Density of the boundary regular set of $2d$ area minimizing currents with arbitrary codimension and multiplicity

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Abstract

In the present work, we consider area minimizing currents in the general setting of arbitrary codimension and arbitrary boundary multiplicity. We study the boundary regularity of $2d$ area minimizing currents, beyond that, several results are stated in the more general context of (C_0, α_0, r_0)-almost area minimizing currents of arbitrary dimension m and arbitrary codimension taking the boundary with arbitrary multiplicity. Furthermore, we do not consider any type of convex barrier assumption on the boundary, in our main regularity result which states that the regular set, which includes one-sided and two-sided points, of any $2d$ area minimizing current T is an open dense set in Γ.

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Contents

1 Introduction

The regularity of area minimizing currents is widely studied by several great mathematicians and beautiful results have been proved about both interior and boundary regularity. In a nutshell:

- **Interior regularity.** In codimension 1, Ennio De Giorgi, Leon Simon, Hebert Federer, Frederick J. Almgren Jr and Wendell Fleming have stated in several different works that the dimension of the singular set is at most $m-7$, where m is the dimension of the current, which is an optimal result taking into consideration the famous example of a cone, with the only singularity at the origin, given by Simon in [23], the so-called, Simon's cone, which Enrico Bombieri, Ennio De Giorgi and Enrico Giusti proved in [3] that it is indeed area minimizing. In higher codimension, Almgren have proved in [2] that in general dimension and codimension we have that $\dim(\mathrm{Sing}_i(T)) \leq m-2$, where m is the dimension of T, which is an optimal result assured by the famous Federer's examples of complex varieties, [16, Section 5.4.19]. Camillo de Lellis and Emanuele Spadaro have generalized this result to Riemannian manifolds with less regularity conditions (namely, $C^{3,\alpha}$), and also gained simplified arguments in the serie of papers [11, 13, 14] producing also some improved estimates.
- **Boundary regularity.** In codimension 1, Hardt and Simon ([17]) stated the $C^{1,\alpha}$ regularity of $\mathrm{Reg}_b(T)$ of an area minimizing current T in Euclidean spaces. In fact they proved much more, namely that $Sing_b(T) = \emptyset$. The generalization of Hardt-Simon work to the Riemannian setting is due to Simone Steinbruechel in [24] where is stated the $C^{1,\frac{1}{4}}$ regularity of the boundary of an area minimizing current in a smooth Riemannian manifold. As it was noticed in [6, Problem 4.19], this result can be extended under the same assumptions but with arbitrary boundary multiplicity using a decomposition argument provided by White in [25]. In higher codimension assuming boundary multiplicity 1, William K. Allard have proved that the boundary of an area minimizing current taking the boundary with multiplicity 1 is regular, however, he needed to impose a crucial condition on $\mathrm{spt}(\partial T)$ which is that $\mathrm{spt}(\partial T)$ is cointained in the boundary of a uniformly convex set named, according to Camillo De Lellis, *convex barrier*. Indeed, what Allard proved is that boundary points with density $\frac{1}{2}$ are regular after he shows that every boundary point in a $C^{1,1}$ submanifold contained in a convex barrier satisfies this last density condition. Since the boundary regularity theorem proved by Allard in 1975, no one attacked the problem of dropping the convex barrier condition until the recent work of De Lellis, De Philippis, Hirsch and Massaccesi, [7], where it is proved that $\mathrm{Reg}_b(T)$ is dense in $\mathrm{spt}(\partial T)$ where T is an area minimizing current taking the boundary with multiplicity 1 in a Riemannian manifold of class $C^{3,\alpha}$. To achieve this regularity result the authors have adapted the whole construction in [2] by Almgren to the boundary case which is even much more involved and needed the introduction of a lot of highly nontrivial new ideas. In the even more recent work [8] by Camillo De Lellis, Stefano Nardulli and Simone Steinbruechel, the authors have generalized [1] to 2-dimensional area minimizing currents taking the boundary with arbitrary multiplicity but still having boundary contained in a convex barrier.

So, our present work fills one of the gaps in the theory providing a boundary regularity theorem to $2d$ area minimizing currents with arbitrary boundary multiplicity and without any convex barrier assumption. An important apparatus for this work is the characterization of two-dimensional tangent cones given in [9] which allow us to proceed in various steps below. The main goal of this text is to prove the density of the regular set of the boundary, Theorem 1.3. So, we need to define regular and singular points, such definitions are standard when considering multiplicity at the boundary equals to one, in the case of higher multiplicity we need to split it in two different definitions based on the local behaviour of the current.

