}{r}\right)^m\right]}^{I}, \forall i \geq i_\varepsilon.$$

In the last inequality, taking in first the limit with respect to i, in second with respect to ε and finally with respect to r leads to

$$\limsup_{i \to +\infty} \Theta^m(T, x_i) \leq \lim_{r \to 0} \limsup_{i \to +\infty} f(x, r) = \Theta^m(T, x).$$

We mention that to prove (ii), the upper semicontinuity at the interior, one may use this very same argument but now using the almost monotonicity formula given in [DSS17a, Proposition 2.1]. Let us turn to the proof of (iii), it is enough to prove that

$$2\Theta(T, x) \geq \limsup{}_{i \to +\infty} \Theta(T, x_i)$$

where $\{x_i\}_{i \in \mathbb{N}} \subset \mathbb{R}^{m+n} \setminus \Gamma$ converges to $x \in \Gamma$. Take y_i the orthogonal projection of x_i into Γ and then define $T_i' = \frac{T - y_i}{|x_i - y_i|}$ and $x_i' = \frac{x_i - y_i}{|x_i - y_i|}$, up to subsequences , we may take $T' := \lim T_i'$ and

$x' := \lim x_i'$. By the almost monotonocity formula (Proposition 18.0.3), for every $r \in (0, r_1)$, the quantity $e^{C_1 r^{\alpha_1}} \frac{\|T\|(\mathbf{B}(y_i,r))}{r^m}$ is monotone nondecreasing. Thus, for any $\rho \in \mathbb{R}_+$,

$$\begin{aligned} e^{C_1 r^{\alpha_1}} \frac{\|T\|(\mathbf{B}(x,r))}{r^m} &\overset{(i)}{\geq} \limsup_{i\to+\infty} e^{C_1 r^{\alpha_1}} \frac{\|T\|(\mathbf{B}(y_i,r))}{r^m} \\ &\geq \limsup_{i\to+\infty} \frac{\|T\|(\mathbf{B}(y_i,\rho|x_i-y_i|))}{(\rho|x_i-y_i|)^m} \\ &= \limsup_{i\to+\infty} \frac{\|T_i'\|(\mathbf{B}(0,\rho))}{\rho^m} \\ &\geq \frac{\|T'\|(\mathbf{B}(0,\rho))}{\rho^m}. \end{aligned}$$

According to [All75, Reflection Principle, 3.2], we can reflect T' in $T_x\Gamma$ to obtain a stationary varifold V'' in $\mathbb{R}^{m+n}$. Thus, we let $r \to 0$ and $\rho \to +\infty$ in the latter equation to obtain

$$\begin{aligned} \Theta^m(T,x) &\geq \limsup_{\rho\to\infty} \frac{\|T'\|(\mathbf{B}(0,\rho))}{\rho^m} \\ &\overset{\text{def of } V''}{=} \frac{1}{2}\limsup_{\rho\to\infty} \frac{\|V''\|(\mathbf{B}(0,\rho))}{\rho^m} \\ &= \frac{1}{2}\limsup_{\rho\to\infty} \frac{\|V''\|(\mathbf{B}(x',\rho))}{\rho^m} \\ &\overset{((ii))}{\geq} \frac{1}{2}\Theta^m(V'',x') \\ &\overset{\text{def of } V''}{=} \frac{1}{2}\Theta^m(T',x') \\ &\overset{((ii))}{\geq} \frac{1}{2}\limsup_{i\to\infty} \Theta^m(T_i',x_i') \\ &= \frac{1}{2}\limsup_{i\to\infty} \Theta^m(T,x_i), \end{aligned}$$

where in the last equality we did a standard rescaling argument. □

Let us enunciate the existence of area minimizing tangent cones to boundary points of an almost area minimizing current, even though this result is very standard, we will prove it here for the sake of completeness.

Proposition 18.0.5 (Existence of area minimizing tangent cones)**.** *Let T and Γ be as in Assumptions 6. If T is (C_0, r_0, α_0)-almost area minimizing in a neighbourhood U of $x \in \Gamma$, then, for any sequence $r_k \to 0$, there exists a blowup $\lim_{k\to+\infty} T_{x,r_k} = T_0 \in \mathbf{I}_m^{loc}(\mathbb{R}^{m+n})$ such that:*

- *(i) $\|T_{x,r_k}\| \to \|T_0\|$ as $k \to +\infty$, in the sense of measures,*
- *(ii) T_0 is area minimizing,*
- *(iii) $\|T_0\|(\mathbf{B}(0,r)) = \Theta^m(T,x)\omega_m r^m, \forall r > 0$,*
- *(iv) T_0 is a tangent cone to T at x.*

Remark 18.0.6. We only need to assume that $\partial T = Q^\star [\![\Gamma]\!]$ and Γ is an $(m-1)$-submanifold of class $C^{1,\alpha}$, $\alpha \in (0,1)$, in order to deduce that $\partial T_0 = Q^\star [\![T_x\Gamma]\!]$.

Proof. Assume that $x = 0$. By the almost monotonicity formula at the boundary and at the interior (i.e., Proposition 18.0.3 and [DSS17a, Proposition 2.1]), we have that

$$\limsup_{k\to+\infty}\|T_{r_k}\|(\mathbf{B}(y,1)) < +\infty,$$

for every $y \in U$. Thus

$$\limsup_{k\to+\infty}\|T_{r_k}\|(K) < +\infty,$$

for all compact set $K \subset \mathbb{R}^{m+n}$. Since the boundary of T_{r_k} is $Q^\star [\![\Gamma/r_k]\!]$ and Γ is the graph of $u \in C^{1,\alpha}, u(0) = 0, Du(0) = 0$, we have

$$\begin{aligned}
\|\partial T_{r_k}\|(K) = \|\iota_{0,r_k\sharp}(Q^\star [\![\Gamma]\!])\|(K) &= \frac{1}{r_k^{m-1}}\|Q^\star [\![\Gamma]\!]\|(r_k K) \\
&= \frac{Q^\star}{r_k^{m-1}}\mathcal{H}^{m-1}((r_k K)\cap\Gamma) \\
&\leq \frac{Q^\star}{r_k^{m-1}}\int_{\mathrm{proj}(r_k K)}\sqrt{1+|Du(z)|^2}dz \\
&\leq \frac{Q^\star}{r_k^{m-1}}\mathcal{H}^{m-1}(\mathrm{proj}(r_k K))\sqrt{1+C_\Gamma(\mathrm{diam}(K)r_k)^{2\alpha}} \\
&\leq Q^\star\omega_{m-1}\,\mathrm{diam}(K)^{m-1}\sqrt{1+C_\Gamma\,\mathrm{diam}(K)^{2\alpha}} \\
&= C(\Gamma, K, \alpha, m, Q^\star),
\end{aligned}$$

and thus we can bound uniformly the mass of the boundary of T_{r_k} in K. Therefore, we can use standard compactness results (one could consult [Fed69, Section 4.2]) to ensure the existence of $T_0 \in \mathbf{I}_m^{loc}(\mathbb{R}^{m+n})$ such that $T_{r_k} \to T_0$, up to a subsequence, in the flat norm.

Proof of ((*i*)): Let us write $T_{r_k} - T_0 = R_{r_k} + \partial\tilde{T}_{r_k}$ in $\mathbf{B}(0, R+2)$ with

$$\limsup_{k\to+\infty}\left(\|R_{r_k}\|(\mathbf{B}(0,R+1)) + \|\tilde{T}_{r_k}\|(\mathbf{B}(0,R+1))\right) = 0.$$

Thus, since all the measures involved are Radon measures, for almost every $s \in (R, R+1)$, it follows that

$$\limsup_{k\to+\infty}\|R_{r_k}\|(\mathbf{B}(0,s)) = 0 \tag{18.0.3}$$

and

$$\limsup_{k\to+\infty}\mathbb{M}\left(\langle\tilde{T}_{r_k}, d, s\rangle\right) = 0. \tag{18.0.4}$$

Note that (18.0.4) follows directly from the formula of the slice and the fact that T_{r_k} converges to T_0 in the flat norm. We may use again the slice formula to get

$$T_{r_k}\llcorner\mathbf{B}(0,s) = T_0\llcorner\mathbf{B}(0,s) + R_{r_k}\llcorner\mathbf{B}(0,s) - \langle\tilde{T}_{r_k}, d, s\rangle + \partial(\tilde{T}_{r_k}\llcorner\mathbf{B}(0,s)). \tag{18.0.5}$$

The almost minimality condition gives

$$\|T_{r_k}\|(\mathbf{B}(0,s)) \leq (1 + C_0(rs)^{\alpha_0})\left\|T_{r_k} + \partial\tilde{T}_{r_k}\right\|(\mathbf{B}(0,s)).$$

Putting into account the latter inequality, the triangle inequality and (18.0.5), we obtain that

$$\|T_{r_k}\|(\mathbf{B}(0,s)) \leq (1 + C_0(rs)^{\alpha_0})\left(\|T_0\|(\mathbf{B}(0,s)) + \|R_{r_k}\|(\mathbf{B}(0,s)) + \mathbb{M}\left(\langle\tilde{T}_{r_k}, d, s\rangle\right) + 2\|\partial\tilde{T}_{r_k}\|(\mathbf{B}(0,s))\right).$$

Note that, by our construction, it follows that $\|\partial \tilde{T}_{r_k}\|(\mathbf{B}(0,s)) \to 0$ as $k \to +\infty$. Finally, by the lower semicontinuity of the mass, (18.0.3), (18.0.4) and the last equation passed through $\lim_{k\to+\infty}$, we conclude the proof of ((*i*)).

Proof of ((*ii*))**:** Fix $R \in (0,+\infty)$, by the lower semicontinuity of the mass, for all $\tilde{T} \in \mathbf{I}_{m+1}(\mathbf{B}(0,R))$ we have that

$$\begin{aligned}\|T_0\|(\mathbf{B}(0,R)) &\leq \liminf_{k\to+\infty}\|T_{r_k}\|(\mathbf{B}(0,R)) \\ &\leq \liminf_{k\to+\infty}\left(1+C_0(r_kR)^{\alpha_0}\right)\|T_{r_k}+\partial\tilde{T}\|\left(\mathbf{B}(0,R)\right) \\ &\overset{((i))}{=} \|T_0+\partial\tilde{T}\|\left(\mathbf{B}(0,R)\right),\end{aligned}$$

for all $R>0$.

Proof of ((*iii*))**:** From ((*i*)) and the almost monotonicity formula, we know that $\Theta^m(T,x)$ exists and, $\forall r>0$, we have that

$$\begin{aligned}\|T_0\|(\mathbf{B}(0,r)) &= \lim_{k\to+\infty}\|T_{r_k}\|(\mathbf{B}(0,r)) \\ &= \lim_{k\to+\infty}\frac{\|T\|(\mathbf{B}(0,r_kr))}{r_k^m} \\ &= \lim_{k\to+\infty}\frac{\omega_m r^m\|T\|(\mathbf{B}(0,r_kr))}{\omega_m(r_kr)^m} \\ &= \Theta^m(T,x)\omega_m r^m.\end{aligned}$$

Proof of ((*iv*))**:** By following closely the argument given in [Sim14, Theorem 3.1, Chapter 7] using ((*iii*)), we prove that T_0 is in fact a cone. Indeed, by [DLNS21, Theorem 3.2] applied to T_0 which is area minimizing, we know that

$$Q^\star\int_{T_p\Gamma\cap\mathbf{B}(p,\rho)}(x-p)\cdot\vec{n}(x)d\mathcal{H}^{m-1}(x)=0,$$

since $\vec{n}\perp T_p\Gamma$. Using ((*ii*)), we obtain that $\vec{H}_T$ is zero a.e., so, by ((*iii*)), we obtain the constancy of the mass ratio and, then using again [DLNS21, Theorem 3.2], we get that

$$\int_{\mathbf{B}(p,r)\setminus\mathbf{B}(p,s)}\frac{\left|(x-p)^\perp\right|^2}{|x-p|^{m+2}}d\|T\|(x)=0.$$

Then, if we fix a cone C such that $\partial C=\partial T_0$, notice that since $\partial T_0 = Q^\star[\![T_p\Gamma]\!]$ we may choose for instance C as a half subspace, we can apply [Sim14, Lemma 2.33, Chapter 6] for $T_0 - C$ and conclude that it is a cone and thus $T_0 = T_0 - C + C$ is a cone. □

Chapter 19

Stratification for (C_0, r_0, α_0)-almost area minimizing currents

19.1 Stratification

Definition 19.1.1 (Conelike functions)**.** An upper semicontinuous function $g : \mathbb{R}^{m+n} \to \mathbb{R}_+$ is called **conelike** provided:

(i) $g(\lambda x) = g(x)$ for all $\lambda > 0$ and for all $x \in \mathbb{R}^{m+n}$,

(ii) If $g(x) = g(0)$, then $g(x + \lambda v) = g(x + v)$ for all $\lambda > 0$ and $v \in \mathbb{R}^{m+n}$.

If g is conelike we also define the **spine of** g as the set

$$\mathrm{spine}(g) := \{x \in \mathbb{R}^{m+n} : g(x) = g(0)\}.$$

By (i) in the last definition and upper semicontinuity, we have that $g(z) \leq g(0)$ for all z. Note that, by [Whi97, Theorem 3.1], spine(g) is a vector subspace and

$$\mathrm{spine}(g) = \{x \in \mathbb{R}^{m+n} : g(x + v) = g(v),\ \forall v \in \mathbb{R}^{m+n}\}.$$

Fix T and Γ as in Assumption 6, and set the class of functions

$$\mathscr{G}(p) := \{g_{p,T_0} : T_0 \text{ is a tangent cone to } T \text{ at } p\},$$

where

$$f_T(x) := \begin{cases} \Theta^m(T, x), & x \notin \Gamma \\ 2\Theta^m(T, x) + 1, & x \in \Gamma \end{cases}, \quad \text{and} \quad g_{p,T_0}(x) := \begin{cases} \Theta^m(T_0, x), & x \notin T_p\Gamma \\ 2\Theta^m(T_0, x) + 1, & x \in T_p\Gamma \end{cases},$$

then f and each g_{p,T_0} are upper semicontinuous from Proprosition 18.0.4.

Definition 19.1.2 (Spine of a cone)**.** Let $p \in \Gamma$, T_0 be an oriented tangent cone with $\partial T_0 = Q^\star [\![T_p\Gamma]\!]$, we define the **spine of** T_0, and denote it by spine(T_0), to be the set of vectors $v \in T_p\Gamma$ such that $(\tau_v)_\sharp T_0 = T_0$ where $\tau_v(w) = w + v$. Clearly the spine(T_0) is always a subspace of $T_p\Gamma$.

We now provide an equivalence for the definition of spine of an oriented tangent cone which follows the ideas furnished by Almgren in his stratification process, [Alm00, Theorem 2.26], and used in [Whi97], [Sim83], [DDHM18].

Lemma 19.1.3 (Spine as constant density set)**.** *Let T and Γ be as in Assumption 6 and T_0 be an oriented tangent cone to T at $p \in \Gamma$. Then* $\mathrm{spine}(T_0) = \{x \in T_p\Gamma : \Theta^m(T_0, x) = \Theta^m(T_0, 0)\}$.

Proof. Take $x \in \mathrm{spine}(T_0)$, by the definition of spine, we have the third equality below

$$\Theta^m(T_0, x) = \lim_{r\to 0} \frac{\|T_0\|(\mathbf{B}(x,r))}{\omega_m r^m} = \lim_{r\to 0} \frac{\|(\tau_x)_\sharp T_0\|(\mathbf{B}(0,r))}{\omega_m r^m} = \lim_{r\to 0} \frac{\|T_0\|(\mathbf{B}(0,r))}{\omega_m r^m} = \Theta^m(T_0, 0).$$

On the other hand, consider x such that $\Theta^m(T_0, x) = \Theta^m(T_0, 0)$, we claim that $x \in \mathrm{spine}(T_0)$. To prove this claim, we apply the monotonicity formula [Bou16, Equation 31] to the cone T_0, which is area minimizing and $\partial T_0 = Q^\star [\![T_p\Gamma]\!]$, to obtain, for $0 < s < r$,

$$\frac{\|T_0\|(\mathbf{B}(x,r))}{\omega_m r^m} - \frac{\|T_0\|(\mathbf{B}(x,s))}{\omega_m s^m} = \int_{\mathbf{B}(x,r)\setminus\mathbf{B}(x,s)} \frac{|(z-x)^\perp|^2}{|z-x|^{m+2}} \,\mathrm{d}\,\|T_0\|(z). \tag{19.1.1}$$

It is well known that when T_0 is a cone, the right hand side of the last equation is equal to 0 (for instance, see [Sim14, Lemma 2.33, Chapter 6]), so letting $s \to 0$ in (19.1.1) we reach

$$\frac{\|T_0\|(\mathbf{B}(x,r))}{\omega_m r^m} = \Theta^m(T_0, x) = \Theta^m(T_0, 0).$$

Therefore, we have that $\|(\tau_x)_\sharp T_0\|(\mathbf{B}(0,r)) = \|T_0\|(\mathbf{B}(x,r)) = \|T_0\|(\mathbf{B}(0,r))$ thus, by measure theory, we get that $\|T_0\| = \|(\tau_x)_\sharp T_0\|$ as measures, which in turn ensures that $T_0 = \pm(\tau_x)_\sharp T_0$. But we know that $\partial T_0 = Q^\star [\![T_p\Gamma]\!]$ and $\partial(\tau_x)_\sharp T_0 = (\tau_x)_\sharp \partial T_0 = Q^\star [\![T_p\Gamma]\!]$, then we conclude that $T_0 = (\tau_x)_\sharp T_0$ which shows that $x \in \mathrm{spine}(T_0)$. □

We would like to show that we are in position to apply [Whi97, Theorem 3.2], to that end, we prove the following lemma.

Lemma 19.1.4. *For each $p \in \Gamma$ and each oriented tangent cone T_0 to T at p, g_{p,T_0} is conelike.*

Proof. Property (i) in Definition 19.1.1 is a direct consequence of the scaling invariance of T_0. To what concerns property (ii), if $g_{p,T_0}(x) = g_{p,T_0}(0)$, and $x \in T_p\Gamma$, we have that $x \in \mathrm{spine}(T_0)$ and thus

$$\begin{aligned}
\Theta^m(T_0, x+v) &= \Theta^m((\tau_x)_\sharp T_0, v) \\
&= \Theta^m(T_0, v) \\
&= \Theta^m(T_0, \lambda v) \\
&= \Theta^m((\tau_x)_\sharp T_0, \lambda v) \\
&= \Theta^m(T_0, x + \lambda v),
\end{aligned}$$

for any $\lambda > 0$ and $v \in \mathbb{R}^{m+n}$.

If $g_{p,T_0}(x) = g_{p,T_0}(0)$ and $x \notin T_p\Gamma$, then, by definition of g_{p,T_0}, we have $\Theta^m(T_0, x) = 2\Theta^m(T_0, 0) + 1$. Since T_0 is a cone we have that $\Theta^m(T_0, x) = \Theta^m(T_0, \lambda x)$, for every $\lambda > 0$. Then $\Theta^m(T_0, \lambda x) = 2\Theta^m(T_0, 0) + 1$, for every $\lambda > 0$. Taking the limsup and recalling Proposition 18.0.4 (*iii*) we get $\limsup_{\lambda \to 0^+} \Theta^m(T_0, \lambda x) = 2\Theta^m(T_0, 0) + 1 \leq 2\Theta^m(T_0, 0)$, which is a contradiction. □

Remark 19.1.5. Note that, a simple consequence of the Lemma 19.1.3 and the proof of Lemma 19.1.4 is that $\mathrm{spine}(g_{p,T_0}) = \mathrm{spine}(T_0)$.

Definition 19.1.6 (Stratum)**.** Let $p \in \Gamma$ and T be a m-current with $\partial T = Q^\star [\![\Gamma]\!]$, we define the **$j$-stratum of Γ with respect to T** as the set

$$\mathscr{P}_j(T, \Gamma) = \{p \in \Gamma : \dim(\mathrm{spine}(T_0)) \leq j, \text{ for all tangent cone } T_0 \text{ to } T \text{ at } p\}.$$

Now, we shall directly apply Proposition 18.0.5 to check the conditions (1) and (2) of [Whi97, Theorem 3.2] which in turn furnishes

Theorem 19.1.7 (Stratification Theorem, Theorem 3.2 of [Whi97])**.** *For T and Γ as in Assumption 6, let*

$$\Sigma_i := \{x : f_T(x) > 0 \text{ and } \sup\{\dim(\mathrm{spine}(g)) : g \in \mathscr{G}(x)\} \leq i\},$$

then the Hausdorff dimension of Σ_i is at most i and Σ_0 is at most countable. In particular, we have the same statements for the stratum $\mathscr{P}_i(T, \Gamma)$, i.e., the Hausdorff dimension of $\mathscr{P}_i(T, \Gamma)$ is at most i, $\mathscr{P}_0(T, \Gamma)$ is at most countable, and

$$\mathscr{P}_0(T, \Gamma) \subset \mathscr{P}_1(T, \Gamma) \subset \cdots \subset \mathscr{P}_{m-1}(T, \Gamma) = \Gamma.$$

19.2 Open books, flat cones, one-sided and two-sided points

The characterization of tangent cones is an important tool in the subsequent theory, in fact, when dealing with two dimensional area minimizing cones we have general structure results, see for instance [HM19, Lemma 3.1] or [DLNS23, Proposition 4.1]. If we consider arbitrary dimensions, there is no general structure theorem for area minimizing tangent cones, however, assuming that the density of the cone is constant along the boundary of the cone which is equivalent to assume that the spine has maximal dimension, as showed in Lemma 19.1.3, we can characterize tangent cones as in [Bro77, Theorem 5.1], or [DDHM18, Lemma 3.17]. We enunciate and prove in Lemma 19.2.5 rigorous statement of the assertions just mentioned. To go further in our treatment we need the following definitions.

Definition 19.2.1 (Open book)**.** Let $T_0 \in \mathbf{I}_m^{\mathrm{loc}}(\mathbb{R}^{m+n})$ be an oriented cone and V is an oriented $(m-1)$-dimensional linear subspace of $\mathbb{R}^{m+n}$. We say that T_0 is an **open book with boundary $[\![V]\!]$ and multiplicity $Q^\star$**, if $\partial T_0 = Q^\star [\![V]\!]$ and there exist $Q_1, \ldots, Q_N \in \mathbb{N} \setminus \{0\}$ and $\pi_1, \ldots, \pi_N$ distinct m-dimensional half-planes with $\pi_i \neq -\pi_j$, for any $1 \leq i, j \leq N$, such that

(*i*) $\partial [\![\pi_i]\!] = [\![V]\!], \forall i \in \{1, \ldots, N\}$,

(*ii*) $T_0 = \sum_{i=1}^N Q_i [\![\pi_i]\!]$ with $Q^\star = \sum_{i=1}^N Q_i$.

If exists $i \neq j$ such that $\pi_i \neq \pi_j$, we say that T_0 is a **genuine open book with boundary $[\![V]\!]$ and multiplicity $Q^\star$**.

Definition 19.2.2 (Flat cones)**.** Let $T_0 \in \mathbf{I}_m^{\mathrm{loc}}(\mathbb{R}^{m+n})$ be an oriented cone and V is an oriented $(m-1)$-dimensional linear subspace of $\mathbb{R}^{m+n}$. We say that T_0 is a **flat cone with boundary** $[\![V]\!]$ **and multiplicity** $Q^\star$, if $\partial T_0 = Q^\star [\![V]\!]$ and there exist a m-dimensional closed plane π, $Q^{int} \in \mathbb{N}$ and $Q \in \mathbb{N} \setminus \{0\}, Q \geq Q^\star$, such that

(*i*) $\operatorname{spt} T_0 = \pi$ is an m-dimensional subspace,

(*ii*) $\partial [\![\pi^+]\!] = -\partial [\![\pi^-]\!] = [\![V]\!]$,

(*iii*) $T_0 = Q^{int} [\![\pi]\!] + Q [\![\pi^+]\!] + (Q - Q^\star) [\![\pi^-]\!]$.

If $Q^{int} = 0$, we say that T_0 is a **boundary flat cone with multiplicity** $Q^\star$. If $Q^{int} = 0$ and $Q > Q^\star$, we will call T_0 a **two-sided boundary flat cone with multiplicity** $Q^\star$. If $Q^{int} = 0$ and $Q = Q^\star$, we will call T_0 a **one-sided boundary flat cone with multiplicity** $Q^\star$.

Note that, T_0 is an open book which is not genuine if, and only if, T_0 is an one-sided boundary flat cone.

Definition 19.2.3. Let $T_0 \in \mathbf{I}_m^{\mathrm{loc}}(\mathbb{R}^{m+n})$ be an oriented cone and V is an oriented $(m-1)$-dimensional linear subspace of $\mathbb{R}^{m+n}$ such that $\partial T_0 = Q^\star [\![V]\!]$. If $p \in \operatorname{spt}(\partial T_0)$, we say that

(i) p is a **boundary flat point** provided T_0 is a flat cone,

(ii) p is a **one-sided boundary flat point** provided T_0 is an open book which is non genuine,

(iii) p is a **two-sided boundary flat point** provided T_0 is a two-sided boundary flat cone.

Lemma 19.2.4 (The set of one-sided points is open)**.** *Let T, Γ and $p \in \Gamma$ as in Assumption 6. If $\Theta^m(T,p) < \frac{Q^\star+1}{2}$, then there exists a neighbourhood U of p such that $\Theta^m(T,q) < \frac{Q^\star+1}{2}$ for every $q \in U \cap \Gamma$.*

Proof. It follows directly from the upper semicontinuity of the density function, see Proposition 18.0.4. □

Note that, if $m = 2$ and the tangent cone T_p is a two-sided boundary flat cone, Theorem 19.3.2 ensures that the least possible density on p is $\frac{Q^\star}{2} + 1$. The following lemma is a generalization of [DDHM18, Lemma 3.17] to the case of higher multiplicity with essentially the same proof.

Lemma 19.2.5. *Let T, Γ and $p \in \Gamma$ as in Assumption 6 with $C_0 = 0$. If T_0 is an oriented tangent cone to T at p with $\dim(\operatorname{spine}(T_0)) = m - 1$, then*

(i) If $\Theta^m(T_0, 0) = \frac{Q^\star}{2}$, T_0 is an open book,

(ii) If $\Theta^m(T_0, 0) > \frac{Q^\star}{2}$, T_0 is a two-sided boundary flat cone.

Remark 19.2.6. We also mention that the assumption that the spine has maximal dimension in Γ was assumed in a similar fashion (see Lemma 19.1.1) in [Bro77, Theorem 5.1].

Proof. By the assumption, we have that $\mathrm{spine}(T_0) = T_p\Gamma$. By [Alm00, Theorem 2.2 (3)], there exists an one-dimensional area minimizing current T_{01} in $(T_p\Gamma)^\perp$ such that $T_0 = [\![T_p\Gamma]\!] \times T_{01}$. This fact that Almgren proved is an application of [Fed69, Theorem 5.4.8] and [Fed69, Section 4.3.15], using that T_0 is an oriented cylinder with direction v for any $v \in T_p\Gamma = \mathrm{spine}(T_0)$. Thus, since $\partial T_0 = Q^\star [\![T_p\Gamma]\!]$, we obtain that $\partial T_{01} = (-1)^{m-1} Q^\star [\![0]\!]$, the fact that T_{01} is invariant under homotheties allows us to write, for some $Q \in \mathbb{N} \setminus \{0\}$,

$$(-1)^{m-1} T_{01} = \sum_{i=1}^{Q} [\![\ell_i^+]\!] + \sum_{j=1}^{Q-Q^\star} [\![\ell_i^-]\!]\,,\; T_{01} = \sum_{i=1}^{Q} \|\, [\![\ell_i^+]\!] \,\| + \sum_{j=1}^{Q-Q^\star} \|\, [\![\ell_i^-]\!] \,\|,$$

where ℓ_i^+, ℓ_j^- are all oriented half-lines such that $\partial\, [\![\ell_i^+]\!] = [\![0]\!]$ and $\partial \left[\!\!\left[\ell_j^-\right]\!\!\right] = -\,[\![0]\!]$. In particular, we have

$$\mathrm{spt}(T_0) \subset \left(\bigcup_{i=1}^{Q} T_p\Gamma + \ell_i^+ \right) \cup \left(\bigcup_{i=1}^{Q-Q^\star} T_p\Gamma + \ell_i^- \right). \tag{19.2.1}$$

Note that, if $Q > Q^\star$, $\partial([\![\ell_i^+]\!] + \left[\!\!\left[\ell_j^-\right]\!\!\right]) = 0$ and $[\![\ell_i^+]\!] + \left[\!\!\left[\ell_j^-\right]\!\!\right]$ is area minimizing for any choice of i and j which ensures that the support of $[\![\ell_i^+]\!] + \left[\!\!\left[\ell_j^-\right]\!\!\right]$ is a straight line ℓ_{ij}. Since the choice of i and j is arbitrary, then we have $\mathrm{spt}([\![\ell_i^+]\!] + \left[\!\!\left[\ell_j^-\right]\!\!\right]) \subset \ell$, where ℓ is a straight line which, by (19.2.1), concludes the proof of (ii). If $Q = Q^\star$, we have that T_{01} is a sum of lines which might be distinct, and this concludes the proof of (i). □

19.3 Two dimensional case

In Lemma 19.2.5, T_0 has dimension m and we assume that the dimension of the spine is maximal. Nevertheless, if $m = 2$, we can drop the hypothesis on the spine since we have a full characterization of tangent cones with boundary being a subspace as stated in the proposition below.

Proposition 19.3.1 (Proposition 4.1, [DLNS23])**.** *Let T_0 be a 2-dimensional area minimizing cone in $\mathbb{R}^{2+n}$ with $\partial T_0 = Q^\star [\![\ell]\!]$ for some positive integer $Q^\star$ and a straight line ℓ containing the origin. Then we can decompose $T_0 = {T_0}^{int} + {T_0}^\flat$ into two area minimizing cones with supports intersecting only at the origin which satisfy*

(i) $\partial {T_0}^{int} = 0$ and thus ${T_0}^{int} = \sum_{i=1}^N Q_i [\![\pi_i]\!]$ where $Q_1, \dots, Q_N$ are positive integers and $\pi_1, \dots, \pi_N$ are distinct oriented 2-dimensional planes such that $\pi_i \cap \pi_j = \{0\}$ for all $i \neq j$,

(ii) ${T_0}^\flat$ is either a two-sided boundary flat cone or an open book.

Let us also recall two pivotal results in the theory which will be used in this work. We also denote d_H for the Hausdorff distance between closed sets and we denote by $e(p, r)$ the usual **spherical excess of a current** T, namely

$$e(p, r) := \frac{\|T\| \left(\mathbf{B}(p, r)\right)}{\pi r^2} - \Theta(T, p),$$

we also define the Holder seminorm used to measure the regularity of Γ, for any open set U,

$$[\Gamma]_{0,\alpha,U} := \sup_{q\neq p\in\Gamma\cap U} \frac{|T_p\Gamma - T_q\Gamma|}{|p-q|^\alpha}.$$

We have the following decay properties.

Theorem 19.3.2 (Uniqueness of tangent cones and speed of convergence, Theorem 2.1, [DLNS23])**.** *Let T and Γ be as in Assumption 6 with $m = 2$. Then there are positive constants ε_0, C and β with the following property. If $p \in \Gamma$ and $e(p, r) \leq \varepsilon_0^2$ for some $r \leq \operatorname{dist}(p, \partial\mathbf{B}(0, 1))$, then there exists a unique tangent cone T_p to T at p which, for every $\rho \in (0, r]$, satisfies:*

$$|e(p,\rho)| \leq C|e(p,r)| \left(\frac{\rho}{r}\right)^{2\beta} + C\left(C_0^2 + [\Gamma]^2_{0,\beta,\mathbf{B}(p,r)}\right)\left(\frac{\rho}{r}\right)^{2\beta},$$

$$\mathbf{d}_{\mathbf{B}(0,1)}\left(T_{p,\rho}, T_p\right) \leq C|e(p,r)|^{\frac{1}{2}} \left(\frac{\rho}{r}\right)^{\beta} + C\left(C_0 + [\Gamma]_{0,\beta,\mathbf{B}(p,r)}\right)\left(\frac{\rho}{r}\right)^{\beta},$$

$$\operatorname{dist}_H\left(\operatorname{spt}\left(T_{p,\rho}\right)\cap\mathbf{B}(0,1), \operatorname{spt}(T_p)\cap\mathbf{B}(0,1)\right) \leq C|e(p,r)|^{\frac{1}{2}} \left(\frac{\rho}{r}\right)^{\beta} + C\left(C_0 + [\Gamma]_{0,\beta,\mathbf{B}(p,r)}\right)\left(\frac{\rho}{r}\right)^{\beta}.$$

We also state the Holder continuity of the map that to each point $p \in \Gamma$ assigns its unique tangent cone T_p.

Lemma 19.3.3 (Holder continuity)**.** *Let T, p, r be as in Theorem 19.3.2 and $q \in \Gamma \cap \mathbf{B}(p, r)$. Then the functions $q \mapsto T_q$ is Holder continuous, i.e., it holds*

$$\mathbf{d}\left(T_q \llcorner \mathbf{B}(0,1), T_p \llcorner \mathbf{B}(0,1)\right) \leq C|q-p|^\beta, \quad \forall q \in \mathbf{B}(p,r). \tag{19.3.1}$$

Proof. For the proof we refer the reader directly to [DLNS21, Equation 4.7] which can be readily adjusted to the almost area minimizing setting. □

Definition 19.3.4 (Two-sided collapsed points)**.** Let T and Γ be as in Assumption 6. A point $p \in \Gamma$ will be called **two-sided collapsed point of** T if

- (i) there exists a tangent cone T_0 to T at p which is a two-sided boundary flat cone,
- (ii) there exists a neighbourhood U of p such that $\Theta(T, q) \geq \Theta(T, p)$ for every $q \in \Gamma \cap U$.

Lemma 19.3.5 (The set of two-sided collapsed points is open)**.** *Let T and p as in Theorem 19.3.2. Assume that $p \in \Gamma$ is a two-sided collapsed point, then there is $\rho > 0$ such that $\Theta^2(T, q) = \Theta^2(T, p)$ for all $q \in \mathbf{B}(p, \rho) \cap \Gamma$. In particular, every such q is two-sided collapsed.*

Proof. Fix $Q \in \mathbb{N}, Q > Q^\star$ such that $\Theta^2(T, p) = Q - \frac{Q^\star}{2}$ and the unique tangent cone to T at p is $T_p = Q [\![\pi^+]\!] + (Q - Q^\star) [\![\pi^-]\!]$. If we choose $r > 0$ small enough we can assume that

$$\frac{\|T\|(\mathbf{B}(p,r))}{\pi r^2} \leq Q - \frac{Q^\star}{2} + \frac{1}{8}.$$

Now, we choose $s \in (0, r)$, in order to hold

$$\|T\|(\mathbf{B}(q,r-s)) \leq \|T\|(\mathbf{B}(p,r)) \leq \pi r^2\left(Q - \frac{Q^\star}{2} + \frac{1}{8}\right) \leq \pi(r-s)^2\left(Q - \frac{Q^\star}{2} + \frac{3}{16}\right),$$

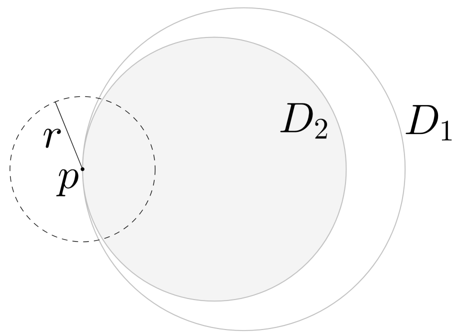


Figure 19.1: *Condition ((ii)) essentially excludes points of tangential intersection of connected parts of* spt T, *i.e., it forbid the existence of one-sided points arbitrarily close to two-sided points. For instance as in the picture p does not verify ((ii)), let $T = [\![D_1]\!] + [\![D_2]\!]$ with D_1 and D_2 being tangential circles at p then $\Theta(T,p) = 1 \geq \frac{1}{2} = \Theta(T,q)$ for all $q \neq p$ which belongs to the outer circumference.*

for every $q \in \mathbf{B}(p,s) \cap \Gamma$. For every $\sigma \in (0, r-s)$, by the almost monotonocity formula, Proposition 18.0.3, we have that

$$\begin{aligned} \frac{\|T\|(\mathbf{B}(q,\sigma))}{\pi\sigma^2} &\leq e^{C_1((r-s)^{\beta_1} - \sigma^{\beta_1})} \frac{\|T\|(\mathbf{B}(q, r-s))}{\pi(r-s)^2} \\ &\leq e^{C_1(r-s)^{\beta_1}} \left(Q - \frac{Q^\star}{2} + \frac{3}{16} \right), \end{aligned} \tag{19.3.2}$$

for every $q \in \mathbf{B}(p,s) \cap \Gamma$. Then take $q \in \mathbf{B}(p,s) \cap \Gamma$ where this ball is chosen to be a subset of the neighbourhood U given by the definition of two-sided collapsed points. Hence, by Proposition 19.3.1 and Lemma 19.2.5, the tangent cone to T at q has to be of the form $T_q = Q' [\![\pi_q^+]\!] + (Q' - Q^\star) [\![\pi_q^-]\!]$, for some integer $Q' > Q^\star$, thus, letting $r \to 0$ in (19.3.2) we obtain

$$Q - \frac{Q^\star}{2} = \Theta^2(T,p) \leq \Theta^2(T,q) = Q' - \frac{Q^\star}{2} \overset{(19.3.2)}{\leq} Q - \frac{Q^\star}{2} + \frac{3}{16}.$$

□

The following theorem allows us to reduce the proof of Theorem 17.0.3 to the proof that any two-sided collapsed point is regular.

Theorem 19.3.6. *Let T and p as in Theorem 19.3.2 and assume that $C_0 = 0$. If $\mathrm{Reg}_b(T)$ is not dense in Γ, then there exists a two-sided collapsed singular point $p \in \Gamma$ with $\Theta^2(T,p) > \frac{Q^\star}{2}$.*

Remark 19.3.7. When we consider the setting of [DLNS21], i.e., when Γ belongs to a $C^{3,\alpha}$ convex barrier, $\alpha \in (0,1)$,, we have that two-sided points do not exists, in particular, $\mathrm{Reg}_b^2(T) = \emptyset$ and, by [DLNS21, Theorem 0.2], we know that $\mathrm{Reg}_b^1(T) = \Gamma$. In other words, the authors in [DLNS21] proved the full regularity of the current at the boundary.

Proof. Assume that $\mathrm{Sing}_b(T)$ has no empty interior, then we can define

$$C_i := \left\{ p \in \Gamma : \Theta^2(T,p) \geq i - \frac{1}{2} \right\} \cap \mathrm{int}(\mathrm{Sing}_b(T)).$$

By Proposition 18.0.4, the density restricted to the boundary is upper semicontinuous, then C_i is relatively closed in $\mathrm{int}(\mathrm{Sing}_b(T))$. Let D_i be the topological interior of C_i and E_i be the relatively open set $D_i \setminus C_{i+1}$ in $\mathrm{int}(\mathrm{Sing}_b(T))$. We fix $p \in \Gamma$ and the natural number i such that

$$i - \frac{1}{2} \leq \Theta^2(T, p) < i + \frac{1}{2}. \tag{19.3.3}$$

Assume that $p \notin \cup_{i \geq 1} E_i$, by the latter inequalities, we have $p \in C_i \setminus D_i$ which leads to

$$\mathrm{int}(\mathrm{Sing}_b(T)) \setminus \cup_i E_i \subset \cup_i C_i \setminus D_i.$$

Observe that $C_i \setminus D_i$ is relatively closed in $\mathrm{int}(\mathrm{Sing}_b(T))$ and then $\mathrm{int}(\mathrm{Sing}_b(T)) \setminus \cup_i E_i$ is the union of countably many closed subsets of $\mathrm{int}(\mathrm{Sing}_b(T))$ which guarantees, by the Baire Category Theorem, that $\cup_i E_i$ cannot be empty. So, there is $E_i \neq \emptyset$ relatively open in Γ, hence, in view of [DLNS21, Theorem 0.2], since E_i contains only singular points, any $p \in E_i$ satisfies $\Theta^2(T, p) \geq \frac{Q^\star + 1}{2}$ and, from Proposition 19.3.1 and Lemma 19.2.5, p is two-sided boundary flat point. Fix $p \in E_i$, we know that there exists $Q \in \mathbb{N}, Q > Q^*$ such that $\Theta^2(T, p) = Q - \frac{Q^\star}{2}$, thus

- if $\Theta^2(T, p) \in \mathbb{N}$, we get $\Theta^2(T, p) = i$. Now, assume by contradiction that there is $q \in E_i$ such that $\Theta^2(T, q) < \Theta^2(T, p)$, then, by (19.3.3), we necessarily have $i - \frac{1}{2} \leq \Theta^2(T, q) < i + \frac{1}{2}$ which ensures, by the classification of tangent cones, $\Theta^2(T, q) = Q - \frac{Q^\star + 1}{2}$. Since $\Theta^2(T, q) = Q' - \frac{Q^\star}{2}$ for some $Q' \in \mathbb{N}$, we obtain $Q' = Q - \frac{1}{2}$ which is a contradiction. We then conclude that p is a singular point which is also two-sided collapsed.
- if $\Theta^2(T, p) \notin \mathbb{N}$, then $\Theta^2(T, p) = i - \frac{1}{2}$ and, since E_i is relatively open, there is a relatively open, in Γ, neighborhood $U \subset E_i$ of p. By definition of E_i, for every $q \in U$, $\Theta^2(T, q) \geq i - \frac{1}{2} = \Theta^2(T, p)$ which ensures that p is two-sided collapsed.

□

We have now reduced our situation to prove the following theorem.

Theorem 19.3.8 (Two-sided collapsed points are regular)**.** *Let T and Γ be as in Assumption 6 with $C_0 = 0$. Then any two-sided collapsed point of T is a two-sided regular point of T.*

In fact, the rest of the paper is devoted to prove Theorem 19.3.8, we also mention that we prove this to the general setting of m-dimensional area minimizing currents with boundary multiplicity $Q^\star \geq 1$. However, our main result (Theorem 17.0.3) is stated for $2d$ area minimizing currents with boundary multiplicity $Q^\star \geq 1$ because we need to apply Theorem 19.3.6 which we proved only in this setting.

Chapter 20

Approximations of currents by multi-valued collapsed Dirichlet minimizers

20.1 Definitions and regularity of collapsed Dirichlet minimizers

We refer the reader to [DS14] and [DS15] for standard definitions and notations about the theory of multiple valued functions. Throughout all this section we will consider an open set $\Omega \subset \mathbb{R}^m$ together with a $(m-1)$-submanifold γ of class $C^{3,\alpha}$ dividing Ω in two disjoint open sets Ω^+ and Ω^-.

Definition 20.1.1. Let $\varphi \in W^{\frac{1}{2},2}(\gamma, \mathcal{A}_{Q^\star}(\mathbb{R}^n))$, $Q, Q^\star \in \mathbb{N}$, $Q \geq Q^\star \geq 1$. A **$(Q - \frac{Q^\star}{2})$-valued function with interface** (γ, φ), consists of a pair (f^+, f^-) satisfying the following properties

(*i*) $f^+ \in W^{1,2}(\Omega^+, \mathcal{A}_Q(\mathbb{R}^n))$, $f^- \in W^{1,2}(\Omega^-, \mathcal{A}_{Q-Q^\star}(\mathbb{R}^n))$,

(*ii*) $f^+|_\gamma = f^-|_\gamma + \varphi$.

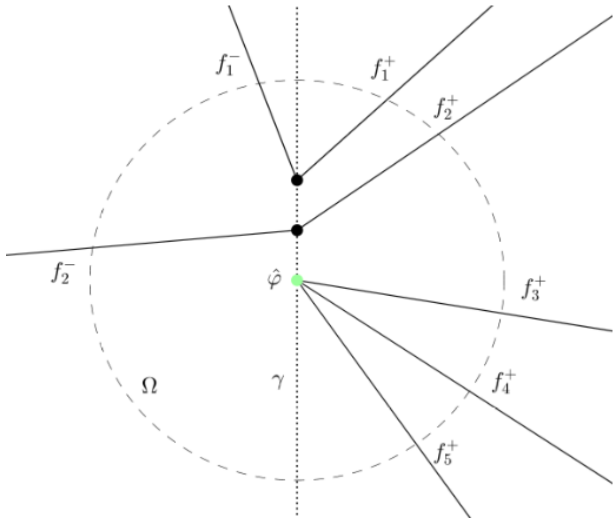
We define the **Dirichlet energy of** (f^+, f^-) as $\mathrm{Dir}(f^+, f^-, \Omega) := \mathrm{Dir}(f^+, \Omega^+) + \mathrm{Dir}(f^-, \Omega^-)$. Such a pair will be called **Dir-minimizing in** Ω, if for all $\left(Q - \frac{Q^\star}{2}\right)$-valued function (g^+, g^-) with interface (γ, φ) which agrees with (f^+, f^-) outside of a compact set $K \subset\subset \Omega$ satisfies $\mathrm{Dir}(f^+, f^-, \Omega) \leq \mathrm{Dir}(g^+, g^-, \Omega)$.

Remark 20.1.2. Note that when $Q^\star$ is an even number, we unfortunately overlap Definition 20.1.1 and Almgren's definition of Q-valued functions. However, since it will not cause any confusion in what follows, so we will allow this abuse of notation.

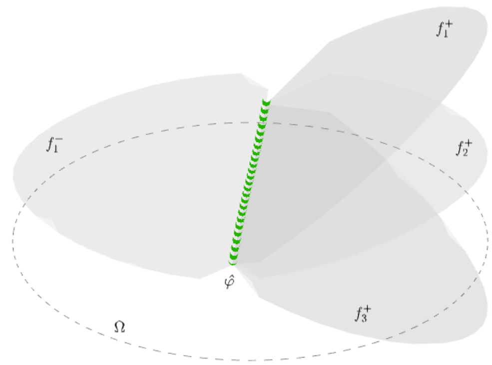
The interesting case to be treated here is when $Q > Q^\star > 1$. When $Q = Q^\star = 1$, the pair (f^+, f^-) consists of a single-valued function f^+ and its Dir-minimality is equivalent to the harmonicity of f^+. The case $Q > Q^\star = 1$ is studied in [DDHM18, Section 4]. The *one-sided* case, i.e. $Q = Q^\star$, has to be treated differently and it is done in dimension 2 in [DLNS21].

Definition 20.1.3. Let (f^+, f^-) be a $\left(Q - \frac{Q^\star}{2}\right)$-valued function with interface (γ, φ) and $\varphi = Q^\star [\![\hat{\varphi}]\!]$ for a single valued function $\hat{\varphi}$. We say that (f^+, f^-) **collapses at the interface**, if $f^+|_\gamma = Q [\![\hat{\varphi}]\!]$.

Notice that, (f^+, f^-) satisfy the properties of the preceding definition, if and only if, $f^-|_\gamma = (Q - Q^\star)\,[\![\hat\varphi]\!]$.



(a) *$n = 1, Q = 5, Q^\star = 3, f^+(x) = \sum_{i=1}^5 [\![f_i^+(x)]\!]$ and $f^+(x) = \sum_{i=1}^2 [\![f_i^-(x)]\!]$, so that the $(Q - \frac{Q^\star}{2})$-valued function (f^+, f^-) has interface $(\gamma, \hat\varphi)$ where $\gamma = \{x = 0\}$ and $(x, \hat\varphi(x))$ is constantly equal to the green point.*



(b) *Assume $\Omega \subset \mathbb{R}$ and $n = 1, Q = 3, Q^\star = 2, f^+(x) = \sum_{i=1}^3 [\![f_i^+(x)]\!]$ and $f^+(x) = [\![f_1^-(x)]\!]$, so that the $(Q - \frac{Q^\star}{2})$-valued function (f^+, f^-) collapses at the interface $(\gamma, \hat\varphi)$ where $\gamma = \{x = 0\}$ and $\hat\varphi$ is represented by the green curve.*

With these definitions settled, we aim to prove the harmonic regularity of collapsed $(Q - \frac{Q^\star}{2})$-valued maps along the same lines for $Q^\star = 1$ as it is done in [DDHM18, Theorem 4.5]. As we mentioned above, this part of the linear theory in our setting is true only when we consider $Q > Q^\star$ since we will construct some competitors in the arguments which need the existence of multi-valued functions defined in both sides of γ.

Theorem 20.1.4 (Regularity of collapsing $(Q - \frac{Q^\star}{2})$-Dir minimizers)**.** *Let $\varphi : \gamma \to \mathcal{A}_{Q^\star}(\mathbb{R}^n)$, where $\varphi = Q^\star\,[\![\hat\varphi]\!]$ for some $\hat\varphi \in C^{1,\alpha}(\gamma, \mathbb{R}^n)$, γ be a $(m-1)$-submanifold of class C^3 in $\mathbb{R}^m$, $Q > Q^\star \geq 1$, and (f^+, f^-) be a $\left(Q - \frac{Q^\star}{2}\right)$-valued Dir-minimizer with interface (γ, φ). If (f^+, f^-) collapses at the interface, then there is a single-valued harmonic function $h : \Omega \to \mathbb{R}^n$ such that $f^+ = Q\,[\![h|_{\Omega^+}]\!]$ and $f^- = (Q - Q^\star)\,[\![h|_{\Omega^-}]\!]$.*

If we do not assume that the pair (f^+, f^-) is collapsed and impose that γ is real analytic, we can obtain that the singular set of the pair is discrete when $Q^\star = 1$, see [LZ19, Theorem 1.6]. Let us turn to the proof of Theorem 20.1.4, firstly we define the tangent function and then we characterize these tangent function.

Definition 20.1.5 (Tangent function)**.** Let (f^+, f^-) be a $\left(Q - \frac{Q^\star}{2}\right)$-valued function with interface $(\gamma, Q^\star\,[\![0]\!])$. Fix $p \in \gamma$ and define a **blowup of f at p at scale r** as follows

$$f^\pm_{p,r}(x) := \frac{f^\pm(p + rx)}{\sqrt{r^{2-m}\,\mathrm{Dir}(f^+, f^-, \mathrm{B}_r(p))}}, \forall r > 0,$$

where we assume that (f^+, f^-) is not identically $(Q\,[\![0]\!], (Q - Q^\star)\,[\![0]\!])$ in every ball $\mathrm{B}_r(0)$. For any sequence $r_k \to 0$, if the limit exists, we say that $g^\pm = \lim_{k\to+\infty} f^\pm_{p,r_k}$ is a **tangent function at p to f**.

Lemma 20.1.6. *Let $Q > Q^\star$, (f^+, f^-) be a $\left(Q - \frac{Q^\star}{2}\right)$ Dir-minimizer which collapses at the interface $(\gamma, Q^\star [\![0]\!])$, where γ is a C^3 $(m-1)$-submanifold in $\mathbb{R}^m$, and fix $p \in \gamma$. Consider a tangent function $\left(h_0^+, h_0^-\right)$ to (f^+, f^-) at p and $\{e_1, \cdots, e_{m-1}\}$ a base of $T_p\gamma$. For each $i \in \{1, \cdots, m-1\}$, we define (h_i^+, h_i^-) to be a tangent to (h_{i-1}^+, h_{i-1}^-) at e_i. Then $(h^+, h^-) := (h_{m-1}^+, h_{m-1}^-)$ is given by $(Q[\![L]\!], (Q - Q^\star)[\![L]\!])$ where L is a nonzero linear function which vanishes on $T_p\gamma$.*

Proof. Assume $T_p\gamma = \{x : x_m = 0\}$. The consequences of [DDHM18, Lemma 4.29, Remark 4.31] readily holds in our higher multiplicity case, so we have the following properties:

(A) (h^+, h^-) is a $\left(Q - \frac{Q^\star}{2}\right)$ Dir-minimizer which collapses at the interface $(T_p\gamma, Q^\star [\![0]\!])$,

(B) (h^+, h^-) depends only on x_m namely there exist Q-valued function $\alpha^+ : \mathbb{R}_+ \to \mathcal{A}_Q(\mathbb{R}^n)$ and a $(Q - Q^\star)$-valued function $\alpha^- : \mathbb{R}_- \to \mathcal{A}_{Q-Q^\star}(\mathbb{R}^n)$ such that $h^\pm(x) = \alpha^\pm(x_m)$,

(C) (h^+, h^-) is an I-homogeneous function for some $I > 0$, namely there is a Q-point P and a $(Q - Q^\star)$-point P' such that $\alpha^+(x_m) = x_m^I P$ and $\alpha^-(x_m) = (-x_m)^I P'$,

(D) $\mathrm{Dir}(h^+, \mathrm{B}_1(0)) + \mathrm{Dir}(h^-, \mathrm{B}_1(0)) = 1$.

Since (h^+, h^-) is a Dir-minimizer and both h^+ and h^- are C^2, both h^+ and h^- are classical harmonic functions, therefore, since they depend only upon one variable, we necessarily have that $I = 1$. So there are coefficients $\beta_1^+, \ldots, \beta_Q^+$ and $\beta_1^-, \ldots, \beta_{Q-Q^\star}^-$ such that

$$h^+(x) = \sum_{i=1}^{Q} [\![\beta_i^+ x_m]\!], \text{if } x_m > 0, \quad \text{and} \quad h^-(x) = \sum_{i=1}^{Q-Q^\star} [\![\beta_i^- x_m]\!], \text{if } x_m < 0.$$

If $Q > Q^\star > 1$, then we can assume that $\beta_{j_0}^+ \neq \beta_{i_0}^-$. Now, we will construct a competitor of (h^+, h^-) with less Dir-energy which is the desired contradiction. Note that, in order to construct a competitor, we have to assure that it has the same interface of (h^+, h^-), i.e. it takes $[\![0]\!]$ at least $Q^\star$ times at $T_p\gamma = \{x_m = 0\}$. For $x = (x', x_m)$, define

$$\hat{h}^+(x) = \begin{cases} \left[\!\left[\hat{\beta} x_m + c(|x'|)\right]\!\right] + \sum_{j=1, j\neq j_0}^{Q} \left[\!\left[\beta_j^+ x_m\right]\!\right], & \text{if } x \in \overline{\mathrm{B}_{\frac{1}{2}}^+(0)}, \\ h^+(x), & \text{if } x \in \mathrm{B}_1^+(0) \setminus \overline{\mathrm{B}_{\frac{1}{2}}^+(0)}. \end{cases}$$

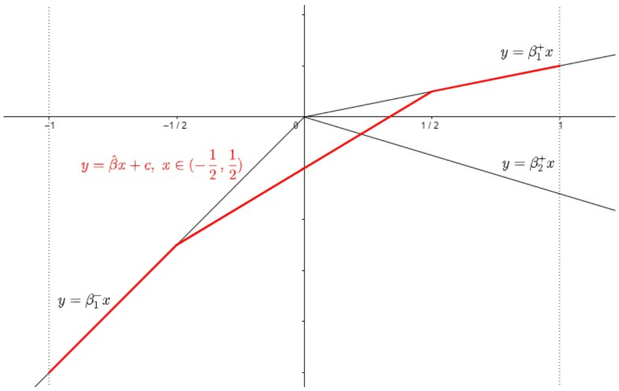
$$\hat{h}^-(x) = \begin{cases} \left[\!\left[\hat{\beta} x_m + c(|x'|)\right]\!\right] + \sum_{i=1, i\neq i_0}^{Q-Q^\star} [\![\beta_i^+ x_m]\!], & \text{if } x \in \overline{\mathrm{B}_{\frac{1}{2}}^-(0)}, \\ h^-(x), & \text{if } x \in \mathrm{B}_1^-(0) \setminus \overline{\mathrm{B}_{\frac{1}{2}}^-(0)}. \end{cases}$$

where $\hat{\beta} = \frac{\beta_{j_0}^+ + \beta_{i_0}^-}{2}$, $c(|x'|) = \bar{\beta}\sqrt{1/4 - |x'|^2}$ and $\hat{\beta} = \frac{\beta_{j_0}^+ - \beta_{i_0}^-}{2}$.

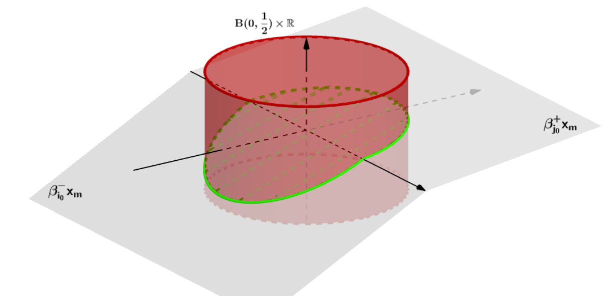
By direct computation, we have

$$\mathrm{Dir}(\hat{h}^+, \mathrm{B}_{1/2}^+(0)) = |\mathrm{B}_{1/2}^+(0)| \left[\sum_{j=1, j\neq j_0}^{Q} |\beta_{j_0}^+|^2 + |\hat{\beta}|^2\right] + \int_{\mathrm{B}^{m-1}(0,1/2)} \int_0^{\sqrt{\frac{1}{4} - |x'|^2}} \frac{|\bar{\beta}|^2 |x'|^2}{\frac{1}{4} - |x'|^2} dx' dx_m,$$

the integral on the right hand side can be bounded by $|\bar{\beta}|^2 |\mathrm{B}_{1/2}^+(0)|$ since $\bar{\beta} \neq 0$ and the integrating function is radial. By the very same computation, we finally have that



(a) *if* $\Omega \subset \mathbb{R}$ *and* $n = 1, Q = 2, Q^\star = 1, h^+(x) = [\![\beta_1^+ x]\!] + [\![\beta_2^+ x]\!]$ *and* $h^-(x) = [\![\beta_1^- x]\!]$*, we define the competitor* $(\hat{h}^+, \hat{h}^-)$ *as the red function which has the same interface of* (h^+, h^-) *and less* Dir*-energy.*



(b) *With* $m = 2$ *competitor* $(\hat{h}^+, \hat{h}^-)$ *represented by the green hypersurface is not linear inside the cylinder, but it also satisfies what we need, i.e. it has the same interface of* (h^+, h^-) *and it has less* Dir*-energy.*

$$\begin{aligned}\mathrm{Dir}(\hat{h}^+, \hat{h}^-, \mathrm{B}_{1/2}(0)) &< |\mathrm{B}^+_{1/2}(0)| \Big[\sum_{j=1, j\neq j_0}^{Q} |\beta_j^+|^2 + \sum_{i=1, i\neq i_0}^{Q-Q^\star} |\beta_i^+|^2 + 2|\hat{\beta}|^2 + 2|\bar{\beta}|^2 \Big] \\ &= \mathrm{Dir}(h^+, h^-, \mathrm{B}_{1/2}(0)).\end{aligned}$$

By construction they have the same Dir-energy outside $\mathrm{B}_{1/2}(0)$, thus every β_j^+ has to coincide with β_i^- and we finish the proof of the lemma. ☐

Definition 20.1.7. Let us denote $\boldsymbol{\eta}(P) = \frac{1}{Q}\sum_{i=1}^{Q} P_i$ the center of the Q-point $P = \sum_{i=1}^{Q} [\![P_i]\!]$.

As a simple corollary of the above lemma we have:

Corollary 20.1.8. *Let* $Q > Q^\star$ *and assume* (f^+, f^-) *is a* $\left(Q - \frac{Q^\star}{2}\right)$ *Dir-minimizer which collapses at* $(\gamma, Q^\star [\![0]\!])$*, where* γ *is a* C^3 $(m-1)$*-submanifold in* $\mathbb{R}^m$*. If* $\boldsymbol{\eta} \circ f^- = \boldsymbol{\eta} \circ f^+ = 0$*, then* $f^+ = Q[\![0]\!]$ *and* $f^- = (Q - Q^\star)[\![0]\!]$*.*

Proof. If (f^+, f^-) is identically $(Q[\![0]\!], (Q-Q^\star)[\![0]\!])$ in a neighborhood U of a point $p \in \gamma$, then, by the interior regularity theory of Dir-minimizer (precisely, [DS11, Proposition 3.22]), (f^+, f^-) is identically $(Q[\![0]\!], (Q-Q^\star)[\![0]\!])$ in the connected component of the domain of (f^+, f^-) which contains p. Thus, if the corollary were false, then there would be a point $p \in \gamma$ such that $\mathrm{Dir}(f^+, \mathrm{B}_r^+(p)) + \mathrm{Dir}(f^-, \mathrm{B}_r^-(p)) > 0$ for every $r > 0$ such that $\mathrm{B}_r(p) \subset \Omega$. If we consider (h^+, h^-) as in Lemma 20.1.6, we conclude that $\boldsymbol{\eta} \circ h^+ = \boldsymbol{\eta} \circ h^- = 0$, since such property is inherited by each tangent map. But then the nonzero linear function L of the conclusion of Lemma 20.1.6 should be equal to $\boldsymbol{\eta} \circ h^+$ on $\{x_m > 0\}$ and $\boldsymbol{\eta} \circ h^-$ on $\{x_m \leq 0\}$. Hence L should vanish identically, contradicting Lemma 20.1.6. ☐

Before the proof of Theorem 20.1.4, we introduce the following notation which will be used throughout the paper.

$$f \oplus \zeta := \sum_i [\![f_i + \zeta]\!],$$

where f is a Q-valued function with a measurable selection of single-valued functions f_i and ζ is a single-valued functions both defined on the same domain.

Proof of Theorem 20.1.4. **The case $\hat{\varphi} \equiv 0$:** Firstly, using the regularity theory of classical elliptic equations, we obtain that the functions $\boldsymbol{\eta} \circ f^{\pm}$ are differentiable up to the boundary γ, i.e., belong to $C^1(\Omega^{\pm} \cup \gamma)$. Let ν be the unit normal to γ. We claim that

$$\partial_\nu \left(\boldsymbol{\eta} \circ f^+\right)(p) = \partial_\nu \left(\boldsymbol{\eta} \circ f^-\right)(p) \quad \text{for all } p \in \gamma \cap \Omega. \tag{20.1.1}$$

In fact, assume by contradiction that, at some point $p \in \gamma \cap \Omega$, we have $\partial_\nu (\boldsymbol{\eta} \circ f^+)(p) \neq \partial_\nu (\boldsymbol{\eta} \circ f^-)(p)$ and consider a tangent function (h^+, h^-) to (f^+, f^-) at p which is the limit of some rescaled sequence $\left(f^+_{p,\rho_k}, f^-_{p,\rho_k}\right)$, where we denote

$$f^{\pm}_{p,\rho_k}(x) := \frac{f^{\pm}(p + \rho_k x)}{\sqrt{\rho_k^{2-m} \operatorname{Dir}(f^+, f^-, \mathrm{B}_{\rho_k}(p))}}.$$

Observe that, since at least one among $\partial_\nu (\boldsymbol{\eta} \circ f^+)(p)$ and $\partial_\nu (\boldsymbol{\eta} \circ f^-)(p)$ differs from 0, we necessarily have

$$c_1 \rho_k^m \geq \operatorname{Dir}\left(\boldsymbol{\eta} \circ f^+, \boldsymbol{\eta} \circ f^-, \mathrm{B}_{\rho_k}(p)\right) \geq c_0 \rho_k^m,$$

for some constants $c_1 = c_1(f^+, f^-) > 0, c_0 = c_0(f^+, f^-) > 0$. Thus, by rescaling, we obtain

$$\begin{aligned} c_1 \frac{\rho_k^m}{\operatorname{Dir}(f^+, f^-, \mathrm{B}_{\rho_k}(p))} &\geq \frac{\operatorname{Dir}(\boldsymbol{\eta} \circ f^+, \boldsymbol{\eta} \circ f^-, \mathrm{B}_{\rho_k}(p))}{\operatorname{Dir}(f^+, f^-, \mathrm{B}_{\rho_k}(p))} \\ &= \frac{\operatorname{Dir}\left(\boldsymbol{\eta} \circ f^+_{p,\rho_k}, \boldsymbol{\eta} \circ f^-_{p,\rho_k}, \mathrm{B}_1(0)\right)}{\operatorname{Dir}\left(f^+_{p,\rho_k}, f^-_{p,\rho_k}, \mathrm{B}_1(0)\right)} \\ &= \operatorname{Dir}\left(\boldsymbol{\eta} \circ f^+_{p,\rho_k}, \boldsymbol{\eta} \circ f^-_{p,\rho_k}, \mathrm{B}_1(0)\right) \\ &\geq c_0 \frac{\rho_k^m}{\operatorname{Dir}(f^+, f^-, \mathrm{B}_{\rho_k}(p))}. \end{aligned} \tag{20.1.2}$$

Therefore, we have the following two alternatives:

(I) If $\limsup_k (\rho_k)^{-m} \operatorname{Dir}(f^+, f^-, \mathrm{B}_{\rho_k}(p)) = +\infty$, by (20.1.2), denoting by (h_0^+, h_0^-) the tangent function to (f^+, f^-) at p, passing to the limit in (20.1.2), we have that $\operatorname{Dir}(\boldsymbol{\eta} \circ h_0^+, \boldsymbol{\eta} \circ h_0^-, \mathrm{B}_1(0)) = 0$ and then $\boldsymbol{\eta} \circ h_0^{\pm} \equiv 0$. By Corollary 20.1.8, $\left(h_0^+, h_0^-\right)$ should be trivial. But this is not possible, because the energy of a tangent function satisfies $\operatorname{Dir}\left(h_0^+, h_0^-, \mathrm{B}_1(0)\right) = 1$, see Lemma 20.1.6.