Definition 1.1 (Regular and singular one-sided boundary points, Definition 0.1 of [8])**.** *Let T be a 2-dimensional integer rectifiable current (i.e. $T \in \mathscr{R}_2^{loc}(\mathbb{R}^{2+n})$) with $\partial T = Q^\star [\![\Gamma]\!], Q^\star \in \mathbb{N} \setminus \{0\}, p \in \Gamma$ and $\Theta^2(T,p) = \frac{Q^\star}{2}$. Then p is called a* **regular one-sided boundary point** *if T consists, in a neighborhood U of p, of the union of finitely many surfaces with boundary Γ, counted with multiplicities, which meet at Γ transversally. More precisely, if there are:*

(i): a finite number $\Sigma_1, \dots, \Sigma_J$ of oriented embedded surfaces in U,

(ii): and a finite number of positive integers $k_1, \dots, k_J$ such that:

(a) $\partial \Sigma_j \cap U = \Gamma \cap U = \Gamma_i \cap U$ (in the sense of differential topology) for every j,

(b) $\Sigma_j \cap \Sigma_l = \Gamma \cap U$ for every $j \neq l$,

(c) for all $j \neq l$ and at each $q \in \Gamma$ the tangent cones to Λ_j and Λ_l are distinct,

(d) $T \llcorner U = \sum_j k_j [\![\Sigma_j]\!]$ and $\sum_j k_j = Q^\star$.

The set $\mathrm{Reg}_b^1(T)$ of **regular boundary one-sided points** *is a relatively open subset of Γ.*

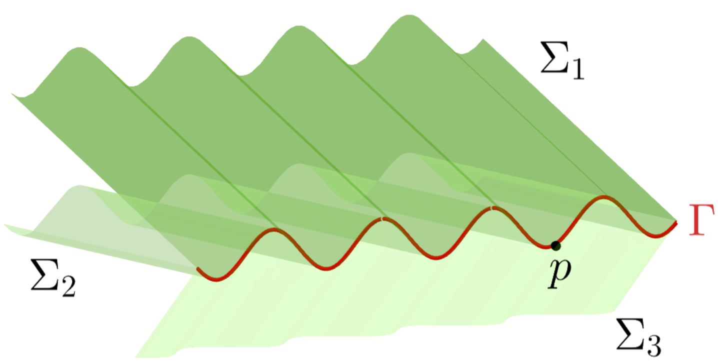


Figure 1: Here $J = 3$ and the current is given by $T = \sum_{j=1}^{3} k_j [\![\Sigma_j]\!]$, then p is a regular one-sided boundary point of T. Note that, each surface Σ_j is taken with an integer multiplicity k_j and the boundary ∂T has multiplicity $Q^\star = k_1 + k_2 + k_3$.

Definition 1.2 (Regular and singular two-sided boundary points, Definition 1.1 of [7])**.** *Let T be a m-dimensional integer rectifiable current (i.e. $T \in \mathscr{R}_2^{loc}(\mathbb{R}^{2+n})$) with $\partial T = Q^\star [\![\Gamma]\!], Q^\star \in \mathbb{N} \setminus \{0\}, p \in \Gamma$ and $\Theta^2(T,p) > \frac{Q^\star}{2}$.*

(i): We say that p is a **regular boundary two-sided point for** *T if there exist a neighborhood $U \ni p$ and a surface $\Sigma \subset U \cap \Sigma$ such that $\mathrm{spt}(T) \cap U \subset \Sigma$. The set of such points will be denoted by $\mathrm{Reg}_{\mathrm{b}}^2(T)$,*

(ii): We also denote $\mathrm{Reg}_b(T) := \mathrm{Reg}_{\mathrm{b}}^1(T) \mathring{\cup} \mathrm{Reg}_{\mathrm{b}}^2(T)$, $\mathrm{Sing}_b(T) := \Gamma \setminus (\mathrm{Reg}_{\mathrm{b}}^1(T) \mathring{\cup} \mathrm{Reg}_{\mathrm{b}}^2(T))$

(iii): We will say that $p \in \mathrm{Sing}_{\mathrm{b}}(T)$ is of **crossing type** *if there is a neighborhood U of p and two currents T_1 and T_2 in U with the properties that:*

(a) $T_1 + T_2 = T$ *and* $\partial T_1 = 0$,

(b) $p \in \mathrm{Reg_b}\,(T_2)$.