(II) If $\limsup_k (\rho_k)^{-m} \operatorname{Dir}(f^+, f^-, \mathrm{B}_{\rho_k}(p)) < +\infty$, by (20.1.2), we have $\operatorname{Dir}(\boldsymbol{\eta} \circ h_0^+, \boldsymbol{\eta} \circ h_0^-, \mathrm{B}_1(0)) > 0$ and thus $\boldsymbol{\eta} \circ h_0^+$ and $\boldsymbol{\eta} \circ h_0^-$ are distinct functions and at least one among them is a nontrivial function. Indeed, they are distinct follows from the fact that the blowup of a differentiable function coincides with its differential and we are assuming that $\partial_\nu (\boldsymbol{\eta} \circ f^+)(p) \neq \partial_\nu (\boldsymbol{\eta} \circ f^-)(p)$. Since case (I) never occurs, we can apply this argument iteratively until we reach the pair (h^+, h^-) of Lemma 20.1.6 and then conclude that $\boldsymbol{\eta} \circ h^+$ and $\boldsymbol{\eta} \circ h^-$ are two distinct linear functions with one of them being non trivial and this contradicts Lemma 20.1.6.

We have verified the validity of (20.1.1) and it is enough to conclude the proof, indeed, it implies that the function

$$\zeta := \begin{cases} \boldsymbol{\eta} \circ f^+ & \text{on } \Omega^+, \\ \boldsymbol{\eta} \circ f^- & \text{on } \Omega^-. \end{cases} \tag{20.1.3}$$

is an harmonic function defined on the entire Ω. Using the notation above, we set

$$\tilde{f}^+ := f^+ \oplus (-\zeta) \text{ and } \tilde{f}^- := f^- \oplus (-\zeta). \tag{20.1.4}$$

By [DS11, Lemma 3.23], it is easy to see that $\left(\tilde{f}^+, \tilde{f}^-\right)$ is a $\left(Q - \frac{Q^\star}{2}\right)$ Dir-minimizer which collapses at the interface $(\gamma, [\![0]\!])$ and that $\boldsymbol{\eta} \circ \tilde{f}^+ = \boldsymbol{\eta} \circ \tilde{f}^- = 0$. Thus we apply Corollary 20.1.8 and conclude that $\tilde{f}^+ = Q\,[\![0]\!]$ and $\tilde{f}^- = (Q - Q^\star)\,[\![0]\!]$, which complete the proof.

The general case: We fix ν as an unit normal to γ. As in the particular case $\hat{\varphi} \equiv 0$, we claim that $\partial_\nu\left(\boldsymbol{\eta} \circ f^+\right) = \partial_\nu\left(\boldsymbol{\eta} \circ f^-\right)$. With this claim, proceeding as in the former case, we can define ζ as in (20.1.3) and conclude that it is a harmonic function. We then define $\left(\tilde{f}^+, \tilde{f}^-\right)$ as in (20.1.4). To this pair, we can apply the former case and conclude the proof of the theorem. To prove the claim, assume by contradiction that, for some $p \in \gamma$, we have that $\partial_\nu(\boldsymbol{\eta} \circ f^+(p)) \neq \partial_\nu\left(\boldsymbol{\eta} \circ f^-\right)(p)$. Without loss of generality we can assume that $p = 0, \hat{\varphi}(0) = 0$ and $D\hat{\varphi}(0) = 0$. Since at least one among $Df^\pm(0)$ does not vanish, we must have

$$\operatorname{Dir}\left(f^+, f^-, \mathrm{B}_\rho(0)\right) \geq \operatorname{Dir}\left(\boldsymbol{\eta} \circ f^+, \boldsymbol{\eta} \circ f^-, \mathrm{B}_\rho(0)\right) \geq c_0 \rho^m, \tag{20.1.5}$$

for some positive constant c_0. It also means that there exist a constant $\eta > 0$ and a sequence $\rho_k \downarrow 0$ such that

$$\operatorname{Dir}\left(f^+, f^-, \mathrm{B}_{\rho_k}(0)\right) \geq \eta\left(\operatorname{Dir}\left(f^+, f^-, \mathrm{B}_{2\rho_k}(0)\right)\right),$$

otherwise we would contradict the lower bound (20.1.5). We see that $f^\pm_{0,\rho_k}$ have finite energy on $\mathrm{B}_2(0)$ and thus there is strong convergence of a subsequence to a $\left(Q - \frac{Q^\star}{2}\right)$ Dir-minimizer (h^+, h^-) with interface $(T_0\gamma, Q^\star\,[\![0]\!])$. The latter must then have Dirichlet energy 1 on $\mathrm{B}_1(0)$. We then have two possibilities:

(I) $\limsup_k (\rho_k)^{-m} \operatorname{Dir}(f^+, f^-, \mathrm{B}_{\rho_k}(0)) = +\infty$. Arguing as in (I) of the former case, this gives that $\boldsymbol{\eta} \circ h^+ = \boldsymbol{\eta} \circ h^- = 0$ and thus we conclude that $\left(h_0^+, h_0^-\right)$ is trivial, which is a contradiction,

(II) $\limsup_k (\rho_k)^{-m} \operatorname{Dir}(f^+, f^-, \mathrm{B}_{\rho_k}(0)) < +\infty$. Assuming in this case that $T_0\gamma = \{x_m = 0\}$, we conclude that $\left(h_0^+, h_0^-\right)$ is a $\left(Q - \frac{Q^\star}{2}\right)$ Dir-minimizer with flat interface $(T_0\gamma, Q^\star\,[\![0]\!])$, but also that $\boldsymbol{\eta} \circ h_0^\pm(x) = C_d \partial_\nu\left(\boldsymbol{\eta} \circ f^\pm\right)(0) x_m$ for some positive constant C_d that in general is not necessarily equal to one, because we use a normalization constant to have $\operatorname{Dir}(h_0^+, h_0^-, \mathrm{B}_1(0)) = 1$. By the particular case $\hat{\varphi} \equiv 0$, we thus conclude that $\partial_\nu\left(\boldsymbol{\eta} \circ f^+\right)(0) = \partial_\nu\left(\boldsymbol{\eta} \circ f^-\right)(0)$.

□

20.2 Harmonic approximations

In this chapter we aim to approximate the area minimizing current T by Q copies of an harmonic function in the right side and $Q - Q^\star$ copies of the same harmonic function in the left side. To this end, we will first approximate the current T by $(Q - \frac{Q^\star}{2})$-Lipschitz functions which, if we do not assume the minimality of T, will not be necessarily minimizers for the Dirichlet energy. Once we consider the minimality condition on T we will be able to upgrade our approximations using the regularity theorem, see Theorem 20.1.4, to furnish the desired harmonic approximations.

For any π, π_0 belonging to $\mathrm{G}_{m,m+n}$, where $\mathrm{G}_{k,l}$ denotes the set of k-dimensional subspaces of $\mathbb{R}^l$, we introduce, for any $p \in \mathbb{R}^{m+n}$ the notation $\mathrm{B}_r(p,\pi)$ for the disks $\mathbf{B}(p,r) \cap (p+\pi)$, if π is omitted, then we assume $\pi = \pi_0 = \mathbb{R}^m \times \{0\}$, and $\mathbf{C}(p,r,\pi)$ for the cylinders $\mathrm{B}_r(p,\pi) + \pi^\perp$, we also fix $\mathbf{C}(x,r) := \mathbf{C}(p,r,\pi_0)$.

Definition 20.2.1. Let $\alpha \in]0,1]$ and integers $m \geq 2$, $Q^\star \geq 1$, and take Γ any $(m-1)$-rectifiable set. Let T be an m-dimensional integral current with boundary $\partial T = Q^\star [\![\Gamma]\!]$ and assume that $p \in \Gamma$. Then

(i) We call the **cylindrical excess relative to the plane** π the quantity

$$\mathbf{E}(T, \mathbf{C}(p,r), \pi) := \frac{1}{\omega_m r^m} \int_{\mathbf{C}(p,r)} \frac{|\vec{T}(x) - \vec{\pi}|^2}{2} \, d\|T\|(x),$$

and the **cylindrical excess** the quantity

$$\mathbf{E}(T, \mathbf{C}(p,r)) := \min\{\mathbf{E}(T, \mathbf{C}(p,r), \pi) : \pi \subset \mathbb{R}^{m+n}\}.$$

(ii) We call the **spherical excess relative to the plane** π the quantity

$$\mathbf{E}(T, \mathbf{B}(p,r), \pi) := \frac{1}{\omega_m r^m} \int_{\mathbf{B}(p,r)} \frac{|\vec{T}(x) - \vec{\pi}|^2}{2} \, d\|T\|(x),$$

and the **spherical excess** the quantity

$$\mathbf{E}(T, \mathbf{B}(p,r)) := \min\{\mathbf{E}(T, \mathbf{B}(p,r), \pi) : \pi \subset \mathbb{R}^{m+n}\}.$$

(iii) We say that the **boundary spherical excess** is the quantity

$$\mathbf{E}^\flat(T, \mathbf{B}(p,r)) := \min\left\{\mathbf{E}(T, \mathbf{B}(p,r), \pi) : \, T_p\Gamma \subset \pi \subset \mathbb{R}^{m+n}\right\}.$$

(iv) The **height of T in a set $G \subset \mathbb{R}^{m+n}$ with respect to a plane** π is defined as

$$\mathbf{h}(T, G, \pi) := \operatorname{diam}(\mathbf{p}_\pi^\perp(\operatorname{spt}(T) \cap G)) = \sup\left\{\left|\mathbf{p}_\pi^\perp(q-p)\right| : q, p \in \operatorname{spt}(T) \cap G\right\},$$

where $\mathbf{p}_\pi^\perp$ denotes the orthogonal projection onto $\pi^\perp$.

(v) If $\operatorname{spt}(T) \subset \mathbf{C}(p,r,\pi)$, we define the **excess measure with respect to** $\mathbf{C}(p,r,\pi)$ as the measure which to each $F \subset \mathrm{B}_r(p,\pi)$ gives

$$\mathbf{e}_T(F) := \frac{1}{2} \int_{F+\pi^\perp} |\vec{T} - \vec{\pi}|^2 d\|T\|.$$

In this subsection we assume that $\pi_0 = \mathbb{R}^m \times \{0\}$ and we use the notation $\mathbf{p}$ and $\mathbf{p}^\perp$ for the orthogonal projections onto π_0 and $\pi_0^\perp$ respectively, whereas $\mathbf{p}_\pi$ and $\mathbf{p}_\pi^\perp$ will denote, respectively, the orthogonal projections onto the plane π and its orthogonal complement $\pi^\perp$. For the remaining part of this work, we will call **dimensional constants** those which depends only on $m, n, Q^\star$ and Q.

Assumption 7. Let $\alpha \in (0,1]$ and integers $m \geq 2$, $Q^\star \geq 1$. Consider Γ a $C^{2,\alpha}$ oriented $(m-1)$-submanifold without boundary. Let T be an m-dimensional integral current in $\mathbf{B}(0,2)$ with boundary $\partial T \llcorner \mathbf{B}(0,2) = Q^\star [\![\Gamma \cap \mathbf{B}(0,2)]\!]$ and assume that $p \in \Gamma$. We also assume $T_p\Gamma = \mathbb{R}^{m-1} \times \{0\} \subset \pi_0, \psi_1 : \mathbb{R}^{m-1} \to \mathbb{R}, \psi : \gamma \subset \mathbb{R}^m \times \{0\} \to \mathbb{R}^n, \psi_2 : \mathbb{R}^{m-1} \to \mathbb{R}^{n+1}, \psi_2(x) = (\psi_1(x), \psi(x, \psi_1(x)))$ with $\mathrm{Gr}\psi_1 = \gamma$ and $\Gamma = \mathrm{Gr}\psi_2$ satisfying the bounds $\|D\psi_2\|_{0,\mathrm{B}_2(0)} \leq c_0$ and $\mathbf{A} := \|A_\Gamma\|_{0,\mathbf{B}(0,2)} \leq c_0$, where A_Γ denotes the second fundamental form of Γ and c_0 is a positive small geometric constant. We assume that

(i) $p \in \Gamma$ is a two-sided collpased point with $Q - \frac{Q^\star}{2} = \Theta^m(T,p)$, for some integers $Q > Q^\star \geq 1$,

(ii) $\gamma = \mathbf{p}(\Gamma)$ divides $\mathrm{B}_{4r}(p) \subset \pi_0$ in two disjoint open sets Ω^+ and Ω^-,

(iii) $\mathbf{p}_\sharp T = Q [\![\Omega^+]\!] + (Q - Q^\star) [\![\Omega^-]\!]$.

Observe that thanks to ((iii)) we have the identities

$$\begin{aligned} \mathbf{E}(T, \mathbf{C}(p,4r)) &= \frac{1}{\omega_m (4r)^m} \left(\|T\| (\mathbf{C}(p,4r)) - \left(Q \left|\Omega^+\right| + (Q-Q^\star)\left|\Omega^-\right| \right) \right), \\ \mathbf{e}_T(F) &= \|T\| (F \times \mathbb{R}^n) - \left(Q \left|\Omega^+ \cap F\right| + (Q-Q^\star)\left|\Omega^- \cap F\right| \right). \end{aligned} \tag{20.2.1}$$

Definition 20.2.2. Given a current T in a cylinder $\mathbf{C}(p,4r,\pi)$, we introduce the noncentered maximal function of $\mathbf{e}_T$ as

$$\mathbf{me}_T(y) := \sup_{y \in \mathrm{B}_s(z,\pi) \subset \mathrm{B}_{4r}(p,\pi)} \frac{\mathbf{e}_T(\mathrm{B}_s(z,\pi))}{\omega_m s^m}.$$

The following theorem allows us to approximate the current by a $(Q - \frac{Q^\star}{2})$-Lipschitz map which coincides with the current in a closed set K which is called the **good set**. Moreover, we prove that the **bad set**, i.e. $\mathrm{B}_{3r}(0) \setminus K$, has small measure. The tricky part of this theorem is to show that we can take such approximation collapsing at the interface. Notice that, no minimality condition are being assumed to prove this result.

Theorem 20.2.3 (Lusin type weak Lipschitz approximation)**.** *There are positive geometric constants $C = C(m,n,Q,Q^\star)$ and $c_0 = c_0(m,n,Q^\star,Q)$ with the following properties. Assume T satisfies Assumption 7 and $\mathbf{E}(T,\mathbf{C}(p,4r)) \leq c_0$. Then, for any $\delta_* \in (0,1)$, there are a closed set $K \subset \mathrm{B}_{3r}(\mathbf{p}(p))$ and a $\left(Q - \frac{Q^\star}{2}\right)$ valued function (u^+, u^-) on $\mathrm{B}_{3r}(\mathbf{p}(p))$ which collapses at the interface $(\gamma, Q^\star [\![\psi]\!])$ satisfying the following properties:*

$$\mathrm{Lip}\left(u^{\pm}\right) \leq C\left(\delta_*^{1/2} + r^{\frac{1}{2}}\mathbf{A}^{\frac{1}{2}}\right), \tag{20.2.2}$$

$$\mathrm{osc}\left(u^{\pm}\right) \leq C\mathbf{h}\left(T, \mathbf{C}(p,4r), \pi_0\right) + Cr\mathbf{E}\left(T, \mathbf{C}(p,4r)\right)^{1/2} + Cr^2\mathbf{A}, \tag{20.2.3}$$

$$K \subset \mathrm{B}_{3r}\left(\mathbf{p}(p)\right) \cap \left\{\mathbf{me}_T \leq \delta_*\right\}, \tag{20.2.4}$$

$$\mathbf{G}_{u^{\pm}} \llcorner \left[\left(K \cap \Omega^{\pm}\right) \times \mathbb{R}^n\right] = T \llcorner \left[\left(K \cap \Omega^{\pm}\right) \times \mathbb{R}^n\right], \tag{20.2.5}$$

$$\left|\mathrm{B}_s\left(\mathbf{p}(p)\right) \setminus K\right| \leq \frac{C}{\delta_*}\mathbf{e}_T\left(\left\{\mathbf{me}_T > \delta_*\right\} \cap \mathrm{B}_{s+r_1 r}\left(\mathbf{p}(p)\right)\right), \quad \forall s \leq \left(3 - r_1\right) r, \tag{20.2.6}$$

$$\frac{\left\|T - \mathbf{G}_{u^+} - \mathbf{G}_{u^-}\right\|\left(\mathbf{C}(p,3r)\right)}{r^m} \leq \frac{C}{\delta_*}\mathbf{E}\left(T, \mathbf{C}(p,4r)\right), \tag{20.2.7}$$

where $r_1 = c_0 \sqrt[m]{\frac{\mathbf{E}(T,\mathbf{C}(p,4r))}{\delta_*}}$.

Proof. The proof of this theorem is a straightforward adaptation of the corresponding statement considering multiplicity $Q^\star = 1$, c.f. [DDHM18, Theorem 5.5], we reiterate that it is not used any minimality condition to prove this weak approximation. □

From now on the approximation of Theorem 20.2.3 is called the **$\delta_*^{\frac{1}{2}}$-approximation of T in $\mathbf{C}(p,3r)$**. If $E := \mathbf{E}\left(T, \mathbf{C}(p,4r)\right)$, actually in the sequel we will choose $\delta_*^{\frac{1}{2}}$ to be E^β for a small suitable constant β. In the following theorem, we will add the minimality condition on T to prove that, if E is taken sufficiently small, then (u^+, u^-) is close to a minimizer of the Dirichlet energy, i.e., a $\left(Q - \frac{Q^\star}{2}\right)$-Dir-minimizer, which collapses at its interface and thus, by Theorem 20.1.4, consists of a single harmonic sheet.

Theorem 20.2.4 (Harmonic approximation)**.** *For every $\eta_* > 0$ and every $\beta \in \left(0, \frac{1}{4m}\right)$ there exist constants $\varepsilon = \varepsilon(m,n,Q^\star,Q,\eta_*,\beta) > 0$ and $C = C(m,n,Q^\star,Q,\eta_*,\beta) > 0$ with the following property. Let T and Γ be as in Assumption 6 with $C_0 = 0$ and under the conditions of Theorem 20.2.3, $E \leq c_0$, let (u^+, u^-) be the E^β-approximation of T in $\mathrm{B}_{3r}\left(\mathbf{p}(p)\right)$ and let K be the good set satisfying all the properties (20.2.2)-(20.2.7). If $E \leq \varepsilon$ and $r\mathbf{A} \leq \varepsilon E^{\frac{1}{2}}$, then*

$$\mathbf{e}_T\left(\mathrm{B}_{5r/2}\left(\mathbf{p}(p)\right) \setminus K\right) \leq \eta_* E, \tag{20.2.8}$$

and

$$\mathrm{Dir}\left(u^+, u^-, \Omega \cap \mathrm{B}_{2r}\left(\mathbf{p}(p)\right) \setminus K\right) \leq C\eta_* E. \tag{20.2.9}$$

Moreover, there exists an harmonic function $h : \mathrm{B}_{2r}\left(\mathbf{p}(p)\right) \to \mathbb{R}^n$ such that $h|_{\{x_m=0\}} \equiv 0$ and *satisfies the following inequalities:*

$$r^{-2}\int_{\mathrm{B}_{2r}(\mathbf{p}(p))\cap\Omega^+}\mathcal{G}\left(u^+,Q\,[\![h]\!]\right)^2+\int_{\mathrm{B}_{2r}(\mathbf{p}(p))\cap\Omega^+}\left(|Du^+|-\sqrt{Q}|Dh|\right)^2\leq\eta_* E r^m, \tag{20.2.10}$$

$$r^{-2}\int_{\mathrm{B}_{2r}(\mathbf{p}(p))\cap\Omega^-}\mathcal{G}\left(u^-,(Q-Q^\star)\,[\![h]\!]\right)^2+\int_{\mathrm{B}_{2r}(\mathbf{p}(p))\cap\Omega^-}\left(|Du^-|-\sqrt{Q-Q^\star}|Dh|\right)^2\leq\eta_* E r^m, \tag{20.2.11}$$

$$\int_{\mathrm{B}_{2r}(\mathbf{p}(p))\cap\Omega^\pm}\left|D\left(\boldsymbol{\eta}\circ u^\pm\right)-Dh\right|^2\leq\eta_* E r^m. \tag{20.2.12}$$

Proof. Without loss of generality we assume that $p=0$, $r=1$, and $\psi(0)=0$.

Proof of (20.2.8) **and** (20.2.9). Firstly we want to note that (20.2.9) is a consequence of (20.2.8). Indeed, since, $\delta_*=E^{2\beta}$, use first (20.2.4), (20.2.6) and (20.2.8) to estimate

$$|\mathrm{B}_2(0)\setminus K|\leq C\eta_* E^{1-2\beta}.$$

Since $\mathrm{Lip}\left(u^\pm\right)\leq CE^{2\beta}$, (20.2.9) follows easily. We now let fixed β,η_*, and we will argue by contradiction. Assuming that the statement is false, we obtain a sequence of area minimizing currents T_k and submanifolds Γ_k as in Assumption 7 satisfying the following properties for all $k\in\mathbb{N}$:

(i) The cylindrical excesses $E_k:=\mathbf{E}\left(T_k,\mathbf{C}(0,4),\pi_0\right)$ satisfy $E_k\leq\frac{1}{k}$,

(ii) Γ_k is the graph of the entire function $\psi_{2k}:\mathbb{R}^{m-1}\to\mathbb{R}^{n+1}$ satisfying the bound

$$\|\psi_{2k}\|_{C^2(\mathrm{B}_8(0))}\leq C\mathbf{A}_k\leq\frac{C}{k}E_k^{1/2}, \tag{20.2.13}$$

(iii) The estimate (20.2.8) fails, i.e., for some positive $c_2>0$,

$$\mathbf{e}_{T_k}\left(\mathrm{B}_{5/2}(0)\setminus K_k\right)>\eta_* E_k=3c_2E_k. \tag{20.2.14}$$

The pair $\left(f_k^+,f_k^-\right)$ are $\left(Q-\frac{Q^\star}{2}\right)$-valued maps defined on $\mathrm{B}_3(0)$ which collapses at its interface $(\gamma_k,Q^\star[\![\psi_k]\!])$ denotes the E_k^β-Lipschitz approximations of the current T_k and K_k the corresponding good set. We denote by $B_{k,r}^\pm$ the domains of the functions $f_k^\pm$ intersected with the ball $\mathrm{B}_r(0)\subset\pi_0$. From the Taylor expansion of the area functional, arguing as in [DS14, Remark 5.5], since $E_k\downarrow 0$, we conclude the following inequalities for every $s\in[5/2,3]$:

$$\begin{aligned}\int_{K_k\cap B_{k,s}^+}\frac{\left|Df_k^+\right|^2}{2}+\int_{K_k\cap B_{k,s}^-}\frac{\left|Df_k^-\right|^2}{2}&\overset{Taylor}{\leq}\left(1+CE_k^{2\beta}\right)\mathbf{e}_{T_k}\left(K_k\cap\mathrm{B}_s(0)\right)\\&\overset{(20.2.14)}{<}\left(1+CE_k^{2\beta}\right)\left(\mathbf{e}_{T_k}\left(\mathrm{B}_s(0)\right)-3c_2E_k\right)\\&\leq\mathbf{e}_{T_k}\left(\mathrm{B}_s(0)\right)-2c_2E_k.\end{aligned} \tag{20.2.15}$$

The last inequality holds when E_k is sufficiently small, i.e., k large enough. The rest of the proof is devoted to show that (20.2.15) contradicts the minimizing property of T_k. We have

$$
\begin{aligned}
\mathrm{Dir}\left(f_k^+, f_k^-, B_{k,3}\right) &\leq \mathrm{Dir}\left(f_k^+, f_k^-, B_{k,3}\cap K_k\right) + \mathrm{Dir}\left(f_k^+, f_k^-, B_{k,3}\setminus K_k\right)\\
&\overset{(20.2.15)}{\leq} \mathbf{e}_{T_k}\left(\mathrm{B}_3\left(0\right)\right) - 2c_2 E_k + \mathrm{Dir}\left(f_k^+, f_k^-, B_{k,3}\setminus K_k\right)\\
&\leq \mathbf{e}_{T_k}\left(\mathrm{B}_3\left(0\right)\right) - 2c_2 E_k + \frac{C}{2} E_k^{1-2\beta+2\beta},
\end{aligned}
\tag{20.2.16}
$$

where in the last inequality we use the fact that $\mathrm{Lip}\left(f_k^{\pm}\right) \leq CE_k^{\beta}$ and $|\mathrm{B}_3\left(0\right)\setminus K_k| \leq CE_k^{1-2\beta}$. Now we define $\left(g_k^+, g_k^-\right)$ as $g_k^{\pm} := E_k^{-\frac{1}{2}} f_k^{\pm}$ which also collapses at the interface $\left(\gamma_k, E_k^{-\frac{1}{2}} Q^\star \left[\!\left[\psi_k\right]\!\right]\right)$. Then if we take γ to be the plane $\{x_m = 0\}\subset \pi_0$, by (20.2.13), we obtain the following convergences

$$
\gamma_k \xrightarrow{C^1} \gamma, \quad \psi_k \xrightarrow{C^1} 0. \tag{20.2.17}
$$

We now want to do an argument based on the interpolation of the sequence $g_k^{\pm}$ using the Interpolation Lemma, c.f. [DDHM18, Lemma 4.9] (we mention that this theorems works line by line in higher multiplicity), but unfortunately they do not have the same interface. To overcome this difficulty, we do the following construction. For each k, we let Φ_k be a diffeomorphism which maps $\mathrm{B}_3\left(0\right)$ onto itself and $\gamma_k \cap \mathrm{B}_3\left(0\right)$ onto $\gamma\cap \mathrm{B}_3\left(0\right)$. By the C^1 convergences above for k large enough, it is not difficult to see that we can assume without loss of generality

$$
\left\|\Phi_k - \mathrm{Id}\right\|_{C^1} \to 0, \quad \Phi_k\left(\partial \mathrm{B}_r\left(0\right)\right) = \partial \mathrm{B}_r\left(0\right), \quad \forall r\in[2,3].
$$

Furthermore, we have that $\left\|\psi_k\circ\Phi_k^{-1}\right\|_{C^1(\gamma)} \to 0$. Now consider, for every $x=(x', x_m)\in\mathbb{R}^m$, then we define $\kappa_k \in C^1(\mathrm{B}_3\left(0\right))$ as follows $\kappa_k(x) = (\psi_k\circ\Phi_k^{-1})(x', 0) + x_m$ then we have that $\kappa_k := \psi_k\circ\Phi_k^{-1}$ on γ and $\|\kappa_k\|_{C^1(\mathrm{B}_3(0))} \to 0$. We fix a measurable selection $f_k^{\pm}(x) = \sum_i \left[\!\left[\left(f_k^{\pm}\right)_i(x)\right]\!\right]$, and we set

$$
\hat{g}_k^{\pm} := \left(g_k^{\pm}\circ\Phi_k^{-1}\right)\oplus(-\kappa_k),
$$

thus $\left(\hat{g}_k^+, \hat{g}_k^-\right)$ are $\left(Q - \frac{Q^\star}{2}\right)$-valued maps which collapses at the same interface $(\gamma, Q^\star\left[\!\left[0\right]\!\right])$ and by straightforward computations

$$
\mathrm{Dir}\left(\hat{g}_k^{\pm}, \Phi_k^{-1}(A)\cap B_{k,3}^{\pm}\right) = (1+o(1))\left(\mathrm{Dir}\left(g_k^+, A\cap B_{k,3}^{\pm}\right) + \mathrm{Dir}\left(g_k^-, A\cap B_{k,3}^{\pm}\right)\right) + o(1), \tag{20.2.18}
$$

for all measurable $A\subset \mathrm{B}_3\left(0\right)$ and $o(1)$ is independent of the set A. From the very definition of $g_k^{\pm}$ and (20.2.16), we conclude that the Dirichlet energy of $\left(\hat{g}_k^+, \hat{g}_k^-\right)$ is uniformly bounded. From this bound and (20.2.17) we may apply the compactness theorem, see [DDHM18, Theorem 4.8], we can find a not relabeled subsequence and a $\left(Q - \frac{Q^\star}{2}\right)$-valued map (g^+, g^-) with interface $(\gamma, Q^\star\left[\!\left[0\right]\!\right])$ such that $\left\|\mathcal{G}\left(\hat{g}_k^{\pm}, g^{\pm}\right)\right\|_{L^2\left(\mathrm{B}_3^{\pm}(0)\right)} \to 0$ and

$$
\mathrm{Dir}\left(g^+, g^-\right) \leq \liminf_{k\to\infty} \mathrm{Dir}\left(\hat{g}_k^+, \hat{g}_k^-\right) \overset{(20.2.18)}{=} \liminf_{k\to\infty} \mathrm{Dir}\left(g_k^+, g_k^-\right).
$$

Moreover, up to extracting a subsequence, we can assume that $\left|D\hat{g}_k^{\pm}\right| \to G^{\pm}$ weakly in $L^2\left(\mathrm{B}_3\left(0\right)\right)$.

Once can then easily check, see for instance the proof of [DS14, Proposition 4.3], that

$$\left|Dg^{\pm}\right| \leq G^{\pm}, \text{ a.e. in } \mathrm{B}_3(0).$$

In particular, since $E_k \to 0$ and it bounds the size of the bad set, we have $|\mathrm{B}_3(0) \setminus K_k| \to 0$, hence for every $s \in (2,3)$:

$$\begin{aligned}\operatorname{Dir}\left(g^{\pm}, \mathrm{B}_s^{\pm}(0)\right) &\leq \liminf_{k\to\infty} \int_{\mathrm{B}_s^{\pm}(0)\cap K_k} \left(G^{\pm}\right)^2 \\ &\leq \liminf_{k\to\infty} \operatorname{Dir}\left(\hat{g}_k^{\pm}, \mathrm{B}_s^{\pm}(0)\cap K_k\right) \\ &\overset{(20.2.18)}{\leq} \liminf_{k\to\infty} \operatorname{Dir}\left(g_k^{\pm}, \mathrm{B}_s^{\pm}(0)\cap \Phi_k^{-1}(K_k)\right) \\ &= \liminf_{k\to\infty} \operatorname{Dir}\left(g_k^{\pm}, \mathrm{B}_s^{\pm}(0)\cap K_k\right).\end{aligned} \tag{20.2.19}$$

Let $\varepsilon > 0$ be a small parameter to be chosen later, we apply [DDHM18, Lemma 5.8] to $(g^+, g^-)|_{\mathrm{B}_3(0)}$ with such an ε to produce a $(Q - \frac{Q^\star}{2})$-Lipschitz multivalued function $(g_\varepsilon^+, g_\varepsilon^-)$ satisfying:

$$\int_{\mathrm{B}_3^{\pm}(0)} \mathcal{G}\left(g^{\pm}, g_\varepsilon^{\pm}\right)^2 + \int_{\mathrm{B}_3^{\pm}(0)} \left(\left|Dg^{\pm}\right| - \left|Dg_\varepsilon^{\pm}\right|\right)^2 + \int_{\mathrm{B}_3^{\pm}(0)} \left|D\left(\boldsymbol{\eta}\circ g^{\pm}\right) - D\left(\boldsymbol{\eta}\circ g_\varepsilon^{\pm}\right)\right|^2 \leq \varepsilon, \tag{20.2.20}$$

$$\int_{\partial\mathrm{B}_3^{\pm}(0)} \mathcal{G}\left(g^{\pm}, g_\varepsilon^{\pm}\right)^2 + \int_{\partial\mathrm{B}_3^{\pm}(0)} \left(\left|Dg^{\pm}\right| - \left|Dg_\varepsilon^{\pm}\right|\right)^2 \leq \varepsilon.$$

Additionally, we would like to interpolate without increasing too much Dirichlet energy in the transition region. To solve this problem, let us define the Radon measures

$$\mu_k(A) = \int_{A\cap \mathrm{B}_3^+(0)} \left|D\hat{g}_k^+\right|^2 + \int_{A\cap \mathrm{B}_3^-(0)} \left|D\hat{g}_k^-\right|^2, \quad A \subset \mathrm{B}_3(0).$$

It is easy to check using (20.2.16) that $\mu_k(A) \leq C$ where C is independent of k and A. So, up to a subsequence, we can assume that $\mu_k \overset{*}{\rightharpoonup} \mu$ for some Radon measure μ. We now choose $r \in (5/2, 3)$ and a subsequence, not relabeled, such that

(A) $\mu(\partial \mathrm{B}_r(0)) = 0$,

(B) $\mathbf{M}\left(\left\langle T_k - \left(\mathbf{G}_{f_k^+} + \mathbf{G}_{f_k^-}\right), |\mathbf{p}|, r\right\rangle\right) \leq C E_k^{1-2\beta}$, where the map $|\mathbf{p}|$ is given by $\pi_0 \times \pi_0^\perp \ni (x,y) \mapsto |x|$.

Indeed, by standard measure theory arguments, (A) is true for all but countably many radii while (B) can be obtained from the estimate (20.2.7) through the slicing theory for currents. In particular, by (A) and the properties of weak convergence of measures, we have

$$\begin{aligned}\limsup_{s\to r} \limsup_{k\to\infty} &\left[\int_{\mathrm{B}_r^+(0)\setminus \mathrm{B}_s^+(0)} \left|D\hat{g}_k^+\right|^2 + \int_{A\cap \mathrm{B}_r^-(0)\setminus \mathrm{B}_s^-(0)} \left|D\hat{g}_k^-\right|^2\right] \\ &\leq \limsup_{s\to r} \mu\left(\overline{\mathrm{B}_r(0)}\setminus \mathrm{B}_s(0)\right) = 0.\end{aligned}$$

Hence, given $r \in (5/2, 3)$ satisfying (A) and (B) above, we can now choose $s \in (5/2, 3)$ such that

$$\limsup_{k\to\infty} \int_{\mathrm{B}_r^+(0)\setminus \mathrm{B}_s^+(0)} \left|D\hat{g}_k^+\right|^2 + \int_{\mathrm{B}_r^-(0)\setminus \mathrm{B}_s^-(0)} \left|D\hat{g}_k^-\right|^2 \le \frac{c_2}{3}. \tag{20.2.21}$$

Finally, as aforementioned, we interpolate the pairs $\left(\hat{g}_k^+, \hat{g}_k^-\right)$ and $(g_\varepsilon^+, g_\varepsilon^-)$ which we can do now because all of them have the same interface $(\gamma, Q^\star [\![0]\!])$ and we have control on their Dirichlet energy. We finally apply, for each k, the interpolation lemma to connect the functions $\left(\hat{g}_k^+, \hat{g}_k^-\right)$ and $(g_\varepsilon^+, g_\varepsilon^-)$ on the annulus $\mathrm{B}_r(0) \setminus \mathrm{B}_s(0)$. This gives sets $\overline{\mathrm{B}_s(0)} \subset V_{\lambda,\varepsilon}^k \subset W_{\lambda,\varepsilon}^k \subset \mathrm{B}_r(0)$ and a $\left(Q - \frac{Q^\star}{2}\right)$ valued interpolation map $\left(\zeta_{k,\varepsilon}^+, \zeta_{k,\varepsilon}^-\right)$ with

$$\begin{aligned}\int_{\left(W_{\lambda,\varepsilon}^k\right)^\pm \setminus V_{\lambda,\varepsilon}^k} \left|D\zeta_{k,\varepsilon}^\pm\right|^2 &\le C\lambda \int_{\left(W_{\lambda,\varepsilon}^k\right)^\pm \setminus V_{\lambda,\varepsilon}^k} \left(\left|D\hat{g}_k^\pm\right|^2 + \left|Dg_\varepsilon^\pm\right|^2\right) + \frac{C}{\lambda} \int_{\left(W_{\lambda,\varepsilon}^k\right)^\pm \setminus V_{\lambda,\varepsilon}^k} \mathcal{G}\left(\hat{g}_k^\pm, g_\varepsilon^\pm\right)^2 \\ &\le C\lambda \int_{\left(W_{\lambda,\varepsilon}^k\right)^\pm \setminus V_{\lambda,\varepsilon}^k} \left(\left|D\hat{g}_k^\pm\right|^2 + \left|Dg_\varepsilon^\pm\right|^2\right) + \frac{C}{\lambda} \int_{\left(W_{\lambda,\varepsilon}^k\right)^\pm \setminus V_{\lambda,\varepsilon}^k} \left(\mathcal{G}\left(\hat{g}_k^\pm, g^\pm\right)^2 + \mathcal{G}\left(g^\pm, g_\varepsilon^\pm\right)^2\right).\end{aligned}$$

Hence

$$\limsup_{\lambda\to 0} \limsup_{\varepsilon\to 0} \limsup_{k\to\infty} \int_{\left(W_{\lambda,\varepsilon}^k\right)^\pm \setminus V_{\lambda,\varepsilon}^k} \left|D\zeta_{k,\varepsilon}^\pm\right|^2 = 0.$$

Thus we can find $\lambda, \varepsilon > 0$ sufficiently small such that

$$\limsup_{k\to\infty} \int_{\left(W_{\lambda,\varepsilon}^k\right)^\pm \setminus V_{\lambda,\varepsilon}^k} \left|D\zeta_{k,\varepsilon}^\pm\right|^2 < \frac{c_2}{3}. \tag{20.2.22}$$

Moreover, up to further reduce ε, by (20.2.20) we can also assume that

$$\int_{\mathrm{B}_r(0)^\pm} \left|Dg_\varepsilon^\pm\right|^2 \le \int_{\mathrm{B}_r(0)^\pm} \left|Dg^\pm\right|^2 + \frac{c_2}{6}. \tag{20.2.23}$$

Now that we have interpolated the functions without adding too much energy, we define the $(Q - \frac{Q^\star}{2})$-Lipschitz function on $\mathrm{B}_r(0)$ with interface $(\gamma, Q^\star [\![0]\!])$ by

$$\hat{h}_{k,\lambda,\varepsilon}^\pm := \begin{cases} \hat{g}_k^\pm & \text{on } \mathrm{B}_r(0) \setminus \left(W_{\lambda,\varepsilon}^k\right)^\pm, \\ \zeta_{k,\varepsilon}^\pm & \text{on } \left(W_{\lambda,\varepsilon}^k\right)^\pm \setminus V_{\lambda,\varepsilon}^k \\ g_\varepsilon^\pm & \text{on } \left(V_{\lambda,\varepsilon}^k\right)^\pm. \end{cases},$$

Let us then consider $\left(Q - \frac{Q^\star}{2}\right)$-valued map $\left(h_{k,\lambda,\varepsilon}^+, h_{k,\lambda,\varepsilon}^-\right)$ defined on $B_{k,3}^\pm$ with interface $(\gamma_k, Q^\star [\![\psi_k]\!])$ given by

$$h_{k,\lambda,\varepsilon}^\pm := \left(\hat{h}_{k,\lambda,\varepsilon}^\pm \circ \Phi_k\right) \oplus (\kappa_k \circ \Phi_k),$$

which satisfies

$$
\begin{aligned}
\liminf_{k\to\infty} \mathrm{Dir}\left(h^+_{k,\lambda,\varepsilon}, h^-_{k,\lambda,\varepsilon}, B_{k,r}\right) &= \liminf_{k\to\infty} \mathrm{Dir}\left(\hat h^+_{k,\lambda,\varepsilon}, \hat h^-_{k,\lambda,\varepsilon}, \mathrm{B}_r(0)\right)\\
&\le \mathrm{Dir}\left(g^+_\varepsilon, g^-_\varepsilon, \mathrm{B}_r(0)\right)\\
&+ \limsup_{k\to\infty} \mathrm{Dir}\left(\zeta^+_{k,\varepsilon}, \zeta^-_{k,\varepsilon}, \left(W^k_{\lambda,\varepsilon}\right)\setminus V^k_{\lambda,\varepsilon}\right)\\
&+ \limsup_{k\to\infty} \mathrm{Dir}\left(\hat g^+_k, \hat g^-_k, \mathrm{B}_r(0)\setminus \mathrm{B}_s(0)\right)\\
&\overset{(20.2.21),(20.2.22),(20.2.23)}{<} \mathrm{Dir}\left(g^+, g^-, \mathrm{B}_r(0)\right) + c_2\\
&\overset{(20.2.19)}{<} \liminf_{k\to\infty}\left(\mathrm{Dir}\left(g^+_k, g^-_k, \mathrm{B}_r(0)\cap K_k\right)\right) + 2c_2.
\end{aligned}
\tag{20.2.24}
$$

Let us consider the function $w^\pm_{k,\lambda,\varepsilon}(x) := E_k^{1/2} h^\pm_{k,\lambda,\varepsilon}(x)$. Observe that, by the constructions of $\hat g^\pm_k$, $w^\pm_{k,\lambda,\varepsilon}|_{\partial \mathrm{B}_r(0)} = f^\pm_k|_{\partial\mathrm{B}_r(0)}$ and $\mathrm{Lip}\left(w^\pm_{k,\lambda,\varepsilon}\right) \le C E_k^\beta$. We are now ready to construct a sequence of competitors one for each T_k which for large k will contradict the almost minimality of the sequence T_k. First of all, by the isoperimetric inequality, [Fed69, Section 4.4.2], there is a current S_k such that

$$
\partial S_{k,r} := \left\langle T_k - \left(\mathbf{G}_{f^+_k} + \mathbf{G}_{f^-_k}\right), |\mathbf{p}|, r\right\rangle \quad \text{and} \quad \mathbf{M}\left(S_{k,r}\right) \le C\left(E_k^{1-2\beta}\right)^{\frac{m}{m-1}} \overset{(\beta<\frac{1}{2m})}{=} o\left(E_k\right).
\tag{20.2.25}
$$

Let $Z_{k,r} := \mathbf{G}_{w^+_k} \llcorner \mathbf{C}(0,r) + \mathbf{G}_{w^-_k} \llcorner \mathbf{C}(0,r) + S_{k,r}$. Since $w^\pm_{k,\lambda,\varepsilon}|_{\partial\mathrm{B}_r(0)} = f^\pm_k|_{\partial\mathrm{B}_r(0)}$, we can see that the boundary of Z_k matches that of $T_k \llcorner \mathbf{C}(0,r)$, thus it is an admissible competitor and will furnishes the desired contradiction. To that end, first we compare the Dirichlet energies of $w^+_{k,\lambda,\varepsilon}$ and f^+_k. To begin with this comparison, we note that, up to a subsequence not relabeled, it holds

$$
\begin{aligned}
\mathrm{Dir}(w^+_{k,\lambda,\varepsilon}, w^-_{k,\lambda,\varepsilon}, B_{k,r}) &= E_k \mathrm{Dir}(h^+_{k,\lambda,\varepsilon}, h^-_{k,\lambda,\varepsilon}, B_{k,r})\\
&\overset{(20.2.24)}{<} E_k \mathrm{Dir}(g^+_k, g^-_k, \mathrm{B}_r(0)\cap K_k) + c_2 E_k\\
&= \mathrm{Dir}(f^+_k, f^-_k, B_{k,r}\cap K_k) + 2c_2 E_k,
\end{aligned}
\tag{20.2.26}
$$

for k large enough. In addition, the latter inequality combined with the second inequality in (20.2.15) implies for k large enough that

$$
\mathrm{Dir}(w^+_{k,\lambda,\varepsilon}, w^-_{k,\lambda,\varepsilon}, B_{k,r}) < \mathbf{e}_{T_k}(\mathrm{B}_r(0)) - c_2 E_k + o(E_k).
\tag{20.2.27}
$$

Finally, we estimate

$$
\begin{aligned}
\mathbf{M}(Z_k) - \mathbf{M}(T_k) &\le \mathbf{M}(\mathbf{G}_{w_k^+} \llcorner \mathbf{C}(0,r)) + \mathbf{M}(\mathbf{G}_{w_k^-} \llcorner \mathbf{C}(0,r)) + \mathbf{M}(S_k) - \mathbf{M}(T_k) \\
&\overset{Taylor}{\le} Q|B^+_{k,r}| + (Q-Q^\star)|B^-_{k,r}| + \mathrm{Dir}(w^+_{k,\lambda,\varepsilon}, w^-_{k,\lambda,\varepsilon}, B_{k,r}) + o(E_k) - \mathbf{M}(T_k) \\
&\le Q|B^+_{k,r}| + (Q-Q^\star)|B^-_{k,r}| + \mathrm{Dir}(w^+_{k,\lambda,\varepsilon}, w^-_{k,\lambda,\varepsilon}, B_{k,r}) + o(E_k) \\
&- Q|B^+_{k,r}| - (Q-Q^\star)|B^-_{k,r}| - \mathbf{e}_{T_k}(\mathrm{B}_r(0)) \\
&\overset{(20.2.27)}{<} -c_2 E_k + o(E_k),
\end{aligned}
\tag{20.2.28}
$$

the expression is negative when k is large enough. In particular Z_k is a competitor with less mass than T_k and this completes the proof of the first part of the theorem.

Proof of (20.2.10)**,** (20.2.11) **and** (20.2.12)**.** As in the first part, we argue by contradiction assuming ((i)), ((ii)), and ((iii)) becomes

(iii)' The E_k^β-Lipschitz approximations (f_k^+, f_k^-) fail to satisfy one among the estimates (20.2.10), (20.2.11) and (20.2.12) for any choice of the function κ.

We use the same notations of the previous step for $\psi_k, g_k^\pm, \Phi_k, \kappa_k, \hat{g}_k^\pm$ and $g^\pm$. Therefore, we now claim that

(A) The convergence of $\hat{g}_k^\pm$ to $g^\pm$ is strong in $W^{1,2}(\mathrm{B}_{5/2}(0))$, namely

$$\lim_{k\to\infty} \mathrm{Dir}(\hat{g}_k^+, \hat{g}_k^-, \mathrm{B}_{5/2}(0)) = \mathrm{Dir}(g^+, g^-, \mathrm{B}_{5/2}(0))\,.$$

(B) (g^+, g^-) is a $(Q - \frac{Q^\star}{2})$-minimizer in $\mathrm{B}_{5/2}(0)$.

Recall that, by Theorem 20.2.3 and the construction, (g^+, g^-) collapses at the interface $(\gamma, Q^\star [\![0]\!])$, thus provided we assume that (A) and (B) are proved, from Theorem 20.1.4 we would then infer the existence of a classical harmonic function $\hat{h}$ on $\mathrm{B}_{5/2}(0)$ which vanishes identically on $\{x_m = 0\}$ such that $g^+ = Q\left[\!\left[\hat{h}\right]\!\right]$ and $g^- = (Q-Q^\star)\left[\!\left[\hat{h}\right]\!\right]$. If we set $h_k = E_k^{1/2}\hat{h}$, the following hold

$$
\begin{aligned}
&\int_{B^+_{k,5/2}} \mathcal{G}(f_k^+, Q[\![h_k]\!])^2 + \int_{B^+_{k,5/2}} \left(|Df_k^+| - \sqrt{Q}|Dh_k|\right)^2 = o(E_k), \\
&\int_{B^-_{k,5/2}} \mathcal{G}(f_k^-, (Q-Q^\star)[\![h_k]\!])^2 + \int_{B^-_{k,5/2}} \left(|Df_k^-| - \sqrt{(Q-Q^\star)}|Dh_k|\right)^2 = o(E_k), \\
&\int_{B^\pm_{k,5/2}} \left|D(\boldsymbol{\eta}\circ f_k^\pm) - Dh_k\right|^2 = o(E_k).
\end{aligned}
$$

But these estimates are incompatible with (iii)' above. Hence, at least one between (A) and (B) needs to fail. As in the previous section we will use this to contradict the minimality of T_k. Note that in both cases there exists a $(Q - \frac{Q^\star}{2})$-valued function $(\bar{g}^+, \bar{g}^-)$ with interface $(\gamma, Q^\star [\![0]\!])$, $\gamma = \{x_m = 0\}$, and a positive constant $c_3 > 0$, such that

$$\mathrm{Dir}(\bar g^+, \bar g^-, \mathrm{B}_s(0)) \le \liminf_{k\to\infty} \mathrm{Dir}(\hat g^+, \hat g^-, \mathrm{B}_s(0)) - 2c_3 \tag{20.2.29}$$

for all $s \in (5/2, 3)$. Indeed this is true with $(\bar g^+, \bar g^-) = (g^+, g^-)$ if (A) fails, while if (B) fails we choose $(\bar g^+, \bar g^-)$ to be a $(Q - \frac{Q^\star}{2})$-minimizer with boundary data $g^\pm$ on $\partial\mathrm{B}_{5/2}(0)$ extended to be equal to $g^\pm$ on $\mathrm{B}_3(0)\setminus\mathrm{B}_{5/2}(0)$. We can now follow the exactly the same argument as in the previous step to find a radius $r \in (5/2, 3)$ and functions $\hat h_k^\pm$ such that

$$\mathbf{M}(\langle T_k - (\mathbf{G}_{f_k^+} + \mathbf{G}_{f_k^-}), |\mathbf{p}|, r\rangle) \le C E_k^{1-2\beta}$$

and, arguing as we have done for (20.2.24),

$$\begin{aligned}\liminf_{k\to\infty} \mathrm{Dir}(h^+, h^-, B_{k,r}) &\le \mathrm{Dir}(\bar g^+, \bar g^-, \mathrm{B}_r(0)) + c_3\\ &\le \liminf_{k\to\infty} \mathrm{Dir}(g^+, g^-, B_{k,r}) - c_3.\end{aligned} \tag{20.2.30}$$

Defining $w_k^\pm$ as above, we again observe that $w_k^\pm|_{\partial\mathrm{B}_r^\pm(0)} = f_k^\pm|_{\partial\mathrm{B}_r^\pm(0)}$. We then construct the same competitor currents to test the minimality of T_k. First we consider a current S_k supported in Σ_k such that

$$\partial S_k = \langle T_k - (\mathbf{G}_{f_k^+} + \mathbf{G}_{f_k^-}), |\mathbf{p}|, r\rangle \text{ and } \mathbf{M}(S_k) \le C(E_k^{1-2\beta})^{\frac{m}{m-1}} = o(E_k)\,, \tag{20.2.31}$$

where we again used $\beta < \frac{1}{4m}$. Then we define, as before, $Z_k := \mathbf{G}_{w_k^+} \llcorner \mathbf{C}(0,r) + \mathbf{G}_{w_k^-} \llcorner \mathbf{C}(0,r) + S_k$, for which the minimality condition guarantees

$$\mathbf{M}(Z_k) \ge \mathbf{M}(T_k \llcorner \mathbf{C}(0,r))\,.$$

Since we proved the first part of the theorem, we use it to show that

$$\mathbf{e}_{T_k}(\mathrm{B}_r(0)) = \mathrm{Dir}(f_k^+, \mathrm{B}_r^+(0)) + \mathrm{Dir}(f_k^-, \mathrm{B}_r^-(0)) + O(\eta_k E_k)\,.$$

Observe that now we can choose $\eta_k \to 0$ as $k \to \infty$. Arguing as in (20.2.26) and relying on (20.2.30) we also have

$$\mathrm{Dir}(w_k^+, w_k^-, \mathrm{B}_r(0)) \le \mathrm{Dir}(f_k^+, f_k^-, \mathrm{B}_r(0)) - c_3 E_k + o(E_k)\,.$$

Accordingly, the latter inequality combined with (20.2.15) implies

$$\mathrm{Dir}(w_k^+, w_k^-, B_{k,r}) < \mathbf{e}_{T_k}(\mathrm{B}_r(0)) - c_3 E_k.$$

As before, see (20.2.28), we complete the proof. $\square$

Chapter 21

Superlinear decays and Lusin type strong Lipschitz approximations

The approximation furnished in the last subsection, Theorem 20.2.3, have sublinear exponents bounding the size of the bad set among other important quantities. In the construction of the center manifold, in order to derive good properties of it we will need superlinear decays in some of the estimates of Theorem 20.2.3. To that end, we need to improve our approximation. In fact, we need an accurate height bound and harmonic approximations to achieve a satisfactory excess decay and therefore provide stronger Lipschitz approximations that will appear in Theorem 21.4.1, c.f. [DDHM18, Chapter 6], with the subtle exponents estimating the error of such approximations. So, as a first step we state the height bound of the current T.

21.1 Height bound

Lemma 21.1.1 (Height bound, Lemma 10.4, [DLNS21])**.** *Let T, $\mathbf{C}(p,4r)$, Γ and $\pi_0 := \mathbb{R}^m \times \{0\}$ be as in Assumption 7 with $C_0 = 0$. Then, there exist positive constants $\varepsilon_h = \varepsilon_h(Q, Q^\star, m, n)$ and $C_h = C_h(Q, Q^\star, m, n)$ such that, if $\mathbf{E}(T, \mathbf{C}(p,4r)) + \mathbf{A} \leq \varepsilon_h$, then*

$$\mathbf{h}(T, \mathbf{C}(p,2r)\,, \pi_0) \leq C_h(r^{-1}\mathbf{E}(T, \mathbf{C}(p,4r)) + \mathbf{A})^{\frac{1}{2}} r^{\frac{3}{2}}.$$

21.2 Improved excess estimate

We will follow a very well known process that using the height bound provided by Lemma 21.1.1 allows to obtain our proof of the improved excess decay. To achieve this result we will firstly prove a milder decay in Lemma 21.2.1 and then iterate it to reach the needed excess superlinear estimate. To prove this milder statement which will be stated for the modified excess function introduced in Definition 20.2.1, we will reduce this milder decay of the current T in some steps until we can rely on a similar decay for harmonic functions, this reduction will be possible thanks to Theorem 20.2.4.

Lemma 21.2.1 (Milder excess decay)**.** *Let T and Γ be as in Assumptions 6 with $C_0 = 0$, $p \in \Gamma \cap U$ is a two-sided collapsed point where U is the neighborhood in Definition 19.3.4. Then, for every $q \in U \cap \Gamma$ and $\varepsilon > 0$, there is an $\varepsilon_0 = \varepsilon_0(\varepsilon, Q, Q^\star, m, n) > 0$ (assume that $\varepsilon_h \leq \varepsilon_0^2$) and a $M_0 =$*

$M_0(\varepsilon, Q, Q^\star, m, n) > 0$ with the following property. We set $\theta(\sigma) := \max\{\mathbf{E}^\flat(T, \mathbf{B}(q,\sigma)), M_0\mathbf{A}^2\sigma^2\}$, and assume

$$\mathbf{A}^2\sigma^2 + E := \|A_\Gamma\|^2\sigma^2 + \mathbf{E}^\flat(T, \mathbf{B}(q, 4\sigma)) < \varepsilon_0, \tag{21.2.1}$$

$$\|T\|(\mathbf{B}(q, 4\sigma)) \leq \left(\Theta^m(T, p) + \frac{1}{4}\right)\omega_m(4\sigma)^m. \tag{21.2.2}$$

Then we have

$$\theta(\sigma) \leq \max\{2^{-4+4\varepsilon}\theta(4\sigma), 2^{-2+2\varepsilon}\theta(2\sigma)\}\,. \tag{21.2.3}$$

This milder statement is not enough for our purposes, since the excess are considered in balls with the same center q. However, it facilitates a lot the proof of the improved excess decay which we enunciate below.

Theorem 21.2.2 (Improved excess decay and height bound)**.** *Let T and Γ be as in Assumption 6 with $C_0 = 0$. If $p \in \Gamma \cap U$ is a two-sided collapsed point with density $\Theta(T, p) = Q - \frac{Q^\star}{2}$, U is a neighborhood of Definition 19.3.4, then there exists $r > 0$ such that $\mathbf{B}(p, r) \subset U$, for all $q \in \mathbf{B}(p, r) \cap U$ there exists a m-dimensional plane $\pi(q)$ which $T_q\Gamma \subset \pi(q)$, and for all $\varepsilon > 0$ there is a constant $C = C(m, n, Q^\star, Q, \varepsilon) > 0$ with*

$$\mathbf{E}^\flat(T, \mathbf{B}(q, \rho)) \leq \mathbf{E}^\flat(T, \mathbf{B}(q, \rho)\,, \pi(q)) \leq C\left(\frac{\rho}{r}\right)^{2-2\varepsilon}\mathbf{E}^\flat(T, \mathbf{B}(p, 2r)) + C\rho^{2-2\varepsilon}r^{2\varepsilon}\mathbf{A}^2, \tag{21.2.4}$$

for all $\rho \in (0, r)$. Moreover, if we take $\rho \in (0, \frac{r}{2\sqrt{2}})$, then

$$\mathbf{h}(T, \mathbf{B}(q, \rho)\,, \pi(q)) \leq C(r^{-1}\mathbf{E}^\flat(T, \mathbf{B}(p, 2r)) + \mathbf{A})^{\frac{1}{2}}\rho^{\frac{3}{2}},\ \forall q \in \Gamma \cap \mathbf{B}(p, r)\,. \tag{21.2.5}$$

Remark 21.2.3. We announce that, in Theorem 21.3.1, we prove that $\pi(q)$ is in fact the support of the unique tangent cone to T at q.

We begin with the proof of the Lemma 21.2.1 which will be used to prove the Theorem 21.2.2.

Proof of Lemma 21.2.1. Without loss of generality by scaling, translating and rotating, we can assume $\sigma = 1$, $q = 0$, $\mathbf{E}^\flat(T, \mathbf{B}(0, 2)) = \mathbf{E}(T, \mathbf{B}(0, 2)\,, \pi_0)$, where $\pi_0 = \mathbb{R}^m \times \{0\}$, and $T_0\Gamma = \mathbb{R}^{m-1} \times \{0\}$. We begin assuming

$$\mathbf{E}^\flat(T, \mathbf{B}(0, 2)) \geq 2^{-m}M_0\mathbf{A}^2 \quad \text{and} \quad \mathbf{E}^\flat(T, \mathbf{B}(0, 2)) \geq 2^{-4-m}\mathbf{E}^\flat(T, \mathbf{B}(0, 4)). \tag{21.2.6}$$

Indeed, note that

$$\theta(1) = \max\{M_0\mathbf{A}^2, \mathbf{E}^\flat(T, \mathbf{B}(0, 1))\} \leq \max\{M_0\mathbf{A}^2, 2^m\mathbf{E}^\flat(T, \mathbf{B}(0, 2))\}\,.$$

So, if the first inequality in (21.2.6) fails, by the latter inequality, we have

$$\theta(1) \leq M_0 \mathbf{A}^2 = 2^{-2}(2^2 M_0 \mathbf{A}^2) \leq 2^{-2}\theta(2)\,,$$

whereas, if the second inequality in (21.2.6) fails, then

$$\theta(1) \leq \max\{M_0 \mathbf{A}^2, 2^{-4}\mathbf{E}^\flat(T, \mathbf{B}(0,4))\} \leq 2^{-4}\theta(4)\,.$$

Hence in both cases the conclusion should hold. Reiterating, under assumption (21.2.6), we need to show the decay estimate:

$$\mathbf{E}^\flat(T, \mathbf{B}(0,1)) \leq 2^{2\varepsilon-2}\mathbf{E}^\flat(T, \mathbf{B}(0,2))\,. \tag{21.2.7}$$

Let us now fix a positive $\eta < 1$, to be chosen sufficiently small later, and consider the cylinder $U_\eta := \mathrm{B}_{4-\eta}(0,\pi_0) + \mathrm{B}_{\sqrt{\eta}}(0,\pi_0^\perp)$, which by abuse of notation we denote by $B_{4-\eta} \times B^n_{\sqrt{\eta}}$. If $\varepsilon_0 > 0$ is sufficiently small, we claim that

$$\mathrm{spt}(T) \cap \partial U_\eta \subset \partial B_{4-\eta} \times B^n_{\sqrt{\eta}} \tag{21.2.8}$$

$$\mathbf{B}(0, 4-\eta) \cap \mathrm{spt}(T) \subset U_\eta\,. \tag{21.2.9}$$

Otherwise, arguing by contradiction, we would have a sequence of currents T_k satisfying the assumptions of the theorem with $\varepsilon_0 = \frac{1}{k}$, but violating either (21.2.8) or (21.2.9). Then T_k would converge, in the sense of currents, to a current T_∞ that is area minimizing whose excess w.r.t. π_0 is identically zero so its support is contained in the plane π_0, $\partial T_\infty = Q^\star [\![T_q\Gamma]\!]$. Thus we are in position to apply the Constancy Lemma, [Fed69, 4.1.17], to asserts

$$T_\infty := Q' [\![B_4^+]\!] + (Q' - Q^\star) [\![B_4^-]\!]\ ,$$

where $B_4^\pm = B_4(0,\pi_0) \cap \{\pm x_m > 0\}$ and $Q' \geq Q^\star$ is a positive integer. Since $\partial T_k = Q^\star [\![\mathbb{R}^{m-1} \times \{0\}]\!]$, $\forall k \in \mathbb{N}$, we can use the area-minimizing property to obtain an uniform bound on $\|T_k\|(B_4)$ and thus be in place to apply [Sim14, Theorem 7.2, Chapter 6] which says that, since T_k converge to T_∞ in the sense of currents, the supports of T_k converge to either $\overline{B}_4$ in case $Q' > Q^\star$ or $\overline{B}_4^+$ otherwise, i.e. if $Q' = Q^\star$, in the Hausdorff sense in every compact subset of $\mathbf{B}(0,4)$. This is a contradiction because the following both inequalities hold

$$\mathrm{dist}_H(\overline{\mathbf{B}(0,4-\eta) \setminus U_\eta}, \mathbf{B}(0,4)) > 0 \text{ and } \mathrm{dist}_H(\partial U_\eta \setminus (\partial B_{4-\eta} \times B^n_{\sqrt{\eta}}), \mathbf{B}(0,4)) > 0.$$

We have therefore proved (21.2.8) and (21.2.9).

We now let T_0 be a tangent cone of T at 0 and $\rho_k \to 0^+$ a sequence such that $T_{0,\rho_k} \to T_0$. By a standard argument using the Constancy Lemma, we know that

$$\mathbf{p}_{\pi_0\sharp} T_0 = Q' [\![\pi_0]\!] + (Q' - Q^\star) [\![\pi_0]\!]\,, \tag{21.2.10}$$

for some natural number Q'. By the lower semicontinuity of the total variation and (21.2.2), we further notice that we necessarily have $\|\mathbf{p}_{\pi_0\sharp} T_0\|(\mathbf{B}(0,4)) \leq (Q - \frac{2Q^\star - 1}{4})\omega_m 4^m$. Hence, by the monotonicity formula

$$\Theta^m(\mathbf{p}_{\pi_0\sharp}T_0, 0) \leq Q - \frac{2Q^\star - 1}{4}. \tag{21.2.11}$$

On the other hand, the upper semicontinuity of the density w.r.t. the convergence of area-minimizing currents [Sim14, Chapter 7, Section 3, Eq. (12)] and the fact that p is a two-sided collapsed point allow us to conclude

$$\Theta^m(\mathbf{p}_{\pi_0\sharp}T_0, 0) \geq \limsup \Theta^m(\mathbf{p}_{\pi_0\sharp}T_{0,\rho_k}, 0) = Q - \frac{Q^\star}{2}. \tag{21.2.12}$$

By equations (21.2.11), (21.2.12), and (21.2.10), we have

$$Q - \frac{2Q^\star - 1}{4} \geq \Theta^m(\mathbf{p}_{\pi_0\sharp}T_0, 0) = Q' - \frac{Q^\star}{2} \geq Q - \frac{Q^\star}{2},$$

since Q' is an integer it turns out that $Q' = Q$. By (21.2.9), we can straightforwardly check that

$$\mathrm{dist}_H(\mathrm{spt}(T), \mathrm{spt}(T_0))) < \eta,$$

which, provided η and ε_0 are small enough, leads to

$$\|T\|(\mathbf{B}(0,r)) = \|T_0\|(\mathbf{B}(0,r)) + O(\eta^{m-1}), \quad \forall r \in (1, 4 - \frac{\eta}{2}).$$

Thus, we can state the following property:

(A) the mass of T in the ball $\mathbf{B}(0,r)$ is $\left(Q - \frac{Q^\star}{2}\right)\omega_m r^m + O(\eta^{m-1})$, for any radius $1 \leq r \leq 4 - \frac{\eta}{2}$.

Next, let us define $S_\eta := T \llcorner U_\eta$. Observe that (21.2.8) and (21.2.9) imply:

(B) $\partial S_\eta \llcorner \mathbf{C}(0, 4-\eta) = Q^\star [\![\Gamma \cap \mathbf{C}(0, 4-\eta)]\!]$;

(C) $T \llcorner \mathbf{B}(0, 4-\eta) = S_\eta \llcorner \mathbf{B}(0, 4-\eta)$.