(iv): If $p \in \mathrm{Sing_b}(T)$ *is not of crossing type, we will then say that* p *is a* **genuine boundary singularity point of** T.

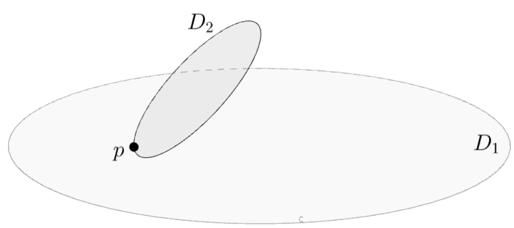


Figure 2: Let $T = [\![D_1]\!] + [\![D_2]\!]$ and $p \in \partial D_2 \cap \mathrm{int}(D_1)$. It is easy to see that p is a crossing type singularity to the $2d$ current T.

Our main theorem will be proved under the following assumptions.

Assumption 1. *Let* $\alpha \in (0, 1]$ *and an integer* $Q^\star \geq 1$. *Consider* $\Gamma \subset \Sigma$ *a* $C^{3,\alpha}$ *oriented curve without boundary. Let* T *be an integral* 2*-dimensional area minimizing current in* $\mathbf{B}(0, 2) \subset \mathbb{R}^{2+n}$ *with boundary* $\partial T \llcorner \mathbf{B}(0, 2) = Q^\star [\![\Gamma \cap \mathbf{B}(0, 2)]\!]$.

We can now state the main theorem which gives the density of the regular set $\mathrm{Reg}_b(T)$ in Γ, where the regular set allows the existence of both one-sided and two-sided points.

Theorem 1.3. *Let* T *and* Γ *be as in Assumptions 1. Then* $\mathrm{Reg}_b(T)$ *is an open dense set in* Γ.

Remark 1.4. We mention that we cite some results directly from [7] where the multiplicity $Q^\star = 1$, even though, we are working with higher multiplicity, the results used follows line by line, otherwise we write the proof here in details. All the results contained in [7], in particular the ones that we use here, are being addressed with all the details in higher multiplicity in the second author Ph.D. Thesis, [20].

1.1 Outline of the proof of Theorem 1.3

Our main issue is the existence of two kind of points which behave completely different, these points are defined as one-sided points and two-sided points, it means that the current is contained in one half space of the ambient space in the one-sided case and, in the case of two-sided points, the current is not contained in any half-space of the ambient space. In [7] the authors consider $Q^\star = 1$ and thus $Q - \frac{1}{2}$ is the density of the two-sided regular points in the boundary, they also allow the existence of one-sided and two-sided points, however, their notion of regularity coincides with the one given by Allard (in the case that $Q = 1$) to the context of one-sided points, in fact, Allard has considered what is called *convex barrier*, according to De Lellis. This equivalence of definitions strongly depends on the fact that the multiplicity at the boundary is 1. If $Q^\star > 1$, as we have defined, the notions of regularity for one-sided and two-sided points given in Definitions 1.1 and 1.2 does not coincide precisely because in the higher multiplicity setting at one-sided points we have currents with many sheets which are regular and this situation is not covered by Definition 1.2 when $Q = Q^\star$. So, similarly to the way that the authors in [7] rely on Allard's regularity result to reduce the analysis to two-sided points, we will use [8] to reduce our analysis to two-sided points. In [8], the authors have proved the analogous of Allard's theorem for higher multiplicity and currents of dimension 2, i.e., assuming that Γ belongs to a *convex barrier*, $2d$ area minimizing currents with $T = Q^\star [\![\Gamma]\!]$ are completely regular at the boundary. This reduction is allowed by Theorem 3.19 where we state, for 2-dimensional (C_0, r_0, α_0)-almost area minimizing currents, that, if every two-sided collapsed point (Definition 3.17) is regular, then $\mathrm{Reg}_b(T)$ is dense, we reiterate that $\mathrm{Reg}_b(T)$ contains both one-sided and two-sided points.