Choose a plane $\overline{\pi}$ which minimizes the boundary excess, i.e., which contains $T_0\Gamma$ and $\mathbf{E}(T, \mathbf{B}(0,4), \overline{\pi}) = \mathbf{E}^\flat(T, \mathbf{B}(0,4))$. Let us observe that, since π_0 is the optimal plane for $\mathbf{E}^\flat(T, \mathbf{B}(0,2))$, we have

$$\begin{aligned} |\overline{\pi} - \pi_0|^2 \|T\|(\mathbf{B}(0,2)) &= \int_{\mathbf{B}(0,2)} |\overline{\pi} - \pi_0|^2 \, d\|T\| \\ &\leq 2\int_{\mathbf{B}(0,2)} |\vec{T} - \pi_0|^2 \, d\|T\| + 2\int_{\mathbf{B}(0,2)} |\vec{T} - \overline{\pi}|^2 \, d\|T\| \\ &\leq 2 \cdot 2^m \omega_m \mathbf{E}^\flat(T, \mathbf{B}(0,2)) + 2 \cdot 4^m \omega_m \mathbf{E}^\flat(T, \mathbf{B}(0,4)) \\ &\leq C\mathbf{E}^\flat(T, \mathbf{B}(0,4)). \end{aligned} \tag{21.2.13}$$

Moreover,

$$
\begin{aligned}
\mathbf{E}(S_\eta, \mathbf{C}(0,4-\eta)) &\leq \mathbf{E}(T, \mathbf{B}(0, 4-{}^{\eta}\!/\!_{2}), \pi_0) \\
&\overset{\text{triangular}}{\leq} \frac{2}{\omega_m (4-{}^{\eta}\!/\!_{2})^m} \mathbf{E}^\flat(T, \mathbf{B}(0,4)) \\
&+ \frac{2}{\omega_m(4-\frac{\eta}{2})^m} |\overline{\pi} - \pi_0|^2 \|T\|(\mathbf{B}(0,4-{}^{\eta}\!/\!_{2})) \\
&\overset{\text{(A)}}{\leq} 2\mathbf{E}^\flat(T,\mathbf{B}(0,4)) + C|\overline{\pi}-\pi_0|^2\|T\|(\mathbf{B}(0,2)) \\
&\overset{(21.2.13)}{\leq} C\mathbf{E}^\flat(T,\mathbf{B}(0,4)),
\end{aligned}
\tag{21.2.14}
$$

Moreover, recalling that $\mathbf{p} : \mathbb{R}^{m+n} \to \pi_0$ is the orthogonal projection, by the Constancy Lemma, [Fed69, 4.1.17],

(D) $\mathbf{p}_\sharp S_\eta = Q_\mathbf{p} [\![\Omega^+]\!] + (Q_\mathbf{p} - Q^\star)[\![\Omega^-]\!]$, where $Q_\mathbf{p}$ is a positive integer and $\Omega^\pm$ are the regions in which $\mathrm{B}_4(0,\pi_0)$ is divided by $\mathbf{p}(\Gamma)$; in particular

$$
\partial [\![\Omega^+]\!] \llcorner \mathbf{C}(0,4-\eta) = -\partial [\![\Omega^-]\!] \llcorner \mathbf{C}(0,4-\eta) = \mathbf{p}_\sharp [\![\Gamma]\!] \llcorner \mathbf{C}(0,4-\eta)\,.
$$

Since $S_\eta = T \llcorner U_\eta$ and $U_\eta \subset \mathbf{B}(0,4-{}^{\eta}\!/\!_{2})$, clearly

$$
\|S_\eta\|(\mathbf{C}(0,4-\eta)) \leq \|T\|(\mathbf{B}(0,4-{}^{\eta}\!/\!_{2})). \tag{21.2.15}
$$

Since projections do not increase mass, we obtain

$$
\|S_\eta\|(\mathbf{C}(0,4-\eta)) \geq \|\mathbf{p}_\sharp S_\eta\|(\mathbf{C}(0,4-\eta)). \tag{21.2.16}
$$

Assuming that the constant ε_0 in the assumption of the theorem is sufficiently small, we conclude that $\mathbf{p}_\sharp [\![\Gamma]\!] \llcorner \mathbf{C}(0,4-\eta)$ is close to $T_0\Gamma = \mathbb{R}^{m-1}\times\{0\}$. In particular, $|\Omega^\pm|$ is close to $|\mathrm{B}^\pm_{4-\eta}(0)|$ and thus $Q_\mathbf{p}|\Omega^+| + (Q_\mathbf{p}-Q^\star)|\Omega^-|$ is close to $(Q_\mathbf{p} - \frac{Q^\star}{2})\omega_m(4-\eta)^m$ too. Therefore, if ε_0 is smaller than a geometric constant, we infer from (21.2.16) that

$$
\|S_\eta\|(\mathbf{C}(0,4-\eta)) \geq (Q_\mathbf{p} - \frac{2Q^\star+1}{4})\omega_m(4-\eta)^m.
$$

In addition, by (A), a sufficiently small ε_0 imply

$$
\begin{aligned}
(Q_\mathbf{p} - \frac{2Q^\star+1}{4})\omega_m(4-\eta)^m &\overset{(21.2.16)}{\leq} \|S_\eta\|(\mathbf{C}(0,4-\eta)) \\
&\overset{(21.2.15)}{\leq} \|T\|(\mathbf{B}(0,4-\eta/2)) \\
&\overset{\text{(A)}}{\leq} (Q - \frac{2Q^\star-1}{4})\omega_m(4-\frac{\eta}{2})^m,
\end{aligned}
$$

we achieve that $Q_\mathbf{p} \leq Q$ provided η is chosen smaller than a geometric constant. On the other hand,

$$
\|S_\eta\|(\mathbf{C}(0,4-\eta)) \leq Q_\mathbf{p}|\Omega^+| + (Q_\mathbf{p}-Q^\star)|\Omega^-| + \mathbf{E}(S_\eta, \mathbf{C}(0,4-\eta))\,.
$$

Using (21.2.14) and the argument above, if ε_0 is sufficiently small we get $\|S_\eta\|(\mathbf{C}(0,4-\eta)) \leq (Q_{\mathbf{p}} - \frac{2Q^\star-1}{4})\omega_m(4-\eta)^m$. Recall that (C) ensures that $\|T\|(\mathbf{B}(0,4-\eta)) \leq \|S_\eta\|(\mathbf{C}(0,4-\eta))$, and, using (A), we also have $\|T\|(\mathbf{B}(0,4-\eta)) \geq (Q - \frac{2Q^\star+1}{4})(4-\eta)^m$. Thus necessarily $Q_{\mathbf{p}} \geq Q$, consequently, we have $Q_{\mathbf{p}} = Q$.

Next, since $T\llcorner\mathbf{B}(0,2) = S_\eta\llcorner\mathbf{B}(0,2)$, then

$$\begin{aligned}\mathbf{A}^2 &\overset{(21.2.6)}{\leq} 2^m M_0^{-1}\mathbf{E}^\flat(T,\mathbf{B}(0,2))\\ &\leq 2^m\left(\frac{4-\eta}{2}\right)^m M_0^{-1}\mathbf{E}(S_\eta,\mathbf{C}(0,4-\eta))\\ &\overset{(21.2.14)}{\leq} CM_0^{-1}\mathbf{E}^\flat(T,\mathbf{B}(0,4))\,.\end{aligned}$$

By the last inequality and $Q_{\mathbf{p}} = Q$, we finally proved that we are in position to apply Theorem 20.2.4 with $\beta = \frac{1}{5m}$ and a sufficiently small parameter η_* to be chosen later, provided ε_0 is sufficiently small and M_0 is sufficiently large.

Reduction to excess decay for graphs

From now on we let (u^+,u^-) and h be as in Theorem 20.2.4. In particular, recall that (u^+,u^-) is the E^β-approximation of Theorem 20.2.3 and h is a single-valued harmonic function. Moreover, denote by E the cylindrical excess $\mathbf{E}(S_\eta,\mathbf{C}(0,4-\eta))$ and record the estimates:

$$\mathbf{A}^2 \leq C_0 M_0^{-1}E \quad\text{and}\quad E \leq C_0\mathbf{E}^\flat(T,\mathbf{B}(0,2)), \tag{21.2.17}$$

where C_0 is a geometric constant and the second inequality follows by combining (21.2.14) and (21.2.6). Next, define π to be the plane given by the graph of the linear function $x \mapsto (Dh(0)x, 0)$. Since, by the Schwarz reflection principle and the unique continuation for harmonic functions, we obtain that h is odd, and $h(x',0) = 0$, so, we have that

$$\pi \supset T_0\Gamma = \mathbb{R}^{m-1}\times\{0\}.$$

Moreover, by elliptic estimates,

$$|\pi| \leq C|Dh(0)| \leq (C\mathrm{Dir}(h, \mathrm{B}_{\frac{5}{2}(4-\eta)}(0)))^{\frac12} \overset{Thm. 20.2.4}{\leq} CE^{\frac12}. \tag{21.2.18}$$

Fix $\overline{\eta}$ to be chosen later. The following inequality is a consequence of the reduction argument given in [DDHM18, Theorem 6.8] where the authors reduce the whole discussion to the analysis of a decay for classical harmonic functions using Theorem 20.2.4

$$\mathbf{E}(\mathbf{G}_{u^+}+\mathbf{G}_{u^-},\mathbf{C}(0,1)\,,\pi) \leq (2-\overline{\eta})^{-(2-\varepsilon)}\mathbf{E}(\mathbf{G}_{u^+}+\mathbf{G}_{u^-},\mathbf{C}(0,2-\overline{\eta})) + C\overline{\eta}E\,. \tag{21.2.19}$$

Now, we claim that this inequality allows us to conclude (21.2.7). First of all, by the Taylor expansion of the mass of a Lipschitz graph, [DS15, Corollary 3.3], and the bound on Dirichlet energy of $u^\pm$ on the bad set, we conclude

$$\begin{aligned}\mathbf{E}(\mathbf{G}_{u^+}+\mathbf{G}_{u^-},\mathbf{C}(0,2-\overline{\eta}))&\leq \mathbf{E}(S_\eta,\mathbf{C}(0,2-\overline{\eta}))+C\int_{\Omega^+\setminus K}|Du^+|^2+C\int_{\Omega^-\setminus K}|Du^-|^2\\&\overset{(20.2.9)}{\leq}\mathbf{E}(S_\eta,\mathbf{C}(0,2-\overline{\eta}))+C\eta_*E.\end{aligned}\tag{21.2.20}$$

In second place, we have

$$\begin{aligned}\mathbf{E}(T,\mathbf{B}(0,1)\,,\pi)&\leq \mathbf{E}(S_\eta,\mathbf{C}(0,1)\,,\pi)\\&\leq \mathbf{E}(\mathbf{G}_{u^+}+\mathbf{G}_{u^-},\mathbf{C}(0,1)\,,\pi)+2\mathbf{e}_T(\mathrm{B}_1\,(0)\setminus K)+2|\pi|^2|\mathrm{B}_1\,(0)\setminus K|\\&\overset{(20.2.8)}{\leq}\mathbf{E}(\mathbf{G}_{u^+}+\mathbf{G}_{u^-},\mathbf{C}(0,1)\,,\pi)+C\eta_*E+2|\pi|^2|\mathrm{B}_1\,(0)\setminus K|\\&\overset{(21.2.18),(21.2.19),(21.2.20)}{\leq}(2-\overline{\eta})^{2-\varepsilon}\mathbf{E}(S_\eta,\mathbf{C}(0,2-\overline{\eta}))+C\eta_*E+C\overline{\eta}E.\end{aligned}\tag{21.2.21}$$

Using the height bound in Theorem 21.1.1, for $\varepsilon<\varepsilon_0$ sufficiently small, we have

$$\mathbf{h}(T,\mathbf{C}(0,2-\overline{\eta})\,,\pi_0)\leq C_h\left(\frac{\mathbf{E}(T,\mathbf{C}(0,4-2\overline{\eta}))}{1-\frac{\overline{\eta}}{2}}+\mathbf{A}\right)^{\frac{1}{2}}(2-\overline{\eta})^{\frac{3}{2}},$$

and thus

$$\operatorname{spt}(T)\cap\mathbf{C}(0,2-\overline{\eta})\subset\mathbf{B}(0,2)\;.\tag{21.2.22}$$

Since $S_\eta\llcorner\mathbf{B}(0,2)=T\llcorner\mathbf{B}(0,2)$, we obtain that

$$\begin{aligned}\mathbf{E}^\flat(T,\mathbf{B}(0,1))&\leq\mathbf{E}(T,\mathbf{B}(0,1)\,,\pi)\\&\overset{(21.2.21),(21.2.22)}{\leq}(2-\overline{\eta})^{-(2-\varepsilon)}\left(\frac{2}{2-\overline{\eta}}\right)^m\mathbf{E}(T,\mathbf{B}(0,2)\,,\pi)+C\eta_*E+\overline{\eta}E\\&=(2-\overline{\eta})^{-(2-\varepsilon)}\left(\frac{2}{2-\overline{\eta}}\right)^m\mathbf{E}^\flat(T,\mathbf{B}(0,2))+C\eta_*E+\overline{\eta}E\\&\overset{(21.2.17)}{\leq}\left[(2-\overline{\eta})^{-(2-\varepsilon)}\left(\frac{2}{2-\overline{\eta}}\right)^m+C(\eta_*+\overline{\eta})\right]\mathbf{E}^\flat(T,\mathbf{B}(0,2)).\end{aligned}$$

Hence, since the constant C in the last inequality is independent of the parameters $\eta_*,\overline{\eta}$, choosing the latter sufficiently small, we conclude (21.2.7). □

Proof of Theorem 21.2.2. Firstly, we want to prove that the assumptions (21.2.1) and (21.2.2) of Lemma 21.2.1 are satisfied. To this end, we notice that, since p is a two-sided collapsed point, by Definition 19.3.4, for every $\delta>0$ there exists $\bar\rho=\bar\rho(\delta)$ small such that

(*i*) $\mathbf{E}^\flat(T,\mathbf{B}(p,2\sigma))+4\mathbf{A}\sigma^2\leq\delta$ for every $\sigma\leq\bar\rho$;

(*ii*) $\Theta(T,q)\geq\Theta(T,p)=Q-\frac{Q^\star}{2}$ for all $q\in\Gamma\cap\mathbf{B}(p,2\bar\rho)$.

Next, since $\Theta(T,p)=Q-\frac{Q^\star}{2}$, if the radius $\bar\rho$ is chosen small enough we can assume that

$$\|T\|(\mathbf{B}(p,4\bar\rho)) \le \omega_m \left(Q - \frac{Q^\star}{2} + \frac{1}{8}\right)(4\bar\rho)^m .$$

By a simple comparison, for η sufficiently small, if $q \in \mathbf{B}(p,\eta)\cap\Gamma$ and $\bar\rho' = \bar\rho - \eta$, then

$$\begin{aligned}\|T\|(\mathbf{B}\big(p,4\bar\rho'\big)) &\le \|T\|(\mathbf{B}(p,4\bar\rho))\\ &\le \omega_m \left(Q - \frac{Q^\star}{2} + \frac{1}{8}\right)(4\bar\rho)^m\\ &\le \omega_m \left(Q - \frac{Q^\star}{2} + \frac{3}{16}\right)(4\bar\rho')^m .\end{aligned}$$

Next, by the latter inequality and by the monotonicity formula

$$\begin{aligned}\sigma^{-m}\|T\|(\mathbf{B}(q,\sigma)) &\le e^{\mathbf{A}(4\bar\rho'-\sigma)}(4\bar\rho')^{-m}\|T\|(\mathbf{B}\big(q,4\bar\rho'\big))\\ &\le e^{\mathbf{A}(4\bar\rho'-\sigma)}\omega_m \left(Q - \frac{Q^\star}{2} + \frac{3}{16}\right)\\ &\le e^{4\mathbf{A}\bar\rho}\omega_m \left(Q - \frac{Q^\star}{2} + \frac{3}{16}\right),\end{aligned}$$

for all $\sigma \le 4\bar\rho'$. In particular, if $\bar\rho$ is chosen sufficiently small, by the last inequality we then conclude

$$\|T\|(\mathbf{B}(q,\sigma)) \le \omega_m \left(Q - \frac{Q^\star}{2} + \frac{1}{4}\right)\sigma^m, \qquad \forall q \in \mathbf{B}(p,\eta)\cap\Gamma \text{ and } \forall \sigma \le 4\bar\rho' . \tag{21.2.23}$$

So, the density of T at q is bounded above by (21.2.23) and below by (ii). Set now $r := \min\{\eta, \bar\rho'\}$. For all points q in $\mathbf{B}(p,r)\cap\Gamma$ we claim that

$$\mathbf{E}^\flat(T,\mathbf{B}(q,r)) \le 2^m\mathbf{E}^\flat(T,\mathbf{B}(p,2r)) + C\mathbf{A}^2r^2 \overset{(i)}{\le} C\delta. \tag{21.2.24}$$

Indeed let π be a plane for which $\mathbf{E}^\flat(T,\mathbf{B}(p,2r)) = \mathbf{E}(T,\mathbf{B}(p,2r),\pi)$. By the regularity of Γ, we find a plane π_q such that $|\pi - \pi_q| \le Cr\mathbf{A}$ and $T_q\Gamma \subset \pi_q$. Then we can estimate

$$\begin{aligned}\mathbf{E}^\flat(T,\mathbf{B}(q,r)) &\le \mathbf{E}(T,\mathbf{B}(q,r),\pi_q) \le C\mathbf{E}(T,\mathbf{B}(p,2r),\pi_q)\\ &\overset{\text{triangle inequality}}{\le} C\mathbf{E}^\flat(T,\mathbf{B}(p,2r)) + Cr^2\mathbf{A}^2 \overset{(i)}{\le} C\delta .\end{aligned} \tag{21.2.25}$$

We will now show that the conclusions of the theorem hold for this particular radius r which, without loss of generality, we assume to be $r = 1$ and we also assume $p = 0$. So, we have proved that we are under the assumptions of Lemma 21.2.1, in fact, (21.2.25) and (21.2.23) ensure the following properties for every $q \in \mathbf{B}(0,1)\cap\Gamma$

(A) $\mathbf{E}^\flat(T,\mathbf{B}(q,1)) + \mathbf{A}^2 \le C\mathbf{E}^\flat(T,\mathbf{B}(0,2)) + C\mathbf{A}^2 \le C\delta$,

(B) $\|T\|(\mathbf{B}(q,s)) \le (Q - \frac{2Q^\star-1}{4})\omega_m s^m$ for every $s \le 1$.

We now fix any point $q \in \Gamma \cap \mathbf{B}(0,1)$ and define $\mathfrak{m}(s) := \mathbf{E}^\flat(T, \mathbf{B}(q,s))$. We claim that

$$\mathfrak{m}(s) \leq Cs^{2-2\varepsilon} \max\{\mathfrak{m}(\tfrac{1}{4}), \mathfrak{m}(\tfrac{1}{2})\} + Cs^2\mathbf{A}^2, \qquad \forall s \in (0, \frac{1}{2}). \tag{21.2.26}$$

In order to prove (21.2.26), we firstly prove for $s = 2^{-k-1}$ and for all $k \in \mathbb{N}$ that

$$\mathfrak{m}(2^{-k-1}) \leq C \max\{2^{(2\epsilon-2)k}\mathfrak{m}(\frac{1}{4}), 2^{(2\epsilon-2)k+2}\mathfrak{m}(\frac{1}{2})\} + C2^{-2k-4}\mathbf{A} \tag{21.2.27}$$

is valid and then we will show how to derive (21.2.26) from (21.2.27). The proof of (21.2.27) will be done by induction. Notice that inequality (21.2.27) is trivially true for $k = 0$, indeed,

$$\mathfrak{m}(\frac{1}{2}) \leq 2^2\mathfrak{m}(\frac{1}{2}) \leq \max\{\mathfrak{m}(\frac{1}{4}), 2^2\mathfrak{m}(\frac{1}{2})\}.$$

If the inequality is true for $k_0 \geq 0$, we want to show it for $k = k_0 + 1$. We set $\sigma = 2^{-k-2}$ and notice that, by inductive assumption, we conclude that

$$\begin{aligned} \mathfrak{m}(2^{-k-1}) \leq \mathfrak{m}(4\sigma) = \mathfrak{m}(2^{-k_0-1}) &\leq \max\{2^{(2\varepsilon-2)k_0}\mathfrak{m}(\frac{1}{4}), 2^{(2\varepsilon-2)k_0+2}\mathfrak{m}(\frac{1}{2})\} \\ &\leq \max\{\mathfrak{m}(\tfrac{1}{4}), \mathfrak{m}(\tfrac{1}{2})\} \leq \mathfrak{m}(1) \overset{(A)}{\leq} C\delta. \end{aligned} \tag{21.2.28}$$

Hence, provided we choose $\delta = \delta(m, n, Q^\star, Q) > 0$ small in (A) and consequently r is sufficiently small too, we are in position to apply Lemma 21.2.1 which assures that

$$\begin{aligned} \mathfrak{m}(2^{-(k+1)-1}) = \mathfrak{m}(\sigma) &\overset{(21.2.3)}{\leq} \max\{2^{-2+2\varepsilon}\theta(2\sigma), 2^{-4+4\varepsilon}\theta(4\sigma)\} \\ &\leq C2^{-4+4\varepsilon}\theta(4\sigma) \\ &= C2^{-4+4\varepsilon} \max\{\mathfrak{m}(4\sigma), M_0\mathbf{A}^2(4\sigma)^2\} \\ &\overset{(21.2.28)}{\leq} C \max\{2^{(2\varepsilon-2)(k_0+1)}\mathfrak{m}(\frac{1}{4}), 2^{(2\varepsilon-2)(k_0+1)+2}\mathfrak{m}(\frac{1}{2})\} \\ &+ M_0\mathbf{A}^2\sigma^2, \end{aligned}$$

where we recall that C and M_0 are both constants that depends on $m, n, Q^\star, Q$ and ε, which finishes our induction steps and proves (21.2.27). To prove (21.2.26), we take $s \in (0, \frac{1}{2})$ and $k_s \in \mathbb{N}$ such that $s \in (2^{-k_s-2}, 2^{-k_s-1})$, hence, by (21.2.27),

$$\mathfrak{m}(s) \leq \mathfrak{m}(2^{-k_s-1}) \leq C \max\{2^{(2\varepsilon-2)k_s}\mathfrak{m}(\frac{1}{4}), 2^{(2\varepsilon-2)k_s+2}\mathfrak{m}(\frac{1}{2})\} + C2^{-2k_s-4}\mathbf{A}^2,$$

taking into account in the last inequality that $s^{2-2\varepsilon} > 4^{2-2\varepsilon} \cdot 2^{(2\varepsilon-2)k_s}$, we finish the proof of (21.2.26). We then conclude, for $\rho \in (0, \frac{1}{2})$, that the following equation holds

$$\begin{aligned} \mathbf{E}(T, \mathbf{B}(q,\rho)) \leq \mathbf{E}^\flat(T, \mathbf{B}(q,\rho)) &\overset{(21.2.26)}{\leq} C\rho^{2-2\varepsilon} \max\{\mathfrak{m}(\frac{1}{2}), \mathfrak{m}(\frac{1}{4})\} + C\rho^2\mathbf{A}^2 \\ &\leq C\rho^{2-2\varepsilon}\mathbf{E}^\flat(T, \mathbf{B}(q,1)) + C\rho^2\mathbf{A}^2 \\ &\overset{(A)}{\leq} C\rho^{2-2\varepsilon}\mathbf{E}^\flat(T, \mathbf{B}(0,2)) + C\rho^{2-2\varepsilon}\mathbf{A}^2. \end{aligned} \tag{21.2.29}$$

Furthermore, by (A), the estimate is trivial for $\frac{1}{2} \le \rho < 1$. For $0 < t < s < 1$, define $\pi(q,s)$ and $\pi(q,t)$ to be the optimal planes for $\mathbf{E}^\flat(T, \mathbf{B}(q,t))$ and $\mathbf{E}^\flat(T, \mathbf{B}(q,s))$, respectively. So, (21.2.29) implies

$$\begin{aligned}
|\pi(q,s) - \pi(q,t)|^2 &= \frac{1}{\|T\|(\mathbf{B}(q,s))} \int_{\mathbf{B}(q,s)} |\pi(q,t) - \pi(q,s)|^2 d\|T\| \\
&\overset{(21.2.23)}{\le} C\mathbf{E}(T, \mathbf{B}(q,s)\,, \pi(q,s)) + C\mathbf{E}(T, \mathbf{B}(q,t)), \pi(q,t)) \\
&\overset{(21.2.29)}{\le} Cs^{2-2\varepsilon}\mathbf{E}^\flat(T, \mathbf{B}(0,2)) + Cs^{2-2\varepsilon}\mathbf{A}^2.
\end{aligned}$$

Letting t goes to 0 in the last equations and thanks to the compactness of $G_m(\mathbb{R}^{m+n})$, we obtain the existence of a limit $\pi(q)$ such that

$$|\pi(q) - \pi(q,\rho)|^2 \le C\rho^{2-2\varepsilon}\mathbf{E}^\flat(T, \mathbf{B}(0,2)) + C\rho^{2-2\varepsilon}\mathbf{A}^2 \quad , \forall\, \rho < 1\,. \tag{21.2.30}$$

Hence, for all $\rho \in (0,1)$, we conclude that

$$\begin{aligned}
\mathbf{E}^\flat(T, \mathbf{B}(q,\rho)) &\le \mathbf{E}^\flat(T, \mathbf{B}(q,\rho)\,, \pi(q)) \\
&\overset{\text{triangular}}{\le} C\mathbf{E}^\flat(T, \mathbf{B}(q,\rho)\,, \pi(q,\rho)) + C|\pi(q,\rho) - \pi(q)|^2 \\
&= C\mathbf{E}^\flat(T, \mathbf{B}(q,\rho)) + C|\pi(q,\rho) - \pi(q)|^2 \\
&\overset{(21.2.29),(21.2.30)}{\le} Cs^{2-2\varepsilon}\mathbf{E}^\flat(T, \mathbf{B}(0,2)) + Cs^{2-2\varepsilon}\mathbf{A}^2.
\end{aligned} \tag{21.2.31}$$

which concludes the proof of (21.2.4).

We now turn to (21.2.5), let

$$S_\rho = T \llcorner \Big(B_\rho(q, \pi(q)) \times B^n_\rho(q, \pi(q)^\perp)\Big)\,.$$

Hence, we immediately have $T \llcorner \mathbf{B}(q,\rho) = S_\rho \llcorner \mathbf{B}(q,\rho)$. Moreover, arguing as in (B), (C) and (D) in the proof of Lemma 21.2.1, we are under Assumption 7, thus we can apply the Height bound (Lemma 21.1.1) to obtain

$$\mathbf{h}(S_\rho, \mathbf{C}(q,\rho)\,, \pi(q)) \le C_h \left(\rho^{-1}\mathbf{E}(S_\rho, \mathbf{C}(q,2\rho)\,, \pi(q)) + \mathbf{A}\right)^{\frac{1}{2}} \rho^{\frac{3}{2}}, \quad \forall \rho \in (0, \frac{1}{2}). \tag{21.2.32}$$

As in (21.2.14), we obtain that

$$\mathbf{E}(S_\rho, \mathbf{C}(q,\rho)\,, \pi(q)) \le C\mathbf{E}^\flat(T, \mathbf{B}\Big(q, \sqrt{2}\rho\Big)\,, \pi(q)), \quad \forall \rho \in (0, \frac{1}{\sqrt{2}}). \tag{21.2.33}$$

We are ready to conclude (21.2.5) as follows, for every $\rho \in (0, \frac{1}{2\sqrt{2}})$,

$$\begin{aligned}
\mathbf{h}(T, \mathbf{B}(q,\rho)\,, \pi(q)) &= \mathbf{h}(S_\rho, \mathbf{B}(q,\rho)\,, \pi(q)) \\
&\overset{(21.2.32)}{\le} C_h \left(\rho^{-1}\mathbf{E}(T, \mathbf{C}(q,2\rho)\,, \pi(q)) + \mathbf{A}\right)^{\frac{1}{2}} \rho^{\frac{3}{2}} \\
&\overset{(21.2.33)}{\le} C_h \left(\rho^{-1}\mathbf{E}(T, \mathbf{C}\Big(q, 2\sqrt{2}\rho\Big)\,, \pi(q)) + \mathbf{A}\right)^{\frac{1}{2}} \rho^{\frac{3}{2}},
\end{aligned} \tag{21.2.34}$$

it is sufficient to apply the improved excess decay, (21.2.4), to conclude the proof. □

21.3 Uniqueness of tangent cones at two-sided collapsed points

In the spirit of Theorem 19.3.2 and Lemma 19.3.3 which state the uniqueness of tangent cones and the Holder continuity of the map $q \mapsto T_q$ for (C_0, r_0, α_0)-almost area minimizing currents of dimension 2, we prove the uniqueness of tangent and the Holder continuity of the same map for area minimizing currents of arbitrary dimension m. We state below the analogous of [DDHM18, Theorem 6.3] when the boundary is taken with multiplicity $Q^\star$.

Theorem 21.3.1 (Uniqueness of tangent cones at two-sided collapsed points)**.** *Let T, p, U and r be as in Theorem 21.2.2. Then for all $q \in \mathbf{B}(p, r) \subset U$, we have that q is a two-sided collapsed point with $\Theta^m(T, q) = \Theta^m(T, p)$ and there is a unique tangent cone $T_q = Q [\![\pi(q)^+]\!] + (Q - Q^\star) [\![\pi(q)^-]\!]$ to T at q, where $\pi(q)$ is an m-dimensional plane. Moreover, for any $\varepsilon > 0$, there is $C = C(\varepsilon) > 0$, such that*

$$|\pi(q) - \pi(z)| \leq C \left(r^{\varepsilon - 1} \left(\mathbf{E}^\flat(T, \mathbf{B}(p, 2r)) \right)^{\frac{1}{2}} + \mathbf{A} r^\varepsilon \right) |z - q|^{1-\varepsilon}, \ \forall z \in \mathbf{B}(p, r)\,. \tag{21.3.1}$$

Remark 21.3.2. Note that, as we have proved in Lemma 19.3.5 in dimension 2 using the characterization of the tangent cones in the $2d$ setting, Theorem 21.3.1 ensures, for arbitrary dimension m, that the set of two-sided collapsed points is relatively open in Γ. Furthermore, it also guarantees that the density is constant in $\mathbf{B}(p, r) \cap \Gamma$.

The following proof goes along the same lines of [DDHM18, Theorem 6.3] and we report it here for completeness's sake.

Proof. We start taking $\pi(q)$ as the plane given by the improved excess decay, see Theorem 21.2.2. Let us now prove that, if T_q is a tangent cone to T at q w.r.t. the sequence $\rho_k \to 0$, then its support is $\pi(q)$. By rescaling we have that

$$\mathbf{E}(T, \mathbf{B}(q, \rho_k)\,, \pi(q)) \leq C\mathbf{E}(T_{q,\rho_k}, \mathbf{B}(0, 2)\,, \pi(q)).$$

The latter rescaling and the improved excess decay, i.e., (21.2.4), furnish

$$\mathbf{E}(T_{q,\rho_k}, \mathbf{B}(0, 2)\,, \pi(q)) \leq C \left(\frac{\rho_k}{r} \right)^{2-2\varepsilon} \mathbf{E}^\flat(T, \mathbf{B}(p, 2r)) + C \rho_k^{2-2\varepsilon} r^{2\varepsilon} \mathbf{A}^2, \ \ \forall \rho_k < r. \tag{21.3.2}$$

We now let $\rho_k \to 0$ in (21.3.2) to conclude that $\mathbf{E}(T_q, \mathbf{B}(0, 2)\,, \pi(q)) = 0$ and hence T_q is supported in $\pi(q)$. We conclude that the tangent cone is unique and, by a standard argument involving the Constancy Lemma as already used many times above, it takes the form

$$T_q = Q(q) [\![\pi(q)^+]\!] + (Q(q) - Q^\star) [\![\pi(q)^-]\!] \ ,$$

for some $Q(q) \in \mathbb{N}$, since the tangent cone is an integral current. By assumption we have that p is a two-sided collapsed point and $q \in U$, $Q(q) - \frac{Q^\star}{2} = \Theta(T, q) \geq \Theta^m(T, p)$. Moreover, by (21.2.2), we obtain $Q(q) - \frac{Q^\star}{2} \leq \Theta^m(T, p) + \frac{1}{4}$ and therefore $\Theta^m(T, q) = \Theta^m(T, p)$. Finally, in order to finish the proof of the theorem, for $0 < t < s < 1$, we let $\pi(q, s)$ and $\pi(q, t)$ such that $\mathbf{E}^\flat(T, \mathbf{B}(q, t)) = \mathbf{E}^\flat(T, \mathbf{B}(q, t)\,, \pi(q, t))$ and $\mathbf{E}^\flat(T, \mathbf{B}(q, s)) = \mathbf{E}^\flat(T, \mathbf{B}(q, s)\,, \pi(q, s))$. We now take $0 < t < \rho := |q - z| < r$ and note that

$$
\begin{aligned}
|\pi(q,t)-\pi(z,t)|^2 &= \fint_{\mathbf{B}(q,t)\cap\mathbf{B}(z,t)} |\pi(q,t)-\pi(z,t)|^2 d\|T\| \\
&\le \frac{C}{\omega_m t^m}\int_{\mathbf{B}(q,t)} |\vec{T}-\pi(q,t)|^2 + \frac{C}{\omega_m t^m}\int_{\mathbf{B}(z,t)} |\vec{T}-\pi(z,t)|^2 d\|T\| \\
&= C(\mathbf{E}^\flat(T,\mathbf{B}(q,t)) + \mathbf{E}^\flat(T,\mathbf{B}(z,t))) \\
&\le C\left(\frac{\rho}{r}\right)^{2-2\varepsilon}\mathbf{E}^\flat(T,\mathbf{B}(p,2r)) + C\rho^{2-2\varepsilon} r^{2\varepsilon}\mathbf{A}^2,
\end{aligned}
$$

where, in the second line, we have used that $\|T\|(\mathbf{B}(p,t)) \ge ct^m$, a simple consequence of the monotonicity formula. Hence, the latter inequality gives

$$
|\pi(q,t)-\pi(z,t)| \le C\left(r^{-2+2\varepsilon}\mathbf{E}^\flat(T,\mathbf{B}(p,2r)) + r^{2\varepsilon}\mathbf{A}^2\right)^{\frac{1}{2}} |q-z|^{1-\varepsilon},
$$

we let t goes to 0 to conclude (21.3.1). □

We state an important corollary of Theorems 21.3.1 and 21.2.2 which will be used often in the remaining chapters and relates the height when we change the reference plane to an optimal plane to the excess in p instead of consider the tangent plane at q.

Corollary 21.3.3 (Height bound relative to tilted optimal planes)**.** *Let $\Gamma, T, p, q, \pi(q)$ and r be as in Theorem 21.3.1 and let π be an optimal plane for $\mathbf{E}^\flat(T,\mathbf{B}(p,2r))$. If we denote by $\mathbf{p}_\pi, \mathbf{p}_\pi^\perp, \mathbf{p}_{\pi(q)}$ and $\mathbf{p}_{\pi(q)}^\perp$ respectively the orthogonal projections onto $\pi, \pi^\perp, \pi(q)$ and $\pi(q)^\perp$, then, for all $q \in \Gamma\cap\mathbf{B}(p,r)$, we have*

$$
|\pi(q)-\pi| \le C(\mathbf{E}^\flat(T,\mathbf{B}(p,2r))^{\frac{1}{2}} + \mathbf{A}r), \tag{21.3.3}
$$

$$
\operatorname{spt}(T)\cap\mathbf{B}\Big(q,\frac{r}{2}\Big) \subset \left\{x\in\mathbb{R}^{m+n} : \left|\mathbf{p}_{\pi(q)}^\perp(x-q)\right| \le C(r^{-1}\mathbf{E}^\flat(T,\mathbf{B}(p,2r))^{\frac{1}{2}} + \mathbf{A})^{\frac{1}{2}} |x-q|^{\frac{3}{2}}\right\}, \tag{21.3.4}
$$

$$
\operatorname{spt}(T)\cap\mathbf{B}\Big(q,\frac{r}{2}\Big) \subset \left\{x\in\mathbb{R}^{m+n} : \left|\mathbf{p}_{\pi}^\perp(x-q)\right| \le C(\mathbf{E}^\flat(T,\mathbf{B}(p,2r))^{\frac{1}{2}} + \mathbf{A}r)^{\frac{1}{2}} |x-q|\right\}. \tag{21.3.5}
$$

Proof. To prove (21.3.3), we proceed as follows

$$
\begin{aligned}
|\pi-\pi(q)|^2 &\le 2|\pi-\pi(p)|^2 + 2|\pi(p)-\pi(q)|^2 \\
&\overset{(21.3.1)}{\le} 2|\pi-\pi(p)|^2 + C(\mathbf{E}^\flat(T,\mathbf{B}(p,2r))^{\frac{1}{2}} + \mathbf{A}r)^2,
\end{aligned} \tag{21.3.6}
$$

and

$$
\begin{aligned}
|\pi-\pi(p)|^2 &\le C\frac{1}{\|T\|(\mathbf{B}(p,2r))}\int_{\mathbf{B}(p,2r)} \left(|\pi-\vec{T}|^2 + |\vec{T}-\pi(p)|^2\right) d\|T\| \\
&\overset{(*)}{\le} C\mathbf{E}^\flat(T,\mathbf{B}(p,2r)) + C\frac{1}{\|T\|(\mathbf{B}(p,2r))}\int_{\mathbf{B}(p,2r)} |\vec{T}-\pi(p)|^2 d\|T\| \\
&\overset{(21.2.4),(*)}{\le} C(\mathbf{E}^\flat(T,\mathbf{B}(p,2r))^{\frac{1}{2}} + \mathbf{A}r)^2,
\end{aligned} \tag{21.3.7}
$$

where in $(*)$ we have used the standard argument with the monotonicity formula to obtain a bound $\|T\|(\mathbf{B}(p,2r)) \geq cr^m$. Therefore (21.3.6) and (21.3.7) prove (21.3.3). Note that (21.3.4) follows immediately from (21.2.5). We next observe that

$$\left|\mathbf{p}_\pi^\perp - \mathbf{p}_{\pi(q)}{}^\perp\right|^2 = \left|\mathbf{p}_\pi - \mathbf{p}_{\pi(q)}\right|^2 \leq C\,|\pi - \pi(q)|^2 . \tag{21.3.8}$$

Furthermore, for $x \in \mathbf{B}(q,r) \cap \operatorname{spt}(T)$, it follows

$$\begin{aligned}
|\mathbf{p}_\pi^\perp(x-q)|^2 &\leq C|x-q|^2|\mathbf{p}_\pi^\perp - \mathbf{p}_{\pi(q)}{}^\perp|^2 + C|\mathbf{p}_{\pi(q)}{}^\perp(x-q)|^2\\
&\leq C|\pi-\pi(q)|^2|x-q|^2 + C|\mathbf{p}_{\pi(q)}{}^\perp(x-q)|^2\\
&\overset{(21.3.3)}{\leq} C(\mathbf{E}(T,\mathbf{B}(p,2r))^{\frac12} + \mathbf{A}r)^2|x-q|^2 + C|\mathbf{p}_{\pi(q)}{}^\perp(x-q)|^2\\
&\overset{(21.3.4)}{\leq} C(\mathbf{E}(T,\mathbf{B}(p,2r))^{\frac12} + \mathbf{A}r)^2|x-q|^2 + C(r^{-1}\mathbf{E}(T,\mathbf{B}(p,2r))^{\frac12} + \mathbf{A})\,|x-q|^{\frac32}\\
&\leq C(\mathbf{E}(T,\mathbf{B}(p,2r))^{\frac12} + \mathbf{A}r)^{\frac12}|x-q|,
\end{aligned}$$

where in the last inequality the fact $|x-q| < r$ took place, thus the latter inequality proves (21.3.5).

□

21.4 Lusin type strong Lipschitz approximations

As we remarked at the beginning of this section, Theorem 20.2.3 provides an approximation which is not enough for our purposes. In the last subsection, we used the harmonic approximation, Theorem 20.2.4, to obtain the superlinear excess decay, Theorem 21.2.2, which will now be used to provide our desired approximation with faster decays and stronger estimates as it is precisely stated in Theorem 21.4.1.

Assumption 8. Let T and Γ be as in Assumption 6 with $C_0 = 0$, π be a m-dimensional subspace, $\{e_i\}_{i=1}^m$ a basis of $\mathbb{R}^m$ and $q \in \Gamma$. We use the following notations $\pi' = \operatorname{span}(\mathbf{p}_\pi(e_1), \cdots, \mathbf{p}_\pi(e_{m-1})), \psi_1 : (q+\pi') \to (q + \operatorname{span}(\mathbf{p}_\pi(e_m))), \psi : \gamma \subset (q+\pi) \to (q+\pi)^\perp, \psi_2 : (q+\pi') \to (q+\operatorname{span}(\mathbf{p}_\pi(e_m))) \times (q+\pi)^\perp, \psi_2(x) = (\psi_1(x), \psi(x,\psi_1(x)))$ with $\operatorname{Gr}\psi_1 = \gamma, \Gamma = \operatorname{Gr}\psi_2$, and ψ is of class $C^{3,\alpha}$. We assume that

(i) In the excess decay, Theorem 21.2.2, $p = 0 \in \Gamma$ and $r = 1$,

(ii) $\mathbf{E}^\flat(T,\mathbf{B}(0,2)) + \mathbf{A} < \varepsilon_1$, where $\varepsilon_1 = \varepsilon_1(m,n,Q^\star,Q) > 0$ is a small constant.

We would like to point out that the approximation and the estimates in the following theorem hold for any point in a suitable ball of the fixed two-sided collapsed point $p = 0$, compare with Theorem 20.2.3 where the approximation and estimates are build in balls centered in the fixed two-sided collapsed point p.

Theorem 21.4.1 (Strong Lipschitz approximation)**.** *Let T, Γ, ψ and $\gamma = \mathbf{p}_\pi(\Gamma)$ be as in Assumption 8, $q \in \Gamma \cap \mathbf{B}(0,1)$, $r < \frac18$ and π be a plane such that $T_q\Gamma \subset \pi$ and $\mathbf{E}(T,\mathbf{C}(q,4r,\pi)) < \varepsilon_1$. Then there* *are a closed set $K \subset \mathrm{B}_r(q,\pi)$ and a $\left(Q - \frac{Q^\star}{2}\right)$-valued map (u^+, u^-) on $\mathrm{B}_r(q,\pi)$ which collapses at the interface $(\gamma, Q^\star [\![\psi]\!])$ satisfying the following estimates:*

$$\mathrm{Lip}(u^\pm) \le C(\mathbf{E}(T, \mathbf{C}(q, 4r, \pi)) + \mathbf{A}^2 r^2)^\sigma \tag{21.4.1}$$

$$\mathrm{osc}(u^\pm) \le C(\mathbf{E}(T, \mathbf{C}(q, 4r, \pi)) + \mathbf{A}^2 r^2)^{\frac{1}{2}} r \tag{21.4.2}$$

$$\mathbf{G}_{u^\pm} \llcorner [(K \cap \Omega^\pm) \times \pi^\perp] = T \llcorner [(K \cap \Omega^\pm) \times \mathbb{R}^n] \tag{21.4.3}$$

$$|\mathrm{B}_r(q,\pi) \setminus K| \le C(\mathbf{E}(T, \mathbf{C}(q, 4r, \pi)) + \mathbf{A}^2 r^2)^{1+\sigma} r^m \tag{21.4.4}$$

$$\mathbf{e}_T(\mathrm{B}_r(q,\pi) \setminus K) \le C(\mathbf{E}(T, \mathbf{C}(q, 4r, \pi)) + \mathbf{A}^2 r^2)^{1+\sigma} r^m \tag{21.4.5}$$

$$\int_{\mathrm{B}_r(q,\pi)\setminus K} |Du|^2 \le C(\mathbf{E}(T, \mathbf{C}(q, 4r, \pi)) + \mathbf{A}^2 r^2)^{1+\sigma} r^m \tag{21.4.6}$$

$$\left| \mathbf{e}_T(F) - \frac{1}{2}\int_F |Du^\pm|^2 \right| \le C(\mathbf{E}(T, \mathbf{C}(q, 4r, \pi)) + \mathbf{A}^2 r^2)^{1+\sigma} r^m, \quad \forall F \subset \Omega^\pm \text{ measurable,} \tag{21.4.7}$$

where $\Omega^\pm$ are the two regions in which $\mathrm{B}_r(q,\pi)$ is divided by γ, $C \ge 1$ and $\sigma \in]0, \frac{1}{4}[$ are two positive constants which depend on $m, n, Q^\star$ and Q.

Proof. Our strategy to prove this theorem is to go back to the interior estimates done in [DS14]. So we will divide the proof into two steps, in Step 1 we will prove further estimates provided by the interior case which are needed to conclude our estimates (21.4.1)-(21.4.7), and in Step 2 we will exhibit how to obtain the theorem from the interior case.

Step 1: If we assume that ε_1 is smaller than the constant of [DS14, Theorem 2.4] (also denoted by ε_1) and the cylinder $\mathbf{C}(x, 4\rho, \pi)$ does not intersect Γ and is contained in $\mathbf{C}(q, 4r, \pi)$. Then, [DS14, Theorem 2.4] provides a map $f : \mathrm{B}_\rho(x,\pi) \to \mathcal{A}_Q(\pi^\perp)$ (or $\mathcal{A}_{Q-Q^\star}(\pi^\perp)$) and a closed set $\bar{K} \subset \mathrm{B}_\rho(x,\pi)$ such that

$$\mathrm{Lip}(f) \le C_{21} \mathbf{E}(T, \mathbf{C}(x, 4\rho))^\sigma, \tag{21.4.8}$$

$$\mathbf{G}_f \llcorner (\bar{K} \times \mathbb{R}^n) = T \llcorner (\bar{K} \times \mathbb{R}^n), \tag{21.4.9}$$

$$|\mathrm{B}_\rho(x,\pi) \setminus \bar{K}| \le C\mathbf{E}(T, \mathbf{C}(x, 4\rho))^{1+\sigma} \rho^m, \tag{21.4.10}$$

$$\left| \|T\|(\mathbf{C}(x,\rho)) - Q\omega_m \rho^m - \frac{1}{2}\int_{\mathrm{B}_\rho(x,\pi)} |Df|^2 \right| \le C\mathbf{E}(T, \mathbf{C}(x, 4\rho))^{1+\sigma} \rho^m, \tag{21.4.11}$$

In order to simplify our notation, we assume that $\pi = \mathbb{R}^m \times \{0\}$ and use the shorthand notation $\mathrm{B}_t(x)$ for $\mathrm{B}_t(x,\pi)$. It remains to prove the analogous of (21.4.5), (21.4.6) and (21.4.7) when q is replaced by the interior point x. Notice that (21.4.8) and (21.4.10) give

$$\int_{F\setminus\bar K} |Df|^2 \le C\mathbf{E}(T,\mathbf{C}(x,4\rho))^{2\sigma}\left|\mathrm{B}_\rho(x)\setminus\bar K\right| \le C\mathbf{E}(T,\mathbf{C}(x,4\rho))^{1+3\sigma}\rho^m,$$

for every $F\subset \mathrm{B}_\rho(x)$ measurable, hence we achieve (21.4.6) taking $F=\mathrm{B}_\rho(x)$. Next recall that either $\|T\|(\mathbf{C}(x,\rho)) - Q\omega_m\rho^m = \mathbf{e}_T(\mathrm{B}_\rho(x))$ or $\|T\|(\mathbf{C}(x,\rho)) - (Q-Q^\star)\omega_m\rho^m = \mathbf{e}_T(\mathrm{B}_\rho(x))$, hence (21.4.11) can be reformulated as

$$\left|\mathbf{e}_T(\mathrm{B}_\rho(x)) - \frac{1}{2}\int_{\mathrm{B}_\rho(x)} |Df|^2\right| \le C\mathbf{E}(T,\mathbf{C}(x,4\rho))^{1+\sigma}\rho^m\,.$$

We also have that

$$\begin{aligned}\frac{1}{2}\int_{\mathrm{B}_\rho(x)}|Df|^2 &\le \left(\mathbf{E}(T,\mathbf{C}(x,4\rho)) + C\mathbf{E}(T,\mathbf{C}(x,4\rho))^{1+\sigma}\right)\rho^m\\ &\le C\mathbf{E}(T,\mathbf{C}(x,4\rho))\rho^m.\end{aligned} \tag{21.4.12}$$

The Taylor expansion of the area functional, [DS15, Corollary 3.3], and (21.4.8) give

$$\left|\mathbf{e}_{\mathbf{G}_f}(F) - \frac{1}{2}\int_F |Df|^2\right| \le C\,\mathrm{Lip}(f)^2\int_F |Df|^2 \le C\mathbf{E}(T,\mathbf{C}(x,4\rho))^{1+2\sigma}\rho^m, \tag{21.4.13}$$

for every $F\subset \mathrm{B}_\rho(x)$ measurable. Therefore, by (21.4.12) and (21.4.13), we obtain

$$\begin{aligned}\mathbf{e}_T(\mathrm{B}_\rho(x)\setminus\bar K) &= \mathbf{e}_T(\mathrm{B}_\rho(x)) - \mathbf{e}_{\mathbf{G}_f}(\mathrm{B}_\rho(x)\cap\bar K)\\ &\le \left|\mathbf{e}_T(\mathrm{B}_\rho(x)) - \frac{1}{2}\int_{\mathrm{B}_\rho(x)}|Df|^2\right|\\ &\quad+\left|\frac{1}{2}\int_{\mathrm{B}_\rho(x)\cap\bar K}|Df|^2 - \mathbf{e}_{\mathbf{G}_f}(\mathrm{B}_\rho(x)\cap\bar K)\right|\\ &\quad+\int_{\mathrm{B}_\rho(x)\setminus\bar K}|Df|^2\\ &\le C\mathbf{E}(T,\mathbf{C}(x,4\rho))^{1+\sigma}\rho^m,\end{aligned}$$

which is (21.4.5). Finally, (21.4.7) is implied by the last inequality, (21.4.6) and (21.4.13) as follows, for every $F\subset\mathrm{B}_\rho(x)$ measurable we have

$$\begin{aligned}\left|\mathbf{e}_T(F) - \frac{1}{2}\int_F|Df|^2\right| &\le \left|\mathbf{e}_{\mathbf{G}_f}(F\cap\bar K) - \frac{1}{2}\int_{F\cap\bar K}|Df|^2\right| + \mathbf{e}_T(F\setminus\bar K) + \frac{1}{2}\int_{F\setminus\bar K}|Df|^2\\ &\le C\mathbf{E}(T,\mathbf{C}(x,4\rho))^{1+\sigma}\rho^m.\end{aligned}$$

Step 2: Without loss of generality we assume that $T_q\Gamma = \mathbb{R}^{m-1}\times\{0\}$, $\pi=\mathbb{R}^m\times\{0\}$. We then use $\mathbf{C}(q,s)$ in place of $\mathbf{C}(q,s,\pi)$, $\mathrm{B}_s(q)$ in place of $\mathrm{B}_s(q,\pi)$, and $\mathbf{p}$ to be the orthogonal projection onto π. By Assumption 8, we have

$$\partial T \llcorner \mathbf{C}(q,4r) = Q^\star [\![\Gamma \cap \mathbf{C}(q,4r)]\!] \qquad \text{and} \qquad \mathbf{p}_\sharp(\partial T \llcorner \mathbf{C}(q,4r)) = Q^\star [\![\gamma \cap \mathrm{B}_{4r}(\mathbf{p}(q))]\!] .$$

As in the previous sections, denote by Ω^+ and Ω^- the two connected components of $\mathrm{B}_{4r}(q) \setminus \gamma$, we have

$$\mathbf{p}_\sharp T \llcorner \mathbf{C}(q,4r) = Q [\![\Omega^+]\!] + (Q - Q^\star) [\![\Omega^-]\!] . \tag{21.4.14}$$

We denote L_0 be the m-cube $q + [-r,r]^m$ and, for any natural number k, let $\mathcal{C}_k$ be a collection of m-cubes given by

$$\mathcal{C}_k = \left\{ L = q + r2^{-k}x + [-2^{-k}r, 2^{-k}r]^m : x \in \mathbb{Z}^m, k \in \mathbb{N}, L \subset L_0, L \cap \mathrm{B}_r(q) \neq \emptyset \right\} .$$

We take a number $N \in \mathbb{N}$ such that the $2^{4-N}\sqrt{m}r$-neighborhood of $\cup_{L \in \mathcal{C}_N} L$ is contained in $\mathbf{C}(q,4r)$ and we will proceed with the construction of a Whitney decomposition of

$$\tilde{\Omega} := \bigcup_{L \in \mathcal{C}_N} L \setminus \gamma.$$

Here and in what follows we set

$$\mathrm{sep}\,(L,\gamma) := \min\{|x-y| : x \in \gamma, y \in L\} .$$

We firstly define the following sets of m-cubes $\mathcal{R}_N = \mathcal{C}_N$,

$$\mathcal{W}_N := \left\{ L \in \mathcal{R}_N : \mathrm{diam}(L) \le \frac{1}{16}\,\mathrm{sep}(L,\gamma) \right\} .$$

If $L \in \mathcal{R}_N \setminus \mathcal{W}_N$, we subdivide L in 2^m m-subcubes of side $2^{-(N+1)}r$ and assign them to $\mathcal{R}_{N+1}$. We proceed inductively to define $\mathcal{W}_k$ and $\mathcal{R}_{k+1}$ for every $k \ge N$. Therefore, we obtain a Whitney decomposition $\mathcal{W} = \cup_{k \ge N} \mathcal{W}_k$ which is a collection of closed dyadic m-cubes such that

$$\mathrm{int}(L) \cap \mathrm{int}(H) = \emptyset, \text{ for all } L, H \in \mathcal{W}, \tag{21.4.15}$$

$$\Omega^+ \cup \Omega^- \subset \cup_{L \in \mathcal{W}} L, \tag{21.4.16}$$

$$\frac{15}{16}\frac{1}{32}\mathrm{sep}\,(L,\gamma) < \ \mathrm{diam}(L) \le \frac{1}{16}\mathrm{sep}\,(L,\gamma), \ L \in \mathcal{W}. \tag{21.4.17}$$

Note that (21.4.15) follows readily from the construction. In regard to (21.4.16) we take $z \in \Omega^\pm$ then, we have two mutually exclusive cases that could appear, namely either there exists $L \in \mathcal{W}_N$ such that $z \in L$, or for every $L \in \mathcal{W}_N$ results $z \notin L$. In the first case we finish readily the proof. In the second case take L such that $L \in \mathcal{R}_N \setminus \mathcal{W}_N$ and $z \in L$. Then we may subdivide it and pass to the next generation and find a new cube $L' \in \mathcal{R}_{N+1}$ such that $z \in L'$. Now we may apply the same reasoning inductively and construct a sequence $L_{k,j_{k,z}}$, $k \ge N$ such that $z \in L_{k,j_{k,z}}$, $L_{k,j_{k,z}} \subseteq L_{k+1,j_{k+1,z}}$, and $L_{k,j_{k,z}} \notin \mathcal{W}$ for every $k \ge N$. If the sequence $(L_{k,j_{k,z}})_k$ is not constant for large values of k, then the diameters of $L_{k,j_{k,z}}$ goes to zero as k goes to infinity, and thus we obtain for sufficiently large k that

$$\mathrm{diam}(L_{k,j_{k,z}}) \le \frac{1}{16}\,\mathrm{sep}(L_{N,j_{N,z}},\gamma) \le \frac{1}{16}\,\mathrm{sep}(L_{k,j_{k,z}},\gamma),$$

since $L_{k,j_{k,z}} \subset L_{N,j_{N,z}}$ which ensures that $L_{k,j_{k,z}} \in \mathcal{W}$ and therefore (21.4.16). To prove (21.4.17), observe that $\operatorname{sep}(L,\gamma) \leq \operatorname{sep}(\tilde{L},\gamma)+\operatorname{diam}(L)$ for every $L \in \mathcal{C}_k$, $\tilde{L} \in \mathcal{C}_{k-1}$ and $L \subseteq \tilde{L}$. By construction for each $L \in \mathcal{W}_k$ there exists $\tilde{L} \in \mathcal{R}_{k-1} \setminus \mathcal{W}_{k-1}$ such that $L \subseteq \tilde{L}$. Thus $\frac{15}{16}\operatorname{sep}(L,\gamma) \leq \operatorname{sep}(\tilde{L},\gamma)$, $2\operatorname{diam}(L) = \operatorname{diam}(\tilde{L}) > \frac{1}{16}\operatorname{sep}(\tilde{L},\gamma)$. So $\operatorname{diam}(L) > \frac{1}{32}\operatorname{sep}(\tilde{L},\gamma) \geq \frac{15}{16}\frac{1}{32}\operatorname{sep}(L,\gamma)$.

Another important property of this family is that if a m-cube stops then its neighbours of next generation must also stop. Precisely, let $H \in \mathcal{W}_j$, $L \in \mathcal{C}_{j+1}$ and $H \cap L \neq \emptyset$, then

$$\operatorname{sep}(L,\gamma) \geq \operatorname{sep}(H,\gamma) - \operatorname{diam}(L) \overset{(21.4.17)}{\geq} 16\operatorname{diam}(H) - \operatorname{diam}(L) \geq 31\operatorname{diam}(L) \geq 16\operatorname{diam}(L). \tag{21.4.18}$$

The chain of inequalities above guarantees that $\operatorname{sep}(L,\gamma) \geq 16\operatorname{diam}(L)$ which is the very definition of the family $\mathcal{W}_{j+1}$, i.e., $L \in \mathcal{W}$.

We denote with c_L the center of the m-cube $L \in \mathcal{W}$ and set $r_L := 3\operatorname{diam}(L)$ so that

$$L \subset \mathrm{B}_{\frac{1}{4}r_L}(c_L)\,. \tag{21.4.19}$$

We claim that for each cube L the current T restricted to the cylinder $\mathbf{C}(c_L, 4r_L)$ satisfies the assumptions of [DS14, Theorem 2.4]. Firstly note that, by (21.4.17), we have $\mathbf{C}(c_L,4r_L) \cap \Gamma = \emptyset$ and thus $\partial T \llcorner \mathbf{C}(c_L,4r_L) = 0$, we also obtain by the choice of N that $\mathbf{B}(c_L,6r_L) \subset \mathbf{B}(q,4r)$. Moreover, either $\mathrm{B}_{4r_L}(c_L) \subset \Omega^+$ or $\mathrm{B}_{4r_L}(c_L) \subset \Omega^-$ and thus by (21.4.14) we have

$$\mathbf{p}_\sharp T \llcorner \mathbf{C}(c_L,4r_L) = \begin{cases} Q\,[\![\mathrm{B}_{4r_L}(c_L)]\!]\,, & \text{if } c_L \in \Omega^+,\\ (Q-Q^\star)\,[\![\mathrm{B}_{4r_L}(c_L)]\!]\,, & \text{if } c_L \in \Omega^-. \end{cases}$$

It remains to prove that the excess is small enough. Towards this goal we will make use of the excess decay, Theorem 21.2.2. In fact, we start distinguishing the two cases $r_L = 2^{-N}r$ and $r_L < 2^{-N}r$. If $r_L = 2^{-N}r$, rescaling the excess and by the assumptions of the theorem, we easily obtain that

$$\mathbf{E}(T,\mathbf{C}(c_L,4r_L)) \leq 2^{mN}\mathbf{E}(T,\mathbf{C}(q,4r)) < \varepsilon_1.$$

Now, for each $L \in \mathcal{W}$ with $r_L < 2^{-N}r$, we let x_L be the orthogonal projection of c_L on γ and $q_L \in \Gamma$ be the point $(x_L, \psi(x_L))$. The first inequality of (21.4.17) implies that

$$\mathbf{C}(c_L,4r_L) \subset \mathbf{C}(q_L,15r_L)\,.$$

From our choice of N, taking ε_1 smaller if necessary, by (21.3.5), we have

$$\operatorname{spt}(T) \cap \mathbf{C}(q_L,16r_L) \subset \mathbf{B}(q_L,17r_L) \subset \mathbf{C}(q,4r)\,. \tag{21.4.20}$$

So, we deduce that

$$\begin{aligned}\mathbf{E}(T,\mathbf{C}(c_L,4r_L)) &\overset{(21.4.20)}{\leq} \mathbf{E}(T,\mathbf{B}(q_L,17r_L)\,,\pi)\\ &\overset{\text{triangular}}{\leq} C\mathbf{E}(T,\mathbf{B}(q_L,17r_L)\,,\pi(q_L)) + C|\pi-\pi(q_L)|\\ &\leq C\mathbf{E}(T,\mathbf{C}(q,4r)) + \mathbf{A}^2r^2 < \varepsilon_1,\end{aligned} \tag{21.4.21}$$

where in the last inequality we have used the excess decay, Theorem 21.2.2, and (21.3.3). So, provided ε_1 is chosen sufficiently small, we can apply Step 1 in every cylinder $\mathbf{C}(c_L, 4r_L)$ and obtain either a Q-valued or a $(Q - Q^\star)$-valued map f_L on each half ball $\mathrm{B}^+_{r_L}(c_L)$ or $\mathrm{B}^-_{r_L}(c_L)$ and a closed set $K_L \subset \mathrm{B}^\pm_{r_L}(c_L)$ such that

$$\operatorname{Lip}(f_L) \le C\mathbf{E}(T, \mathbf{C}(c_L, 4r_L))^\sigma, \tag{21.4.22}$$

$$\mathbf{G}_{f_L} \llcorner (K_L \times \mathbb{R}^n) = T \llcorner (K_L \times \mathbb{R}^n), \tag{21.4.23}$$

$$|\mathrm{B}_{r_L}(c_L) \setminus K_L| \le C\mathbf{E}(T, \mathbf{C}(c_L, 4r_L))^{1+\sigma} r_L^m, \tag{21.4.24}$$

$$\mathbf{e}_T(\mathrm{B}_{r_L}(c_L) \setminus K_L) \le C\mathbf{E}(T, \mathbf{C}(c_L, 4r_L))^{1+\sigma} r_L^m, \tag{21.4.25}$$

$$\int_{\mathrm{B}_{r_L}(c_L)\setminus K_L} |Df_L|^2 \le C\mathbf{E}(T, \mathbf{C}(c_L, 4r_L))^{1+\sigma} r_L^m, \tag{21.4.26}$$

$$\left| \mathbf{e}_T(F) - \frac{1}{2}\int_F |Df_L|^2 \right| \le C\mathbf{E}(T, \mathbf{C}(c_L, 4r_L))^{1+\sigma} r_L^m, \quad \forall F \subset \mathrm{B}_{r_L}(c_L) \text{ measurable.} \tag{21.4.27}$$

Next, for each L we let $\mathcal{N}_\ge(L)$ be what we call the neighboring m-cubes in $\mathcal{W}$ with bigger radius, i.e.

$$\mathcal{N}_\ge(L) = \{H \in \mathcal{W} \colon H \cap L \ne \emptyset, r_H \ge r_L\}.$$

We use the good sets provided by the interior approximation to define

$$K'_L = K_L \cap \bigcap_{H \in \mathcal{N}_\ge(L)} K_H, \quad K^+ = \bigcup_{L \in \mathcal{W}, L \subset \Omega^+} K'_L \cap L, \quad K^- = \bigcup_{L \in \mathcal{W}, L \subset \Omega^-} K'_L \cap L.$$

Furthermore, we set up two functions defined on K^+ and K^-, respectively, given by $\tilde{u}^+(x) := f_L(x)$, for any $x \in L \cap K^+, L \in \mathcal{W}$, and $\tilde{u}^-(x) := f_L(x)$, for all $x \in L \cap K^-, L \in \mathcal{W}$, thanks to (21.4.23) these functions are well defined. Note that these functions are defined on each square in $\mathcal{W}$ which are away from the boundary γ, it means that we have to properly extend these functions in order to have a $(Q - \frac{Q^\star}{2})$-valued map which collapses at the interface. Indeed, we will refine this idea in the sequel.

Note that, if $H \in \mathcal{N}_\ge(L)$, we already know by (21.4.19) that $L \subset \mathrm{B}_{\frac{1}{4}r_L}(c_L)$ and then, since $H \cap L \ne \emptyset$, we deduce that $L \subset B_{r_H}(c_H)$. Hence

$$\begin{aligned}
|L \setminus K'_L| &\le |L \setminus K_L| + \sum_{H \in \mathcal{N}_\ge(L)} |L \setminus K_H| \\
&\le |L \setminus K_L| + \sum_{H \in \mathcal{N}_\ge(L)} |\mathrm{B}_{r_H}(c_H) \setminus K_H| \\
&\overset{(21.4.24)}{\le} C\mathbf{E}(T, \mathbf{C}(q, 4r))^{1+\sigma} r_L^m < C\varepsilon_1 r_L^m,
\end{aligned} \tag{21.4.28}$$

in the last inequality we also use that the cardinality of $\mathcal{N}_{\geq}(L)$ is bounded by a geometric constant C', which allow us to bound r_H by r_L. We now claim the validity of the following:

$$\mathrm{Lip}(\tilde{u}^{\pm}) \leq C(\mathbf{E}(T, \mathbf{C}(q, 4r)) + \mathbf{A}^2 r^2)^{\sigma}, \tag{21.4.29}$$

$$\mathbf{G}_{\tilde{u}^{\pm}} \llcorner (K^{\pm} \times \mathbb{R}^n) = T \llcorner (K^{\pm} \times \mathbb{R}^n), \tag{21.4.30}$$

$$\mathbf{e}_T(L \setminus K'_L) \leq C(\mathbf{E}(T, \mathbf{C}(q, 4r)) + \mathbf{A}^2 r^2)^{1+\sigma} r_L^m, \tag{21.4.31}$$

$$\int_{L \setminus K'_L} \left|D\tilde{u}^{\pm}\right|^2 \leq C(\mathbf{E}(T, \mathbf{C}(q, 4r)) + \mathbf{A}^2 r^2)^{1+\sigma} r_L^m. \tag{21.4.32}$$

Before the proof of this claim, we show how to prove (21.4.1)-(21.4.7) from it. Define the good set as $K = K^+ \cup K^-$ and notice that in view of these last inequalities, up to now, we have finished the proof of (21.4.1), (21.4.2), and (21.4.3). Observe that

$$\sum_{L \in \mathcal{W}} r_L^m \leq C r^m \,, \tag{21.4.33}$$

which furnishes (21.4.4), (21.4.5) and (21.4.6) by summing over $L \in \mathcal{W}$, respectively, (21.4.28), (21.4.31) and (21.4.32). Regarding (21.4.7), we proceed as follows, fix a measurable set $F \subset \Omega^{\pm}$ and observe that, for any m-cube L in the Whitney decomposition $\mathcal{W}$ of $\Omega^{\pm}$ we have that

$$\begin{aligned}
\left|\mathbf{e}_T(F \cap L) - \frac{1}{2}\int_{F \cap L} |D\tilde{u}^+|^2\right| &\overset{\text{triangular}}{\leq} \left|\mathbf{e}_T(F \cap L \cap K^{\pm}) - \frac{1}{2}\int_{F \cap L \cap K^{\pm}} |D\tilde{u}^+|^2\right| \\
&\quad + \mathbf{e}_T(L \setminus K^{\pm}) + \mathrm{Lip}(\tilde{u}^+)^2 |L \setminus K^{\pm}| \\
&\overset{(21.4.28),(21.4.29),(21.4.31)}{\leq} \left|\mathbf{e}_T(F \cap L \cap K^{\pm}) - \frac{1}{2}\int_{F \cap L \cap K^{\pm}} |Df_L|^2\right| \\
&\quad + C(\mathbf{E}(T, \mathbf{C}(q, 4r)) + \mathbf{A}^2 r^2)^{1+\sigma} r_L^m \\
&\overset{(21.4.27)}{\leq} C(\mathbf{E}(T, \mathbf{C}(q, 4r)) + \mathbf{A}^2 r^2)^{1+\sigma} r_L^m \,.
\end{aligned}$$

By (21.4.33), summing over $L \in \mathcal{W}$, we obtain (21.4.7). Now, we turn our attention to the proof of the claim.