Now the main goal becomes to prove that two-sided collapsed points are regular, Theorem 3.21, and, to this end, we follow the framework given in [7]. Moreover, this prove is done for area minimizing currents with arbitrary dimension m, codimension n and arbitrary multiplicity $Q^\star$. Firstly, we construct a linear theory for (f^+, f^-) where we define this pair as $(Q - \frac{Q^\star}{2})$-valued functions (Definition 4.1), and we study the regularity that we can extract once we assume that it minimizes the Dirichlet energy. If we take $\Omega \subset \mathbb{R}^m$ an open set and γ a $(m-1)$-submanifold, called *interface*, which splits Ω into Ω^+ and Ω^-, we define $(Q - \frac{Q^\star}{2})$-valued functions as a Q-valued function f^+ defined in Ω^+ and a $(Q - Q^\star)$-valued function f^- defined in Ω^- in the sense of Almgren, when these functions glue at γ we say that (f^+, f^-) collapses at the interface as in Definition 4.1. So, if we assume that (f^+, f^-) is a $(Q - \frac{Q^\star}{2})$-Dir minimizer which collapses at the interface, then it is given by Q copies of κ in Ω^+ and $(Q - Q^\star)$ copies of κ in Ω^- where κ is a classical harmonic function, Theorem 4.4.

The next step is to approximate the current T by Lipschitz maps, indeed, we can do it without assuming any minimizing property of the current and the approximations, Theorem 4.11. Furthermore, when we add the condition that T is area minimizing, the approximation becomes minimizers of the Dirichlet energy and thus, by Theorem 4.4, we obtain an harmonic approximation of the current T, Theorem 4.12. This harmonic approximation is a key point to prove a milder excess decay for the area minimizing current T, Lemma 5.2, after that we do a simple argument to get a superlinear decay on the excess of T, Theorem 5.3. The superlinear decay is an important tool to prove the uniqueness of tangent cones at two-sided collapsed points of T (Theorem 5.5) and also to improve our Lipschitz approximation in the sense of getting better estimates, i.e. superlinear, on the errors of the approximation, Theorem 5.8. These constructions allow us to build the center manifolds.

We first construct the Whitney decomposition with the suitable stopping condition to then prove the existence of the so-called $C^{3,\kappa}$ center manifolds $\mathcal{M}^+$ and $\mathcal{M}^-$, see Theorem 6.12. After that we build in Theorem 6.19 the $\mathcal{M}$-normal approximation which is multivalued Lipschitz maps $\mathcal{N}^+$ and $\mathcal{N}^-$ defined on the center manifold $\mathcal{M}^+$ and $\mathcal{M}^-$, respectively, and taking values in the normal bundle of $\mathcal{M}$ which approximate the m-current T in the desired fashion. After that, we provide a blowup argument in Subsection 7.2 which ensures that $\mathcal{N}^\pm \equiv 0$ and hence the m-dimensional area minimizing current T has to coincide with $\mathcal{M}^+$ in the right portion and with $\mathcal{M}^-$ in the left portion. This gives that any two-sided collapsed point is a two-sided regular point, i.e., Theorem 3.21, which finishes the proof of our main theorem as aforementioned, i.e., Theorem 1.3.

2 Fundamental concepts and results

For basic definitions and standard notations, we refer the reader to the textbooks [16] and [22]. Let us set up some notation that will be used in this work.

Definition 2.1. *We define for $x \in \mathbb{R}^{m+n}$ and $r > 0$ the function $\iota_{x,r}(y) := \frac{y-x}{r}$. For any current $T \in \mathcal{D}'_m(\mathbb{R}^{m+n})$ let us define the* **rescaled current** T **at** x **at scale** r *as* $\iota_{x,r\sharp}T =: T_{x,r}$ *and* $T_r =: \iota_{0,r\sharp}T$. *We call a current T_x a* **blowup of** T **at** x, *if there exists a sequence of radii $r_j \to 0$ such that $T_{x,r_j} \to T_x$ in the weak topology.*

We also fix the notation of the flat distance between two m-dimensional integer rectifiable currents T and S, i.e., $T, S \in \mathscr{R}^{loc}_m(U)$, U open and $A \Subset U$ as follows:

$$\mathbf{d}_A(T,S) = \inf\left\{\|R\|(A) + \|\tilde{T}\|(A) : T - S = R + \partial\tilde{T} \text{ with } R \in \mathbf{I}_m(U) \text{ and } \tilde{T} \in \mathbf{I}_{m+1}(U)\right\}.$$