We start with the proof of the Lipschitz bound in (21.4.29), we let $H, L \in \mathcal{W}$ with $\mathrm{diam}(H) \geq \mathrm{diam}(L)$ and $x \in H, y \in L$, hence

- If $H \cap L \neq \emptyset$, by the very definitions of $\tilde{u}$ and $K^{\pm}$, we know that $\tilde{u}^{\pm} = f_H$ on $K^{\pm} \cap H$. Take any $z \in H \cap L$, thus the Lipschitz bound on each m-cube, i.e., (21.4.22), ensures

$$\mathcal{G}(\tilde{u}^{\pm}(x), \tilde{u}^{\pm}(y)) \leq \mathcal{G}(\tilde{u}^{\pm}(x), \tilde{u}^{\pm}(z)) + \mathcal{G}(\tilde{u}^{\pm}(z), \tilde{u}^{\pm}(y)) \leq C\mathbf{E}(T, \mathbf{C}(q, 4r))^{\sigma} |x - y|.$$

- If $H \cap L = \emptyset$, let $x_\gamma, y_\gamma \in \gamma$ such that $\mathrm{d}(x, \gamma) = \mathrm{d}(x, x_\gamma)$ and $\mathrm{d}(y, \gamma) = \mathrm{d}(y, y_\gamma)$. We directly obtain that

$$\begin{aligned}
\mathcal{G}(\tilde{u}^{\pm}(x), Q^{\pm} [\![\psi(x_\gamma)]\!]) &\leq C\mathbf{E}(T, \mathbf{C}(q, 4r))^{\frac{1}{2}} |x - x_\gamma|, \\
\mathcal{G}(\tilde{u}^{\pm}(y), Q^{\pm} [\![\psi(y_\gamma)]\!]) &\leq C\mathbf{E}(T, \mathbf{C}(q, 4r))^{\frac{1}{2}} |y - y_\gamma|,
\end{aligned} \tag{21.4.34}$$

where $Q^+ = Q$ and $Q^- = Q - Q^\star$. Indeed, both inequalities are due to the fact that $\mathrm{d}(x,\gamma)$ and r_L are comparable, e.g., (21.4.17), and that we have the height bound

$$\mathrm{spt}(T) \cap \mathbf{B}(0,2) \subset \left\{ x \in \mathbb{R}^{m+n} : |\mathbf{p}_\pi^\perp(x)| \le C\varepsilon_1^{\frac{1}{2}}|x| \right\},$$

as in Corollary 21.3.3, in the cylinder $\mathbf{C}(x_\gamma, 16r_L)$. Note also that, by the regularity of Γ, we obtain $|\psi(x_\gamma) - \psi(y_\gamma)| \le C\mathbf{A}^{1/2}|x_\gamma - y_\gamma|$. As a consequence, by (21.4.34), we can estimate

$$\begin{aligned}\mathcal{G}(\tilde{u}^\pm(x), \tilde{u}^\pm(y)) &\le \mathcal{G}(\tilde{u}^\pm(x), Q^\pm [\![\psi(x_\gamma)]\!]) + (Q^\pm)^{\frac{1}{2}}|\psi(x_\gamma) - \psi(y_\gamma)| \\ &\quad + \mathcal{G}(\tilde{u}^\pm(y), Q^\pm [\![\psi(y_\gamma)]\!]) \le C\left(\mathbf{E}(T, \mathbf{C}(q,4r)) + \mathbf{A}^2 r^2\right)^\sigma |x-y|\end{aligned}$$

where we have used that $\sigma \le \frac{1}{4}$ and that

$$|x - x_\gamma| + |y_\gamma - y| = \mathrm{d}(x,\gamma) + \mathrm{d}(y,\gamma) \le C(r_L + r_H) \le Cr_H \le C\frac{3\sqrt{m}}{2}|x-y|.$$

To see the last inequality observe that when $H \cap L = \emptyset$ by (21.4.18) we have that $|x-y| \ge \frac{2r_H}{3\sqrt{m}}$.

In particular that we have also proved that $(\tilde{u}^+, \tilde{u}^-)$ has a Lipschitz extension to $(K^\pm \cup \gamma) \cap \mathrm{B}_r(q)$ which, by (21.4.34), collapses at the interface $(\gamma \cap \mathrm{B}_r(q), Q^\star [\![\psi]\!])$. We next extend $\tilde{u}^\pm$ to the whole $\Omega^\pm$, and denote by $u^\pm$, keeping the Lipschitz estimate (21.4.29) up to a multiplicative geometric constant, c.f., Kirszbraun Theorem. Finally, inequality (21.4.32) follows directly by the estimates on the bad set and the Lipschitz bound, i.e., (21.4.28) and (21.4.29). Concerning inequality (21.4.31), we obtain it by (21.4.27) and (21.4.29). To conclude the proof of the theorem, we notice that equation (21.4.30) follows from (21.4.23).

□

Chapter 22

Center manifolds $\mathcal{M}^{\pm}$ with boundary Γ

In this section we work under the assumption that $0 \in \mathbb{R}^{m+n}$ is a two-sided collapsed point and that $T_0\Gamma = \mathbb{R}^{m-1} \times \{0\}$ and therefore, from Theorem 21.3.1, the tangent cone of T at 0 is $Q\,[\![\pi_0^+]\!] + (Q - Q^\star)\,[\![\pi_0^-]\!]$, where

$$\pi_0^{\pm} = \{x \in \mathbb{R}^{m+n} : \pm x_m > 0, x_{m+1} = \ldots = x_{n+m} = 0\}\,.$$

Following the notation that we have used up to know, we denote by γ the projection onto π_0 of Γ and, given any sufficiently small open set $\Omega \subset \pi_0$ in $\mathbb{R}^m$ which is contractible and contains 0, we denote by $\Omega^{\pm}$ the two regions in which Ω is divided by γ, i.e., the portions on the right and left of γ. In this section, we build two distinct m-dimensional submanifolds $\mathcal{M}^{\pm}$ of class C^3 which will be called, respectively, **left and right center manifolds**. Both center manifolds will have $\Gamma \cap \mathbf{C}(0, 3/2, \pi_0)$ as their boundary, when considered as submanifolds in the cylinder $\mathbf{C}(0, 3/2, \pi_0)$ and they will be $C^{3,\kappa}$ for a suitable positive κ up to the boundary. Additionally, at each point $p \in \Gamma \cap \mathbf{C}(0, 3/2, \pi_0)$ the tangent space to both manifolds will be the same which is the tangent cone to T at p denoted by $\pi(p)$ as in Theorem 21.3.1. In particular $\mathcal{M} = \mathcal{M}^+ \cup \mathcal{M}^-$ will be a $C^{1,1}$ submanifold in $\mathbf{C}(0, 3/2, \pi_0)$ without boundary.

Our aim in this section is to provide a new approximation of the current T, the way we will do this is building the center manifolds $\mathcal{M}^{\pm}$ which can be understood as an average of the sheets of the current T on each side of Γ, in the construction of the center manifold we will fabricate maps $\mathcal{N}^{\pm}$ which are defined in $\mathcal{M}^{\pm}$ and show that these maps $\mathcal{N}^{\pm}$ approximate the current in the sense of the Lipschitz approximations furnished in the previous chapters, e.g., Theorem 21.4.1. With respect to the final argument of this work to conclude the proof of Theorem 17.0.3 using Theorem 19.3.8, we desire to prove that $\mathcal{N}^{\pm}$ is identically zero and thus the current has to satisfies $T \llcorner \mathbf{C}(0, 3/2, \pi_0) = Q\,[\![\mathcal{M}^+]\!] + (Q - Q^\star)\,[\![\mathcal{M}^-]\!]$ which assures that 0 is a two-sided regular point of T. This strategy will be developed in the remaining part of this work where we begin with the construction of the center manifolds and the approximating maps, after that we use the theory of $(Q - \frac{Q^\star}{2})$-Dir minimizing maps, see Section 20, to obtain that $\mathcal{N}^{\pm}|_{\mathcal{M}^{\pm}} \equiv 0$.

22.1 Construction of the Whitney decomposition

Since the algorithm is the same for both sides of γ, it means that we can repeat the same frame to build both center manifolds. We can focus without loss of generality on the construction of $\mathcal{M}^+$. We start by describing a procedure which furnishes a suitable Whitney-type decomposition of $\mathrm{B}^+_{3/2}(0)$ with cubes whose sides are parallel to the coordinate axes and have sidelength $2\ell(L)$. The center of any such cube L will be denoted by c_L and its sidelength will be denoted by $2\ell(L)$. We start by introducing a family of dyadic cubes $L \subset \pi_0$ in the following way: for $j \geq N_0$, where N_0 is an integer whose choice will be specified below, we introduce the families

$$\mathscr{C}_j := \{L : L \text{ is a dyadic cube of side } \ell(L) = 2^{-j} \text{ and } \mathrm{B}^+_{3/2}(0) \cap L \neq \emptyset\},$$

For each L define a radius

$$r_L := M_0 \sqrt{m} \ell(L),$$

with $M_0 \geq 1$ to be chosen later. We then subdivide $\mathscr{C} := \cup_j \mathscr{C}_j$ into, respectively, **boundary cubes** and **non-boundary cubes**

$$\mathscr{C}^\flat := \{L \in \mathscr{C} : \mathrm{d}(c_L, \gamma) < 64 r_L\}, \quad \mathscr{C}^\flat_j = \mathscr{C}^\flat \cap \mathscr{C}_j,$$
$$\mathscr{C}^\natural := \{L \in \mathscr{C} : \mathrm{d}(c_L, \gamma) \geq 64 r_L\}, \quad \mathscr{C}^\natural_j = \mathscr{C}^\natural \cap \mathscr{C}_j.$$

Observe that some boundary cubes can be completely contained in $\mathrm{B}^+_{3/2}(0)$. For this reason we prefer to use the term "non-boundary" rather than "interior" for the cubes in $\mathscr{C}^\natural$. Indeed in what follows, without mentioning it any further, we will often use the same convention for several other subfamilies of $\mathscr{C}$.

Definition 22.1.1. If $H, L \in \mathscr{C}$ we say that:

- H is a **descendant** of L and L is an **ancestor** of H, if $H \subset L$;
- H is a **child** of L and L is the **parent** of H, if $H \subset L$ and $\ell(H) = \frac{1}{2}\ell(L)$;
- H and L are **neighbors** if $\frac{1}{2}\ell(L) \leq \ell(H) \leq \ell(L)$ and $H \cap L \neq \emptyset$.

Note, in particular, the following elementary consequence of the subdivision of $\mathscr{C}$:

Lemma 22.1.2. *Let H be a boundary cube. Then any ancestor L and any neighbor L with $\ell(L) = 2\ell(H)$ is necessarily a boundary cube. In particular: the descendant of a non-boundary cube is a non-boundary cube.*

Proof. For the case of ancestors it suffices to prove that if L is the parent of a boundary cube H, then L is a boundary cube. Since the parent of H is a neighbor of H with $\ell(L) = 2\ell(H)$, we only need to show the second part of the statement of the lemma. The latter is a simple consequence of the following chain of inequalities:

$$\begin{aligned}\mathrm{d}(c_L, \gamma) &\leq \mathrm{d}(c_H, \gamma) + |c_H - c_L| = \mathrm{d}(c_H, \gamma) + 3\sqrt{m}\ell(H) \\ &< 64 r_H + 3\frac{r_H}{M_0} \leq \left(64 + 3M_0^{-1}\right)\frac{r_L}{2} \leq \frac{67}{2} r_L < 64 r_L.\end{aligned}$$

□

Definition 22.1.3 (Satellite balls)**.** Following the notations above, we set:

(i) If $L \in \mathscr{C}^\natural$, then we define the **non-boundary satellite ball** $\mathbf{B}_L = \mathbf{B}(p_L, 64r_L)$ where $p_L \in \operatorname{spt}(T)$ such that $\mathbf{p}_{\pi_0}(p_L) = c_L$, such p_L is a priori not unique, and π_L is a plane which minimizes the excess in $\mathbf{B}_L$, namely $\mathbf{E}(T, \mathbf{B}_L) = \mathbf{E}(T, \mathbf{B}_L, \pi_L)$,

(ii) If $L \in \mathscr{C}^\flat$, then we define the **boundary satellite ball** $\mathbf{B}_L^\flat = \mathbf{B}\big(p_L^\flat, 2^7 64 r_L\big)$ where $p_L^\flat$ is such that $|\mathbf{p}_{\pi_0}(p_L^\flat) - c_L| = \mathrm{d}(c_L, \gamma)$. Note that in this case the point $p_L^\flat$ is uniquely determined because Γ is regular and $\mathbf{A}$ is assumed to be sufficiently small. Likewise π_L is a plane which minimizes the excess $\mathbf{E}^\flat$, namely such that $\mathbf{E}^\flat(T, \mathbf{B}_L^\flat) = \mathbf{E}(T, \mathbf{B}_L^\flat, \pi_L)$ and $T_{p_L^\flat}\Gamma \subset \pi_L$.

A simple corollary of Theorem 21.2.2 and Corollary 21.3.3 is the following lemma.

Lemma 22.1.4. *Let T and Γ be as in Assumption 6. Then there is a positive dimensional constant $C(m,n)$ such that, if the starting size of the Whitney decomposition satisfies $2^{N_0} \geq C(m,n)M_0$, then the satellite balls $\mathbf{B}_L^\flat$ and $\mathbf{B}_L$ are all contained in $\mathbf{B}_2$. Moreover, there exists ε_1 such that, for any choice of $M_0, \alpha_{\mathbf{e}} > 0$ and $\alpha_{\mathbf{h}} < \frac{1}{2}$, if*

$$\mathbf{E}^\flat(T, \mathbf{B}_2) + \|\psi\|^2_{C^{3,a_0}} < \varepsilon_1 \,, \tag{22.1.1}$$

then for every cube $L \in \mathscr{C}^\flat$ we have

$$\mathbf{E}^\flat(T, \mathbf{B}_L^\flat) \leq C_0 \varepsilon_1 r_L^{2-2\alpha_{\mathbf{e}}}, \tag{22.1.2}$$

$$\mathbf{h}(T, \mathbf{B}_L^\flat, \pi_L) \leq C_0 \varepsilon_1^{1/4} r_L^{1+\alpha_{\mathbf{h}}}, \tag{22.1.3}$$

$$|\pi_L - \pi_0| \leq C_0 \varepsilon_1^{1/2}, \tag{22.1.4}$$

$$\Big|\pi_L - \pi(p_L^\flat)\Big| \leq C_0 \varepsilon_1^{1/2} r_L^{1-\alpha_{\mathbf{e}}}, \tag{22.1.5}$$

where, $\pi(p_L^\flat)$ is the m-dimensional tangent plane supporting the tangent cone to T at $p_L^\flat$ and C_0 depends only upon $\alpha_{\mathbf{e}}$, $\alpha_{\mathbf{h}}$, m and n.

Proof. Take $x \in \mathbf{B}_L^\natural$, using the height bound in (iii) of Assumption 8 we obtain that

$$|x| \leq 64 r_L + |p_L| \leq 64\sqrt{m} M_0 2^{-N_0} + |c_L| + C\varepsilon_0^{1/2} |p_L| \,,$$

recalling that $c_L \in \mathbf{B}(0, 3/2)$, possibly choosing ε_0 small enough and taking the constants $C(m,n), N_0$ big enough, we certainly obtain that $\mathbf{B}_L^\natural \subset \mathbf{B}(0,2)$. The proof that $\mathbf{B}_L^\flat \subset \mathbf{B}(0,2)$ is analogous with the exception that we might multiply the dimensional constant by 2^7. Inequality (22.1.4) is a direct consequence of (21.3.3). In this proof we will use the improved excess decay, i.e., Theorem 21.2.2, with $q = p_L^\flat, p = 0, r = 1, \rho = 2^7 64 r_L$. To prove estimate (22.1.2), we do as follows

$$\mathbf{E}^\flat(T, \mathbf{B}_L^\flat) = \mathbf{E}^\flat(T, \mathbf{B}\Big(p_L^\flat, 2^7 64 r_L\Big)) \overset{(21.2.4)}{\leq} C(2^7 64 r_L)^{2-2\alpha_{\mathbf{e}}} \mathbf{E}^\flat(T, \mathbf{B}(0,2)) + C(2^7 64 r_L)^{2-2\alpha_{\mathbf{e}}} \mathbf{A}^2,$$

and thus (22.1.1) concludes the proof of (22.1.2). With the same argument we prove (22.1.3) using in this time the height bound given by the excess decay, i.e., (21.2.5). It remains to prove (22.1.5), by the monotonicity formula, and recalling that $\Theta(T, p_L^\flat) = Q - \frac{Q^\star}{2} \geq \frac{3}{2}$, we have

$$\|T\|(\mathbf{B}_L^\flat) \geq \omega_m (2^7 64 r_L)^m .$$

Therefore, by the improved excess decay, we obtain

$$\mathbf{E}(T, \mathbf{B}_L^\flat, \pi_L) \leq \mathbf{E}(T, \mathbf{B}_L^\flat, \pi(p_L^\flat)) \overset{(21.2.4),(22.1.1)}{\leq} C_0 \varepsilon_1 r_L^{2-2\alpha_{\mathbf{e}}} .$$

Hence

$$|\pi(p_L^\flat) - \pi_L|^2 \leq C_0 \big(\mathbf{E}(T, \mathbf{B}_L^\flat, \pi_L) + \mathbf{E}(T, \mathbf{B}_L^\flat, \pi(p_L^\flat))\big) \leq C_0 \varepsilon_1 r_L^{2-2\alpha_{\mathbf{e}}} .$$

□

22.2 Stopping conditions of the Whitney decomposition

We will now start to refine our Whitney decomposition putting into account the properties of small excess and height bound of the current, in the sense of Lemma 22.1.4, to then obtain further stronger information about the current on each cube of the decomposition. To this end let $C_{\mathbf{e}}, C_{\mathbf{h}}$ be two large positive constants that will be fixed later. We take a cube $L \in \mathscr{C}_{N_0}$ and we **do not** subdivide it if either the excess "is too big" or the current "is too high", precisely if it belongs to one of the following sets:

(1) $\mathscr{W}_{N_0}^{\mathbf{e}} := \{L \in \mathscr{C}_{N_0}^\natural : \mathbf{E}(T, \mathbf{B}_L) > C_{\mathbf{e}} \varepsilon_1 \ell(L)^{2-\alpha_{\mathbf{e}}}\}$;

(2) $\mathscr{W}_{N_0}^{\mathbf{h}} := \{L \in \mathscr{C}_{N_0}^\natural : \mathbf{h}(T, \mathbf{B}_L, \pi_L) > C_{\mathbf{h}} \varepsilon_1^{1/2m} \ell(L)^{1+\alpha_{\mathbf{h}}}\}$.

We then define

$$\mathscr{S}_{N_0} := \mathscr{C}_{N_0} \setminus \left(\mathscr{W}_{N_0}^{\mathbf{e}} \cup \mathscr{W}_{N_0}^{\mathbf{h}}\right) .$$

The cubes in $\mathscr{S}_{N_0}$ will be subdivided in their childs, it means that we are subdividing the cube whenever the current is well behaved in it. In what follows we aim to show that the current is well behaved in the whole ball, it means that we will ensure that $\mathscr{W}_{N_0} := \mathscr{W}_{N_0}^{\mathbf{e}} \cup \mathscr{W}_{N_0}^{\mathbf{h}} = \emptyset$, and therefore $\mathscr{C}_{N_0} = \mathscr{S}_{N_0}$, by choosing $C_{\mathbf{e}}$ and $C_{\mathbf{h}}$ large enough, depending only upon $\alpha_{\mathbf{h}}, \alpha_{\mathbf{e}}, M_0$ and N_0, see Proposition 22.3.1 below.

We next describe the refining procedure assuming inductively that for a certain step $j \geq N_0 + 1$ we have defined the families $\mathscr{W}_{j-1}$ and $\mathscr{S}_{j-1}$. In particular we consider all the cubes L in $\mathscr{C}_j$ which are contained in some element of $\mathscr{S}_{j-1}$. Among them we select and set aside in the classes $\mathscr{W}_j := \mathscr{W}_j^{\mathbf{e}} \cup \mathscr{W}_j^{\mathbf{h}} \cup \mathscr{W}_j^{\mathbf{n}}$ those cubes where the following stopping criteria are met:

(1) $\mathscr{W}_j^{\mathbf{e}} := \{L \text{ child of } K \in \mathscr{S}_{j-1}^\natural : \mathbf{E}(T, \mathbf{B}_L) > C_{\mathbf{e}} \varepsilon_1 \ell(L)^{2-2\alpha_{\mathbf{e}}}\}$,

(2) $\mathscr{W}_j^{\mathbf{h}} := \{L \text{ child of } K \in \mathscr{S}_{j-1}^\natural : L \notin \mathscr{W}_j^{\mathbf{e}} \text{ and } \mathbf{h}(T, \mathbf{B}_L, \pi_L) > C_{\mathbf{h}} \varepsilon_1^{1/2m} \ell(L)^{1+\alpha_{\mathbf{h}}}\}$,

(3) $\mathscr{W}_j^{\mathbf{n}} := \{L \text{ child of } K \in \mathscr{S}_{j-1} : L \notin \mathscr{W}_j^{\mathbf{e}} \cup \mathscr{W}_j^{\mathbf{h}} \text{ but } \exists L' \in \mathscr{W}_{j-1} \text{ with } L \cap L' \neq \emptyset\}$.

We keep refining the decomposition in the set

$$\mathscr{S}_j := \{L \in \mathscr{C}_j \text{ child of } K \in \mathscr{S}_{j-1}\} \setminus \mathscr{W}_j\,.$$

Observe that it might happen that the child of a cube in $\mathscr{S}_{j-1}$ does not intersect $\mathrm{B}^+_{3/2}(0)$: in that case, according to our definition, the cube does not belong to $\mathscr{S}_j$ neither to $\mathscr{W}_j$: it is simply discarded. As already mentioned, we use the notation $\mathscr{S}^\flat_j$ and $\mathscr{S}^\natural_j$ respectively for $\mathscr{S}_j \cap \mathscr{C}^\flat$ and $\mathscr{S}_j \cap \mathscr{C}^\natural$. Furthermore we set

$$\begin{aligned}
\mathscr{W} &:= \bigcup_{j\geq N_0} \mathscr{W}_j,\\
\mathscr{S} &:= \bigcup_{j\geq N_0} \mathscr{S}_j,\\
\mathbf{S}^+ &:= \bigcap_{j\geq N_0} \Big(\bigcup_{L\in\mathscr{S}_j} L\Big) = \mathrm{B}^+_{3/2}(0) \setminus \bigcup_{H\in\mathscr{W}} H\,.
\end{aligned}$$

Note, in particular, that the refinement of boundary cubes can *never* be stopped because of the conditions (1) and (2), as we state in the following.

Lemma 22.2.1. *$\mathscr{C}^\flat_j \cap \mathscr{W}_i = \emptyset$ for every $i, j \geq N_0$ and in particular $\gamma \cap \mathrm{B}^+_{3/2}(0) \subset \mathbf{S}^+$. Thus boundary cubes always belong to $\mathscr{S}$.*

Proof. Assume there is a boundary cube in $\mathscr{W}_i$, then let L be a boundary cube in $\mathscr{W}_i$ with largest side length. The latter must then belong to $\mathscr{W}^{\mathbf{n}}_i$ because Lemma 22.1.4 excludes the possibility of L to belong to either $\mathscr{W}^{\mathbf{e}}_j$ or $\mathscr{W}^{\mathbf{h}}_j$. However, by definition of the family, this would imply the existence of a neighbor $L' \in \mathscr{W}_i$ with $\ell(L') = 2\ell(L)$. By Lemma 22.1.2, L' would be a boundary cube in $\mathscr{W}$, contradicting the maximality of L. □

Clearly, descendants of boundary cubes might become non-boundary cubes and so their refining cubes can be stopped. We finally set $\mathscr{W}_j := \mathscr{W}^{\mathbf{e}}_j \cup \mathscr{W}^{\mathbf{h}}_j \cup \mathscr{W}^{\mathbf{n}}_j$. From now on we specify a set of assumptions on the various choices of the constants involved in the construction.

Assumption 9. T and Γ are as in Assumptions 6 and we also assume that

(a) $\alpha_{\mathbf{h}}$ is smaller than $\frac{1}{2m}$ and $\alpha_{\mathbf{e}}$ is positive but small, depending only on $\alpha_{\mathbf{h}}$,

(b) M_0 is larger than a suitable constant, depending only upon $\alpha_{\mathbf{e}}$,

(c) $2^{N_0} \geq C(m, n, M_0)$, in particular it satisfies the condition of Lemma 22.1.4,

(d) $C_{\mathbf{e}}$ is sufficiently large depending upon $\alpha_{\mathbf{e}}$, $\alpha_{\mathbf{h}}$, M_0 and N_0,

(e) $C_{\mathbf{h}}$ is sufficiently large depending upon $\alpha_{\mathbf{e}}, \alpha_{\mathbf{h}}, M_0, N_0$ and $C_{\mathbf{e}}$,

(f) (22.1.1) holds with an ε_1 sufficiently small depending upon all the other parameters.

Finally, there is an exponent $\alpha_{\mathbf{L}}$, which depends only on $m, n, Q^\star$ and Q and which is independent of all the other parameters, in terms of which several important estimates in Theorem 22.6.5 will be stated.

We are ensuring that there is a nonempty set of parameters satisfying all the requirements, since the parameters are chosen following a precise hierarchy. The hierarchy is consistent with that of [DS16a], the reader could compare Assumption 9 with [DS16a, Assumption 1.9].

22.3 Tilting optimal planes and L-interpolating functions

In this section, we will define the interpolating functions which will give rise to the function $\boldsymbol{\varphi}^+$ whose graphs will define the center manifold $\mathcal{M}^+$. In order to begin with the construction of these objects, we shall notice that an important fact is that up to now, in the construction of the decomposition, we have nice local information about the current, i.e., inside each square of the decomposition we can do a good analysis, however, we do not know how to work among those cubes, i.e., how to compare quantities as the excess and height between two different cubes of the decomposition. To that end, we enunciate the following crucial result which is the analogous of [DDHM18, Proposition 8.24] which is stated for $Q^\star = 1$, we mention that the proof of this proposition readily works for currents with boundary multiplicity equal to $Q^\star \geq 1$.

Proposition 22.3.1 (Tilting and optimal planes, Proposition 8.24, [DDHM18])**.** *Under the Assumptions 8 and 9, we have $\mathscr{W}_{N_0} = \emptyset$. Then the following estimates hold for any couple of neighbors $H, L \in \mathscr{S} \cup \mathscr{W}$ and for every $H, L \in \mathscr{S} \cup \mathscr{W}$ with H descendant of L:*

(a) denoting by π_H, π_L the optimal planes for the excess in $\mathbf{B}_H$ and $\mathbf{B}_L$, respectively, we have

$$|\pi_H - \pi_L| \leq \bar{C}\varepsilon_1^{1/2}\ell(L)^{1-\alpha_{\mathbf{e}}}, \qquad |\pi_H - \pi_0| \leq \bar{C}\varepsilon_1^{1/2},$$

(b)$^\natural$ $\mathbf{h}(T, \mathbf{C}_{48r_H}(p_H, \pi_0)) \leq C\varepsilon_1^{1/2m}\ell(H)$ *and* $\operatorname{spt}(T) \cap \mathbf{C}_{48r_H}(p_H, \pi_0) \subset \mathbf{B}_H$ *if* $H \in \mathscr{C}^\natural$,

(b)$^\flat$ $\mathbf{h}(T, \mathbf{C}_{2^7 48 r_H}(p_H^\flat, \pi_0)) \leq C\varepsilon_1^{1/4}\ell(H)$ *and* $\operatorname{spt}(T) \cap \mathbf{C}_{2^7 48 r_H}(p_H^\flat, \pi_0) \subset \mathbf{B}_H^\flat$ *if* $H \in \mathscr{C}^\flat$,

(c)$^\natural$ $\mathbf{h}(T, \mathbf{C}_{36 r_L}(p_L, \pi_H)) \leq C\varepsilon_1^{1/2m}\ell(L)^{1+\alpha_{\mathbf{h}}}$ *and* $\operatorname{spt}(T) \cap \mathbf{C}_{36 r_L}(p_L, \pi_H) \subset \mathbf{B}_L$ *if* $H, L \in \mathscr{C}^\natural$,

(c)$^\flat$ $\mathbf{h}(T, \mathbf{C}_{2^7 36 r_L}(p_L^\flat, \pi_H)) \leq C\varepsilon_1^{1/4}\ell(L)^{1+\alpha_{\mathbf{h}}}$ *and* $\operatorname{spt}(T) \cap \mathbf{C}_{2^7 36 r_L}(p_L^\flat, \pi_H)) \subset \mathbf{B}_L^\flat$ *if* $L \in \mathscr{C}^\flat$,

where $\bar{C} = \bar{C}(\alpha_{\mathbf{e}}, \alpha_{\mathbf{h}}, M_0, N_0, C_{\mathbf{e}}) > 0$ *and* $C = C(\alpha_{\mathbf{e}}, \alpha_{\mathbf{h}}, M_0, N_0, C_{\mathbf{e}}, C_{\mathbf{h}}) > 0$.

We now state the following results which is the analogous of [DDHM18, Proposition 8.7] will allow us to locally approximate the current by $(Q - \frac{Q^\star}{2})$-Lipschitz maps in the sense of Theorem 21.4.1. We also noticed that the proof given in [DDHM18, Proposition 8.7] for $Q^\star = 1$ readily works for currents with boundary multiplicity equal to $Q^\star \geq 1$.

Proposition 22.3.2 (Proposition 8.7, [DDHM18])**.** *Under the Assumptions 8 and 9 the following holds for every couple of neighbors $H, L \in \mathscr{S} \cup \mathscr{W}$ and any $H, L \in \mathscr{S} \cup \mathscr{W}$ with H descendant of L:*

$$\begin{aligned} &\operatorname{spt}(T) \cap \mathbf{C}_{36 r_L}(p_L, \pi_H) \subset \mathbf{B}_L && \text{when } L \in \mathscr{C}^\natural, \\ &\operatorname{spt}(T) \cap \mathbf{C}_{2^7 36 r_L}(p_L^\flat, \pi_H) \subset \mathbf{B}_L^\flat && \text{when } L \in \mathscr{C}^\flat, \end{aligned}$$

and the current T satisfies the assumptions of [DS14, Theorem 2.4] in the cylinder $\mathbf{C}_{36 r_L}(p_L, \pi_H)$ (resp. of Theorem 21.4.1 in the cylinder $\mathbf{C}_{2^7 36 r_L}(p_L^\flat, \pi_H)$).

We will now construct the "interpolating functions" g_L for each cube L. To begin with the construction of this interpolation, we approximate the current T by $(Q-\frac{Q^\star}{2})$-Lipschitz functions (Proposition 22.3.2) that will determine the boundary condition of an elliptic system which comes from the linearization of the mean curvature condition for minimal surfaces. The solution of this elliptic system will be further represented by a function g_L defined in the tilted ball contained in π_0 taking values in $\pi_0^\perp$, i.e., we changed to our new reference coordinate system. Since the construction will be local, i.e., in each cube, over the set $\mathrm{B}^+_{3/2}(0) \setminus \mathbf{S}^+$ we will patch every g_L together with a partition of the unity to obtain the function $\boldsymbol{\varphi}^+$, whose graph will be the center manifold, defined in the whole ball. So we need to define $\boldsymbol{\varphi}^+$ over $\mathbf{S}^+$ as well, to that end we introduce all the machinery needed for all cubes in $\mathscr{S} \cup \mathscr{W}$.

Definition 22.3.3 (π_L-approximations)**.** Under the Assumptions 8 and 9, we set

(i) If $L \in \mathscr{S}_j^\flat$ for some j, take the Lipschitz approximation (f_L^+, f_L^-) in the cylinder $\mathbf{C}\big(p_L^\flat, 2^7 9 r_L, \pi_L\big)$ given by Proposition 22.3.2, we call (f_L^+, f_L^-) **a π_L-approximation of T in the cylinder $\mathbf{C}\big(p_L^\flat, 2^7 9 r_L, \pi_L\big)$**.

(ii) If $L \in \mathscr{S}_j^\natural \cup \mathscr{W}_j$ for some j, we take the Lipschitz approximation f_L in the cylinder $\mathbf{C}(p_L, 9r_L, \pi_L)$ given by Proposition 22.3.2, we call f_L **a π_L-approximation of T in the cylinder $\mathbf{C}(p_L, 9r_L, \pi_L)$**.

Definition 22.3.4 (L-tilded harmonic interpolations)**.** Under the Assumptions 8 and 9, we define

(i) if L is a nonboundary cube, let $h_L : \mathrm{B}_{5r_L}(p_L, \pi_L) \to \mathbb{R}$ to be an harmonic function with boundary condition $h_L|_{\partial \mathrm{B}_{5r_L}(p_L,\pi_L)} = \boldsymbol{\eta} \circ f_L|_{\partial \mathrm{B}_{5r_L}(p_L,\pi_L)}$,

(ii) if L is a boundary cube, let $h_L : \mathrm{B}_{2^7 5 r_L}(p_L^\flat, \pi_L) \to \mathbb{R}$ to be an harmonic function with boundary condition $h_L|_{\partial \mathrm{B}_{2^7 5 r_L}(p_L^\flat,\pi_L)} = \boldsymbol{\eta} \circ f_L^+|_{\partial \mathrm{B}_{2^7 5 r_L}(p_L^\flat,\pi_L)}$.

We call h_L **L-tilted harmonic interpolating function**.

We now are ready to define the final function, g_L, on our "reference coordinate system", i.e., the domain of g_L is contained in π_0 and g_L takes values in $\pi_0^\perp$, with the property that its graph coincides with a suitable portion of the graph of h_L. The function g_L is furnished by [DS16a, Lemma B.1] which we state below.

Proposition 22.3.5 (L-interpolating functions)**.** *Under the assumptions of Proposition 22.3.2, we have*

(i) If L is a boundary cube, the function h_L is Lipschitz on $\mathrm{B}^{\pm}_{2^7 \cdot 9 r_L/2}(p_L^\flat, \pi_L)$ and we can define a function $g_L : \mathrm{B}^+_{2^7 4 r_L}(p_L^\flat, \pi_0) \to \pi_0^\perp$ such that $\mathbf{G}_{g_L} = \mathbf{G}_{h_L} \llcorner \mathrm{B}^+_{2^7 4 r_L}(p_L^\flat, \pi_0) \times \mathbb{R}^n$,

(ii) If L is a non-boundary cube, the function h_L is Lipschitz on $\mathrm{B}_{9r_L/2}(p_L, \pi_L)$ and we can define a function $g_L : \mathrm{B}_{4r_L}(p_L, \pi_0) \to \pi_0^\perp$ such that $\mathbf{G}_{g_L} = \mathbf{G}_{h_L} \llcorner \mathbf{C}_{4r_L}(p_L, \pi_0)\big)$.

*The functions g_L is called L-**interpolating function**.*

22.4 Glueing L-interpolations

We now define another set of cubes, the Whitney cubes at the step j, which will be similar to what we have constructed until now but we are including all the ancestors with respect to step j in the same family as follows

$$\mathscr{P}_j := \mathscr{S}_j \cup \bigcup_{i=N_0+1}^{j} \mathscr{W}_i.$$

Note that $\mathscr{P}_j$ is a "Whitney family of dyadic cubes" in the sense that if $K, L \in \mathscr{P}_j$ and $K \cap L \neq \emptyset$, then $\frac{1}{2}\ell(L) \leq \ell(K) \leq 2\ell(L)$. We fix a mollifier function ϑ satisfying

$$\vartheta \in C_c^{\infty}\left(\left[-\frac{17}{16}, \frac{17}{16}\right]^m, [0,1]\right),\ \vartheta|_{[-1,1]^m} \equiv 1,\ \text{and, for each cube } L,\ \tilde{\vartheta}_L(y) := \vartheta\left(\frac{y - c(L)}{\ell(L)}\right).$$

We thus set a partition of unity of $\mathrm{B}^+_{3/2}(0)$ defined as

$$\vartheta_L : \mathrm{B}^+_{3/2}(0) \to \mathbb{R},\ \vartheta_L(y) := \frac{\tilde{\vartheta}_L(y)}{\sum_{H \in \mathscr{P}_j} \tilde{\vartheta}_H(y)}.$$

Definition 22.4.1 (Glued interpolation at the step j)**.** We set $\varphi_j := \sum_{L \in \mathscr{P}_j} \vartheta_L g_L$, this maps φ_j are called **glued interpolation at the step j.**

22.5 Existence of a $C^{3,\kappa}$-center manifold

We are now ready to state the main theorem regarding the construction of the center manifolds needed in this paper, i.e., Theorem 22.5.1, which ensures that $(\varphi_j)_j$ is a sequence that converges to a $C^{3,\kappa}$ map, $\kappa > 0$, whose graph will be called the center manifold. This limit map has good properties as the smallness of the $C^{3,\kappa}$ norm, which is bounded by ε_1. After the main theorem, we will start the construction of the normal approximation in the sense of Theorem 21.4.1 but now the approximations will be defined on the center manifold and will take values on the normal bundle of the center manifold. The normal approximations will enjoy some good estimates relying in the estimates of Theorem 21.4.1 and the estimate obtained in the construction of our Whitney decomposition. The main theorem of this section is stated below and is the version adapted to our setting of Theorem 8.13 of [DDHM18].

Theorem 22.5.1 (Theorem 8.13 ,[DDHM18])**.** *Under Assumptions 8 and 9, there is a* $\kappa := \kappa(\alpha_{\mathbf{e}}, \alpha_{\mathbf{h}}) > 0$*, such that*

(i) $\varphi_j \in C^{3,\kappa}$*, moreover* $\|\varphi_j\|_{3,\kappa,\mathrm{B}^+_{3/2}(0)} \leq C\varepsilon_1^{1/2}$*, for some* $C := C(\alpha_{\mathbf{e}}, \alpha_{\mathbf{h}}, M_0, C_{\mathbf{e}}, C_{\mathbf{h}}) > 0$*,*

(ii) If $i \leq j$*,* $L \in \mathscr{W}_{i-1}$ *and* H *is a cube concentric to* L *with* $\ell(H) = \frac{9}{8}\ell(L)$*, then* $\varphi_j = \varphi_i$ *on* H*,*

(iii) φ_j *converges in* C^3 *to a map* $\varphi^+ : \mathrm{B}^+_{3/2}(0) \to \mathbb{R}^n$*, whose graph is a* $C^{3,\kappa}$ *submanifold* $\mathcal{M}^+$*, which will be called the* ***right center manifold****;*

(iv) $\varphi^+ = \psi$ *on* $\gamma \cap B_{3/2}$*, i.e.,* $\partial\mathcal{M}^+ \cap \mathbf{C}(0, 3/2) = \Gamma \cap \mathbf{C}(0, 3/2)$*;*

(v) *For any $q \in \partial\mathcal{M}^+ \cap \mathbf{C}(0,3/2)$, we have $T_q\mathcal{M}^+ = \pi(q)$ where $\pi(q)$ is the support of the unique tangent cone to T at q.*

We will omit the proof of 22.5.1 because goes mutatis mutandis along the same lines of [DDHM18, Theorem 8.13] and even if this last theorem is stated just when $Q^\star = 1$, a careful inspection of its proof will reveal that it also holds when $Q^\star \geq 1$.

Remark 22.5.2. The construction of $\mathcal{M}^+$ made in Theorem 22.5.1 is based on the decomposition of $\mathrm{B}^+_{3/2}(0)$. Under Assumption 9, the same construction can be made for $\mathrm{B}^-_{3/2}(0)$ and gives a $C^{3,\kappa}$ map $\varphi^- : \mathrm{B}^-_{3/2}(0) \to \mathbb{R}^n$ which agrees with ψ on $\gamma \cap B_{3/2}$. The graph of φ^- is a $C^{3,\kappa}$ submanifold $\mathcal{M}^-$, which will be called the **left center manifold**. Its boundary in the cylinder $\mathbf{C}(0,3/2)$, namely $\mathbf{C}(0,3/2) \cap \partial\mathcal{M}^-$, coincides, in a set-theoretical sense, with $\mathbf{C}(0,3/2) \cap \partial\mathcal{M}^+$, but it has opposite orientation, and moreover its tangent plane $T_q\mathcal{M}^-$ coincides with $\pi(q)$ for every point $q \in \mathbf{C}(0,3/2) \cap \partial\mathcal{M}^-$.

In particular, the union $\mathcal{M} := \mathcal{M}^+ \cup \mathcal{M}^-$ of the two submanifolds is a $C^{1,1}$ submanifold in $\mathbf{C}(0,3/2)$ without boundary in $\mathbf{C}(0,3/2)$, which will be called the **center manifold**. Moreover, we will often state properties of the center manifold related to cubes L in one of the collections $\mathscr{W}_j$ described above. Therefore, we will denote by $\mathscr{W}^+$ the union of all $\mathscr{W}_j$ and by $\mathscr{W}^-$ the union of the corresponding classes of cubes which lead to the left center manifold $\mathcal{M}^-$. We emphasize again that so far we can only conclude the $C^{1,1}$ regularity of $\mathcal{M}$, because we do not know that the traces of the second derivatives of φ^+ and φ^- coincide on γ.

Definition 22.5.3. Let us define the graph parametrization map of $\mathcal{M}^+$ as $\mathbf{\Phi}^+(x) := (x, \varphi^+(x))$. We will call **right contact set** the subset $\mathbf{K}^+ := \mathbf{\Phi}^+(\mathbf{S}^+)$. For every cube $L \in \mathscr{W}^+$ we associate a **Whitney region $\mathcal{L}$ on $\mathcal{M}^+$** as follows:

- $\mathcal{L} := \mathbf{\Phi}^+(H \cap \mathrm{B}_1(0))$ where H is the cube concentric to L such that $\ell(H) = \frac{17}{16}\ell(L)$.

Analogously we define the map $\mathbf{\Phi}^-$, the **left contact set $\mathbf{K}^-$** and the **Whitney regions on the left center manifold $\mathcal{M}^-$**.

To keep the text flow, we postpone the proof of the Theorem 22.5.1 to the last part of this section.

22.6 The $\mathcal{M}$-Lipschitz approximations defined on the center manifolds

Since the two portions $\mathcal{M}^-$ and $\mathcal{M}^+$ are $C^{3,\kappa}$ and they join with $C^{1,1}$ regularity along Γ, in a sufficiently small normal neighborhood of $\mathcal{M}$ there is a well defined orthogonal projection $\mathbf{p}$ onto $\mathcal{M}$. The thickness of the tubular neighborhood is inversely proportional to the norm of the second derivatives of $\varphi^\pm$ and hence, for ε_1 sufficiently small, we can assume that the thickness is 2 which leads to the next assumption.

Assumption 10. Under Assumptions 8 and 9. We let $\mathcal{M} := \mathcal{M}^+ \cup \mathcal{M}^-$ and ε_1 be sufficiently small so that, if

$$\mathbf{U} := \{q \in \mathbb{R}^{m+n} : \exists! q' = \mathbf{p}(q) \in \mathcal{M} \text{ s.t. } |q - q'| < 1 \text{ and } q - q' \in T^\perp_{q'}\mathcal{M}\},$$

where $T^{\perp}_{q'}\mathcal{M} := (T_{q'}\mathcal{M})^{\perp}$, then the map $\mathbf{p}$ extends to a Lipschitz map to the closure $\overline{\mathbf{U}}$ which is $C^{2,\kappa}$ on $\mathbf{U} \setminus \mathbf{p}^{-1}(\Gamma)$ and

$$\mathbf{p}^{-1}(q') = q' + \overline{\mathrm{B}_1(0,(T_{q'}\mathcal{M})^{\perp})} \text{ for all } q' \in \overline{\mathcal{M}}.$$

As highlighted before, we construct the center manifold $\mathcal{M}$ and also a function defined on $\mathcal{M}$ that approximates, with the desired superlinear exponents for the error, the current T (in the sense of Theorem 21.4.1). This approximation will take values on the normal bundle of $\mathcal{M}$, we define precisely this type of approximations. But before that, we shall also define the space of Q-tuples on a manifold analogously to what is done for Euclidean spaces in the Introduction of [DS11].

Definition 22.6.1. Let M be an m-dimensional manifold, and, for each $P \in M$, we denote $[\![P]\!]$ the current with support equal to P, i.e., the current associated with the Dirac measure concentrated in P. Then we define the space of unordered Q-tuples in M, for any $Q \in \mathbb{N}, Q \geq 1$, as follows

$$\mathcal{A}_Q(M) := \left\{ \sum_{i=1}^{Q} [\![P_i]\!] : P_i \in M \text{ for every } i \in \{1,\dots,Q\} \right\}.$$

Definition 22.6.2. Let $\mathcal{M}$ be the center manifold as in Theorem 22.5.1 without loss of generality we can assume that we are under Assumption 10, $Q^+ = Q$ and $Q^- = Q - Q^\star$. We say that $(\mathcal{K}, F^+, F^-)$ is an **$\mathcal{M}$-normal approximation of T**, if

(i) there exist Lipschitz functions $\mathcal{N}^+ : \mathcal{M}^+ \cap \mathbf{C}(0,1) \to \mathcal{A}_Q(T^{\perp}\mathcal{M}^+)$, $\mathcal{N}^+(x) \in \mathcal{A}_Q(T^{\perp}_x\mathcal{M}^+)$ and $\mathcal{N}^- : \mathcal{M}^- \cap \mathbf{C}(0,1) \to \mathcal{A}_{Q-Q^\star}(T^{\perp}\mathcal{M}^-)$, $\mathcal{N}^-(x) \in \mathcal{A}_{Q-Q^\star}(T^{\perp}_x\mathcal{M}^-)$, where $T^{\perp}\mathcal{M}^{\pm} := \sqcup_{x\in\mathcal{M}^{\pm}} T^{\perp}_x\mathcal{M}^{\pm}$ denotes the normal bundle of $\mathcal{M}^{\pm}$ and is seen as a subset of $\mathbb{R}^{m+n}$,

$$\begin{array}{rccl} \mathcal{N}^{\pm} : & \mathcal{M}^{\pm} \cap \mathbf{C}(0,1) & \to & \mathcal{A}_{Q^{\pm}}(T^{\perp}\mathcal{M}^{\pm}) \\ & x & \mapsto & \mathcal{N}^{\pm}(x) := \sum_{i=1}^{Q^{\pm}} [\![\mathcal{N}^{\pm}_i(x)]\!] , \end{array}$$

where $(\mathcal{N}^{\pm}_i : \mathcal{M}^+ \cap \mathbf{C}(0,1) \to T^{\perp}\mathcal{M})_{i\in\{1,\dots,Q^{\pm}\}}$ are measurable sections of the normal bundle, i.e., each $\mathcal{N}^{\pm}_i$ is a classical 1-valued measurable function satisfying $\mathcal{N}^{\pm}_i(x) \in T^{\perp}_x\mathcal{M}^{\pm}$. We then define the Lipschitz function given by

$$\begin{array}{rccl} F^{\pm} : & \mathcal{M}^{\pm} \cap \mathbf{C}(0,1) & \to & \mathcal{A}_{Q^{\pm}}(T^{\perp}\mathcal{M}^{\pm}) \\ & x & \mapsto & (\mathcal{N}^{\pm} \oplus \mathrm{id})\,(x). \end{array}$$

(ii) $\mathcal{K} \subset \mathcal{M}$ is closed and $\mathbf{T}_{F^{\pm}} \llcorner \mathbf{p}^{-1}(\mathcal{K} \cap \mathcal{M}^{\pm}) = T \llcorner \mathbf{p}^{-1}(\mathcal{K} \cap \mathcal{M}^{\pm})$, where $\mathbf{T}_{F^{\pm}} := (F^{\pm})_\sharp [\![\mathcal{M}]\!]$, according to [DS15, Definition 1.3],

(iii) $\mathbf{K}^+ \cup \mathbf{K}^- \subset \mathcal{K}$, $\mathcal{N}^{\pm}|_{\mathcal{K}} \equiv 0$, and then $F^+(x) = Q\,[\![x]\!]$ on $\mathcal{K}^+$ and $F^-(x) = (Q - Q^\star)\,[\![x]\!]$ on $\mathcal{K}^-$.

Observe that the pairs (F^+, F^-) and $(\mathcal{N}^+, \mathcal{N}^-)$ are intuitively $(Q - \frac{Q^\star}{2})$-valued maps in the spirit of Definition 20.1.1. Although this is very intuitive, these functions are defined on manifolds, so, we make the precise definition of it as follows.

Definition 22.6.3. Firstly, we let Z be an m-dimensional manifold and Υ be a $(m-1)$-submanifold of the m-manifold M which splits M into the two connected components M^+ and M^-. Let $\Phi \in W^{\frac{1}{2},2}(\Upsilon, \mathcal{A}_{Q^\star}(Z))$, $Q, Q^\star \in \mathbb{N}$, $Q \geq Q^\star \geq 1$. A $(Q - \frac{Q^\star}{2})$**-valued function with interface** (Υ, Φ), consists of a pair (F^+, F^-) satisfying the following properties (*i*) $F^+ \in W^{1,2}(M^+, \mathcal{A}_Q(Z))$, $F^- \in W^{1,2}(M^-, \mathcal{A}_{Q-Q^\star}(Z))$,

(*ii*) $F^+|_\Upsilon = F^-|_\Upsilon + \Phi$.

We define the **Dirichlet energy of** (F^+, F^-) as $\mathrm{Dir}(F^+, F^-, M) := \mathrm{Dir}(F^+, M^+) + \mathrm{Dir}(F^-, M^-)$. Such a pair will be called **Dir-minimizing in** M, if for all $\left(Q - \frac{Q^\star}{2}\right)$-valued function (G^+, G^-) with interface (Υ, Φ) which agrees with (F^+, F^-) outside of a compact set $K \subset\subset M$ satisfies $\mathrm{Dir}(F^+, F^-, M) \leq \mathrm{Dir}(G^+, G^-, M)$.

Definition 22.6.4. Let (F^+, F^-) be a $\left(Q - \frac{Q^\star}{2}\right)$-valued function with interface (Υ, Φ) and $\Phi = Q^\star \left[\!\left[\hat{\Phi}\right]\!\right]$ for the single valued function $\hat{\Phi} \in W^{\frac{1}{2},2}(\Upsilon, Z)$. We say that (F^+, F^-) **collapses at the interface**, if $F^+|_\Upsilon = Q \left[\!\left[\hat{\Phi}\right]\!\right]$.

The following theorem ensures the existence of an $\mathcal{M}$-normal approximation suitable for our purposes, i.e., with the desired exponents at the bound on the Lipschitz constant, the Dirichlet energy and the size of the complement set of $\mathcal{K}$. It is the same as [DDHM18, Theorem 8.19] but of course adapted to our context where $Q^\star$ is taken any arbitrary positive integer. We omit its proof because goes precisely as in the proof of [DDHM18, Theorem 8.19].

Theorem 22.6.5 (Local behaviour of the $\mathcal{M}$-normal approximation on Whitney regions. Theorem 8.19 of [DDHM18])**.** *Under Assumption 10 there is a constant* $\alpha_{\mathbf{L}} := \alpha_{\mathbf{L}}(m, n, Q^\star, Q) > 0$ *such that there exists an* $\mathcal{M}$*-normal approximation* $(\mathcal{K}, F^+, F^-)$ *satisfying the following estimates on any Whitney region* $\mathcal{L} \subset \mathcal{M}$ *associated to a cube* $L \in \mathscr{W}^+ \cup \mathscr{W}^-$*:*

$$\mathrm{Lip}(\mathcal{N}^\pm|_\mathcal{L}) \leq C\varepsilon_1^{\alpha_{\mathbf{L}}} \ell(L)^{\alpha_{\mathbf{L}}}, \tag{22.6.1}$$

$$\|\mathcal{N}^\pm|_\mathcal{L}\|_0 \leq C\varepsilon_1^{1/2m} \ell(L)^{1+\alpha_{\mathbf{h}}}, \tag{22.6.2}$$

$$\mathcal{H}^m(\mathcal{L} \setminus \mathcal{K}) + \|\mathbf{T}_F - T\|(\mathbf{p}^{-1}(\mathcal{L})) \leq C\varepsilon_1^{1+\alpha_{\mathbf{L}}} \ell(L)^{m+2+\alpha_{\mathbf{L}}}, \tag{22.6.3}$$

$$\int_\mathcal{L} |D\mathcal{N}^\pm|^2 \leq C\varepsilon_1 \ell(L)^{m+2-2\alpha_{\mathbf{e}}}, \tag{22.6.4}$$

for a constant $C = C(\alpha_{\mathbf{e}}, \alpha_{\mathbf{h}}, M_0, N_0, C_{\mathbf{e}}, C_{\mathbf{h}}) > 0$*. Moreover, for any* $a > 0$ *and any Borel* $\mathcal{V} \subset \mathcal{L}$*,*

$$\begin{aligned} \int_\mathcal{V} |\eta \circ \mathcal{N}^\pm| \mathrm{d}\mathcal{H}^m &\leq C\varepsilon_1 \left(\ell(L)^{m+3+\alpha_{\mathbf{h}}/3} + a\ell(L)^{2+\alpha_{\mathbf{L}}/2}\mathcal{H}^m(\mathcal{V})\right) \\ &\quad + \frac{C}{a} \int_\mathcal{V} \mathcal{G}(\mathcal{N}^\pm, Q\left[\!\left[\eta \circ \mathcal{N}^\pm\right]\!\right])^{2+\alpha_{\mathbf{L}}} \mathrm{d}\mathcal{H}^m. \end{aligned} \tag{22.6.5}$$

Chapter 23

Blowup argument by the frequency function

In this section we finish the proof of the main theorem of this work, i.e., Theorem 17.0.3, which is a consequence of Theorem 19.3.8, for $m = 2$, as noticed in Subsection 19.3. We use the frequency function in the center manifold $\mathcal{M}$, c.f. [DDHM18, Chapter 9], originally introduced by Almgren and more recently totally reformulated and adapted to the boundary case by De Lellis, De Philippis, Hirsch, and Massaccesi ([DDHM18]). In Theorem 23.1.3, we show that the m-dimensional area minimizing current has to satisfies at most one of two conditions, where the first one essentially says that 0 is a two-sided regular point of T and the second one is an estimate with the frequency function. Although, there are two alternatives in Theorem 23.1.3, we use a blowup argument in Theorem 23.2.1 to show that the second alternative never occurs, thus implying that the only possible situation is 0 being a two-sided regular boundary point.

23.1 Frequency function w.r.t. $\mathcal{M}$

In order to define our main quantities, we start with the following lemma that shows that exists a good perturbation function of the distance function on the center manifold .

Lemma 23.1.1 (Lemma 9.2 ,[DDHM18])**.** *There exist positive continuous functions $d^\pm : \mathcal{M}^\pm \to \mathbb{R}_+$ which belong to $C^2(\mathcal{M}^\pm \setminus \{0\})$ and satisfies the following properties*

(a) $d^\pm(x) = \mathrm{d}_{\mathcal{M}^\pm}(x, 0) + O(\mathrm{d}_{\mathcal{M}^\pm}(x, 0)^2) = |x| + O(|x|^2)$,

(b) $|\nabla d^\pm(x)| = 1 + O(d^\pm)$, *where ∇ is the gradient on the manifold $\mathcal{M}$,*

(c) $\frac{1}{2}\nabla^2(d^\pm)^2(x) = g + O(d^\pm)$, *where ∇^2 denotes the covariant Hessian on $\mathcal{M}$ (which we regard as a $(0, 2)$ tensor) and g is the induced metric on $\mathcal{M}$ as a submanifold of $\mathbb{R}^{m+n}$,*

(d) $\nabla d^\pm(p) \in T_p\Gamma$ *for all $p \in \Gamma$, i.e.*

$$\nabla d^\pm \cdot \vec{n}^\pm = 0 \text{ on } \Gamma, \tag{23.1.1}$$

where $\vec{n}^\pm$ denotes the outer unit normal to $\mathcal{M}^\pm$ inside $\mathcal{M}$.

In particular this implies

$$\nabla^2 d^\pm(x) = \frac{1}{d}\Big(g - \nabla d^\pm(x) \otimes \nabla d^\pm(x)\Big) + O(1) \tag{23.1.2}$$

and

$$\Delta\, d^\pm = \frac{m-1}{d^\pm} + O(1) \tag{23.1.3}$$

where Δ denotes the Laplace-Beltrami operator on $\mathcal{M}$, namely the trace of the Hessian ∇^2. Moreover:

(S) All the constants estimating the $O(\cdot)$ error terms in the above estimates can be made smaller than any given $\eta > 0$, provided the parameter ε_1 in Assumption 9 is chosen appropriately small (depending on η).

Let us define three functions that we be used in the definition of the frequency function as follows

$$\phi(t) := \begin{cases} 1, & \text{for } 0 \le t \le \frac{1}{2}, \\ 2(1-t), & \text{for } \frac{1}{2} \le t \le 1, \\ 0, & \text{for } t \ge 1, \end{cases} \tag{23.1.4}$$

$$\mathrm{D}_{\phi,d^\pm}(\mathcal{N}^\pm, r) := \int_{\mathcal{M}^\pm} \phi\left(\frac{d^\pm(x)}{r}\right) |D\mathcal{N}^\pm|^2(x)\, \mathrm{d}\,\mathrm{Vol}^\pm, \tag{23.1.5}$$

$$\mathrm{H}_{\phi,d^\pm}(\mathcal{N}^\pm, r) := -\int_{\mathcal{M}^\pm} \phi'\left(\frac{d^\pm(x)}{r}\right) |\nabla d^\pm(x)|^2 \frac{|\mathcal{N}^\pm(x)|^2}{d^\pm(x)}\, \mathrm{d}\,\mathrm{Vol}^\pm, \tag{23.1.6}$$

where $\mathrm{Vol}^\pm$ denotes the standard volume form on $\mathcal{M}^\pm$.

Definition 23.1.2. The **frequency function** is defined as the ratio

$$\mathrm{I}_{\phi,d^\pm}(\mathcal{N}^\pm, r) := \frac{r\mathrm{D}_{\phi,d^\pm}(\mathcal{N}^\pm, r)}{\mathrm{H}_{\phi,d^\pm}(\mathcal{N}\pm, r)}.$$

We also set the notation

$$\mathcal{C}^\pm := \Big\{y \in \mathbf{B}(0,1) : \mathbf{p}(y) \in \mathcal{M}^\pm \text{ and } |y - \mathbf{p}(y)| \le \mathrm{d}(y,\Gamma)^{3/2}\Big\}$$

for the horned neighborhoods of $\mathcal{M}^\pm$ in which T is supported, compare with Corollary 21.3.3 and item (v) of Theorem 22.5.1.

Theorem 23.1.3 (Theorem 9.3, [DDHM18])**.** *Let T and Γ be as in Assumption 10, $Q^+ = Q, Q^- = Q - Q^\star$ and consider ϕ and $d^\pm$ as above. Then only one of the following alternatives holds*

(a) $T \llcorner \mathcal{C}^\pm = Q^\pm [\![\mathcal{M}^\pm]\!]$ *in a neighborhood of* 0,

(b) $\lim_{r\to 0} \mathrm{I}_{\phi,d^\pm}(\mathcal{N}^\pm, r) > 0$.

23.2 Blowup argument

Letting $Q^+ := Q$ and $Q^- := Q - Q^\star$, we define the multivalued maps with domain and codomain in Euclidean spaces,

$$N^\pm(x) = \sum_{i=1}^{Q^\pm} [\![(N^i)^\pm(x)]\!] ,$$

such selections $\{N^i\}_{i=1}^{Q^\pm}$ are given by the formulas

$$\begin{array}{rrcl} (N^i)^\pm : & \mathrm{B}_1^\pm(0) \subset \mathbb{R}^m & \to & \mathbb{R}^n \\ & x & \mapsto & \mathbf{p}_{\{0\}\times\mathbb{R}^n}\big((N^i)^\pm(x, \boldsymbol{\varphi}^\pm(x))\big). \end{array}$$

Observe that the pair (N^+, N^-) is a $\left(Q - \frac{Q^\star}{2}\right)$-valued function with interface $(\gamma, Q^\star [\![0]\!])$. We now set the following notation for the Dirichlet energy

$$\mathrm{Dir}(r) := \frac{1}{2}\int_{\mathrm{B}_1^+(0)} |DN^+|^2 + \frac{1}{2}\int_{\mathrm{B}_1^-(0)} |DN^-|^2 := \mathrm{Dir}^+(r) + \mathrm{Dir}^-(r),$$

and the corresponding rescaling of $N^\pm$

$$N_r^\pm(x) := \sum_i \left[\!\!\left[r^{m/2-1}\,\mathrm{Dir}^\pm(r)^{-1/2}(N^i)^\pm(rx)\right]\!\!\right].$$

Finally, we can state the key result to give our final contradiction argument.

Theorem 23.2.1. *Let T and Γ be as in Assumption 10. If it holds*

$$\lim_{r\to 0} \mathrm{I}_{\phi,d^\pm}(N^\pm, r) > 0, \tag{23.2.1}$$

for at least one of the regions $\mathcal{C}^\pm$, then there exists a sequence $\rho_k \to 0$ as $k \to +\infty$ such that the sequence of pairs $(N^+_{\rho_k}, N^-_{\rho_k})$ would converge in $\mathrm{B}_1(0)$ locally strongly in L^2 to a $\left(Q - \frac{Q^\star}{2}\right)$ Dir-minimizer (N_0^+, N_0^-) where $N_0^+ : \mathrm{B}_1^+(0) \to \mathcal{A}_Q(\mathbb{R}^n)$ and $N_0^- : \mathrm{B}_1^-(0) \to \mathcal{A}_{Q-Q^\star}(\mathbb{R}^n)$, it holds that

$$\lim_{k\to\infty}\left(\int_{\mathrm{B}_R^+(0)} |DN^+_{\rho_k}|^2 + \int_{\mathrm{B}_R^-(0)} |DN^-_{\rho_k}|^2\right) = \int_{\mathrm{B}_R^+(0)} |DN_0^+|^2 + \int_{\mathrm{B}_R^-(0)} |DN_0^-|^2, \forall R \in (0,1), \tag{23.2.2}$$

(N_0^+, N_0^-) collapses at the interface $(T_0\gamma, Q^\star [\![0]\!])$, we have the following properties

- *(i)* (N_0^+, N_0^-) *is nontrivial and in particular* $\mathrm{Dir}(N_0^+, N_0^-, \mathrm{B}_1(0)) = 1$*;*
- *(ii)* $\boldsymbol{\eta} \circ N_0^\pm \equiv 0$.

As it is explained in Subsection 19.3, Theorem 17.0.3 for currents of dimension 2 follows from Theorem 19.3.8 which we are now able to prove for area minimizing currents of dimension $m \geq 2$, codimension $n \geq 2$ and multiplicity $Q^\star \geq 1$.

Theorem I (Theorem 19.3.8)**.** *Let T and Γ be as in Assumption 6 with $C_0 = 0$. Then any two-sided collapsed point of T is a two-sided regular point of T.*

Proof. Now, since we are under Assumption 10, we can apply Theorem 23.1.3 and we show that (b) of Theorem 23.1.3 never occurs, then we are always in the case (a) of Theorem 23.1.3 which ensures that 0 is a boundary two-sided regular point of T. With this aim in mind observe that by the harmonic regularity of $\left(Q - \frac{Q^\star}{2}\right)$-Dir minimizers which collapse at the interface, i.e., Theorem 20.1.4, we have that $N_0^\pm = Q^\pm \llbracket h \rrbracket$ for some classical 1-valued harmonic function $h : \mathrm{B}_1(0) \to \mathbb{R}^n$, hence we necessarily have

$$N_0^+ = Q \left\llbracket \boldsymbol{\eta} \circ N_0^+ \right\rrbracket \quad \text{and} \quad N_0^- = (Q - Q^\star) \left\llbracket \boldsymbol{\eta} \circ N_0^- \right\rrbracket .$$

By (ii) of Theorem 23.2.1, we have that $h \equiv 0$, but this is contradiction with $\mathrm{Dir}(N_0^+, N_0^-, \mathrm{B}_1(0)) = 1$. Thus (b) of Theorem 23.1.3 never occurs. This last fact surely completes the proof of the theorem. □

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