Definition 2.2. *Given three real numbers $C_0 \geq 0, r_0, \alpha_0 > 0$, we say that an m-dimensional integer rectifiable current T (i.e. $T \in \mathscr{R}^{loc}_m(\mathbb{R}^{m+n})$) with $\partial T = Q^\star [\![\Gamma]\!], Q^\star \in \mathbb{N}\setminus\{0\}$, is (C_0, r_0, α_0)-***almost area minimizing at** $x \in \operatorname{spt}(T)$, *if we have*

$$\|T\|(\mathbf{B}(x,r)) \leq (1 + C_0 r^{\alpha_0}) \|T + \partial\tilde{T}\|(\mathbf{B}(x,r)), \tag{2.1}$$

*for all $0 < r < r_0$ and all integral $(m+1)$-dimensional currents $\tilde{T}$ supported in $\mathbf{B}(x,r)$, i.e., for all $\tilde{T} \in \mathbf{I}_{m+1}(\mathbf{B}(x,r))$. The current is called (C_0, r_0, α_0)-***almost area minimizing in** $\mathbb{R}^{m+n}$, *if the current T is (C_0, r_0, α_0)-almost area minimizing at each $x \in \operatorname{spt}(T)$ with the same constants $C_0, r_0, \alpha_0 > 0$ independently of the point x. If T is $(0, r_0, \alpha_0)$-almost area minimizing, we say that T is* ***area minimizing****.*

Assumption 2. *Let $\alpha \in]0,1]$ and integers $m \geq 2$, $Q^\star \geq 1$. Consider $\Gamma \subset \Sigma$ a $C^{3,\alpha}$ oriented $(m-1)$-submanifold without boundary. Let T be an integral m-dimensional (C_0, r_0, α_0)-almost area minimizing current in $\mathbf{B}(0,2)$ with boundary $\partial T \llcorner \mathbf{B}(0,2) = Q^\star [\![\Gamma \cap \mathbf{B}(0,2)]\!]$.*

Let us enunciate the almost monotonicity formula which will often used in what follows. The proof of this formula can be found in [18, Lemma 2.1]. Although, it is done for boundary multiplicity one currents, the proof can be readily adapted to the higher multiplicity case as it is done in [9].

Proposition 2.3 (Almost monotonicity formula for boundary points, Proposition 3.3 of [9])**.** *Let T and Γ be as in Assumption 2, and $x \in \operatorname{spt} T \cap \Gamma$. Set $\alpha_1 := \min\{\alpha_0, \alpha\}$, $0 < r_1 < \min\{r_0, r'\}$. Then there is a constant $C_1 = C_1(m, n, C_0, r_0, r', \alpha_0, \alpha, \theta, \|\Gamma\|_{1,\alpha}) > 0$, such that*

$$e^{C_1 r^{\alpha_1}} \frac{\|T\|(\mathbf{B}(x,r))}{r^m} - e^{C_1 s^{\alpha_1}} \frac{\|T\|(\mathbf{B}(x,s))}{s^m} \geq \int_{\mathbf{B}(x,r)\setminus\mathbf{B}(x,s)} e^{C_1|z-x|^{\alpha_1}} \frac{\left|(z-x)^\perp\right|^2}{2|z-x|^{m+2}} d\|T\|(z), \tag{2.2}$$

for every $0 < s < r < r_1$.

The upper semicontinuity of the density function is well known when restricted to either the interior or the boundary of an area minimizing current, we give a short proof of the validity of this fact at the boundary of almost area minimizing currents. We would like to remark that the upper semicontinuity holds for the restriction of the density function to boundary or interior points, however, it does not hold when it is considered defined on the whole $\operatorname{spt}(T)$, therefore, to get around that we state (iii) as B. White does in the context of area minimizing currents, c.f. [26].

Proposition 2.4 (Upper semicontinuity of the density function)**.** *Let Γ and T be as in Assumption 2. Then*

(i) *The function $x \mapsto \Theta^m(T,x)$ is upper semicontinuous in $\operatorname{spt}(T) \cap \Gamma$,*

(ii) *The function $x \mapsto \Theta^m(T,x)$ is upper semicontinuous in $\operatorname{spt}(T) \setminus \Gamma$,*

(iii) *The function $x \mapsto \begin{cases} \Theta^m(T,x), & x \notin \Gamma \\ 2\Theta^m(T,x), & x \in \Gamma \end{cases}$ is upper semicontinuous.*

Proof. We define $f(x,r) = e^{C_1 r^{\alpha_1}} r^{-m} \|T\|(\mathbf{B}(x,r))$ and take $x_i \to x, x_i \in \operatorname{spt}(T) \cap \Gamma$, and assume $\mathbf{B}(x_i,r) \subset \mathbf{B}(x, r+\varepsilon)\,, \forall i \in \mathbb{N}, \varepsilon > 0$. Applying the almost monotonocity formula (Proposition 2.3), we already know that $f(x_i,\cdot)$ is monotone nondecreasing and so we obtain

$$f(x_i,t) \le f(x_i,r) \le e^{C_1 r^{\alpha_1}} r^{-m} \|T\|(\mathbf{B}(x,r+\varepsilon)) = f(x,r+\varepsilon) \overbrace{\left[e^{-C_1\varepsilon^{\alpha_1}}\left(1+\frac{\varepsilon}{r}\right)^m\right]}^{I},$$

for any $0 < t < r$, $i \ge i_\varepsilon$ and $\varepsilon > 0$. First let $t \to 0$ we get that for every fixed positive $\varepsilon > 0$ it holds

$$f(x_i,0^+) = \Theta^m(T,x_i) \le f(x,r+\varepsilon) \overbrace{\left[e^{-C_1\varepsilon^{\alpha_1}}\left(1+\frac{\varepsilon}{r}\right)^m\right]}^{I}, \forall i \ge i_\varepsilon.$$

In the last inequality, taking in first the limit with respect to i, in second with respect to ε and finally with respect to r leads to

$$\limsup_{i\to+\infty} \Theta^m(T,x_i) \le \lim_{r\to 0} \limsup_{i\to+\infty} f(x,r) = \Theta^m(T,x).$$

We mention that to prove (ii), the upper semicontinuity at the interior, one may use this very same argument but now using the almost monotonicity formula given in [15, Proposition 2.1]. Let us turn to the proof of (iii), it is enough to prove that

$$2\Theta(T,x) \ge \limsup_{i\to+\infty}\Theta(T,x_i)$$

where $\{x_i\}_{i\in\mathbb{N}} \subset \mathbb{R}^{m+n} \setminus \Gamma$ converges to $x \in \Gamma$. Take y_i the orthogonal projection of x_i into Γ and then define $T_i' = \frac{T-y_i}{|x_i-y_i|}$ and $x_i' = \frac{x_i-y_i}{|x_i-y_i|}$, up to subsequences , we may take $T' := \lim T_i'$ and $x' := \lim x_i'$. By the almost monotonocity formula (Proposition 2.3), for every $r \in (0,r_1)$, the quantity $e^{C_1 r^{\alpha_1}} \frac{\|T\|(\mathbf{B}(y_i,r))}{r^m}$ is monotone nondecreasing. Thus, for any $\rho \in \mathbb{R}_+$,

$$\begin{aligned} e^{C_1 r^{\alpha_1}} \frac{\|T\|(\mathbf{B}(x,r))}{r^m} &\overset{(i)}{\ge} \limsup_{i\to+\infty} e^{C_1 r^{\alpha_1}} \frac{\|T\|(\mathbf{B}(y_i,r))}{r^m} \\ &\ge \limsup_{i\to+\infty} \frac{\|T\|(\mathbf{B}(y_i,\rho|x_i-y_i|))}{(\rho|x_i-y_i|)^m} \\ &= \limsup_{i\to+\infty} \frac{\|T_i'\|(\mathbf{B}(0,\rho))}{\rho^m} \\ &\ge \frac{\|T'\|(\mathbf{B}(0,\rho))}{\rho^m}. \end{aligned}$$

According to [1, Reflection Principle, 3.2], we can reflect T' in $T_x\Gamma$ to obtain a stationary varifold V'' in $\mathbb{R}^{m+n}$. Thus, we let $r \to 0$ and $\rho \to +\infty$ in the latter equation to obtain

$$\begin{aligned}
\Theta^m(T,x) &\geq \limsup_{\rho\to\infty} \frac{\|T'\|(\mathbf{B}(0,\rho))}{\rho^m} \\
&\overset{\text{def of } V''}{=} \frac{1}{2}\limsup_{\rho\to\infty} \frac{\|V''\|(\mathbf{B}(0,\rho))}{\rho^m} \\
&= \frac{1}{2}\limsup_{\rho\to\infty} \frac{\|V''\|(\mathbf{B}(x',\rho))}{\rho^m} \\
&\overset{\text{(ii)}}{\geq} \frac{1}{2}\Theta^m(V'',x') \\
&\overset{\text{def of } V''}{=} \frac{1}{2}\Theta^m(T',x') \\
&\overset{\text{(ii)}}{\geq} \frac{1}{2}\limsup_{i\to\infty}\Theta^m(T_i',x_i') \\
&= \frac{1}{2}\limsup_{i\to\infty}\Theta^m(T,x_i),
\end{aligned}$$

where in the last equality we did a standard rescaling argument. □

Let us enunciate the existence of area minimizing tangent cones to boundary points of an almost area minimizing current, even though this result is very standard, we will prove it here for the sake of completeness.

Proposition 2.5 (Existence of area minimizing tangent cones)**.** *Let T and Γ be as in Assumptions 2. If T is (C_0, r_0, α_0)-almost area minimizing in a neighbourhood U of $x \in \Gamma$, then, for any sequence $r_k \to 0$, there exists a blowup $\lim_{k\to+\infty} T_{x,r_k} = T_0 \in \mathbf{I}_m^{loc}(\mathbb{R}^{m+n})$ such that:*

(i) $\|T_{x,r_k}\| \to \|T_0\|$ *as* $k \to +\infty$, *in the sense of measures,*

(ii) T_0 *is area minimizing,*

(iii) $\|T_0\|(\mathbf{B}(0,r)) = \Theta^m(T,x)\omega_m r^m, \forall r > 0,$

(iv) T_0 *is a tangent cone to* T *at* x.

Remark 2.6. We only need to assume that $\partial T = Q^\star [\![\Gamma]\!]$ and Γ is an $(m-1)$-submanifold of class $C^{1,\alpha}$, $\alpha \in (0,1)$, in order to deduce that $\partial T_0 = Q^\star [\![T_x\Gamma]\!]$.

Proof. Assume that $x = 0$. By the almost monotonicity formula at the boundary and at the interior (i.e., Proposition 2.3 and [15, Proposition 2.1]), we have that

$$\limsup_{k\to+\infty} \|T_{r_k}\|(\mathbf{B}(y,1)) < +\infty,$$

for every $y \in U$. Thus

$$\limsup_{k\to+\infty} \|T_{r_k}\|(K) < +\infty,$$

for all compact set $K \subset \mathbb{R}^{m+n}$. Since the boundary of T_{r_k} is $Q^\star [\![\Gamma/r_k]\!]$ and Γ is the graph of $u \in C^{1,\alpha}, u(0) = 0, Du(0) = 0$, we have

$$\begin{aligned}
\|\partial T_{r_k}\|(K) = \|\iota_{0,r_k\sharp}(Q^\star [\![\Gamma]\!])\|(K) &= \frac{1}{r_k^{m-1}}\|Q^\star [\![\Gamma]\!]\|(r_k K) \\
&= \frac{Q^\star}{r_k^{m-1}}\mathcal{H}^{m-1}((r_k K)\cap\Gamma) \\
&\leq \frac{Q^\star}{r_k^{m-1}}\int_{\mathrm{proj}(r_k K)}\sqrt{1+|Du(z)|^2}dz \\
&\leq \frac{Q^\star}{r_k^{m-1}}\mathcal{H}^{m-1}(\mathrm{proj}(r_k K))\sqrt{1+C_\Gamma(\mathrm{diam}(K)r_k)^{2\alpha}} \\
&\leq Q^\star\omega_{m-1}\,\mathrm{diam}(K)^{m-1}\sqrt{1+C_\Gamma\,\mathrm{diam}(K)^{2\alpha}} \\
&= C(\Gamma, K, \alpha, m, Q^\star),
\end{aligned}$$

and thus we can bound uniformly the mass of the boundary of T_{r_k} in K. Therefore, we can use standard compactness results (one could consult [16, Section 4.2]) to ensure the existence of $T_0 \in \mathbf{I}_m^{loc}(\mathbb{R}^{m+n})$ such that $T_{r_k} \to T_0$, up to a subsequence, in the flat norm.

Proof of (i): Let us write $T_{r_k} - T_0 = R_{r_k} + \partial\tilde{T}_{r_k}$ in $\mathbf{B}(0, R+2)$ with

$$\limsup_{k\to+\infty}\Big(\|R_{r_k}\|(\mathbf{B}(0,R+1)) + \|\tilde{T}_{r_k}\|(\mathbf{B}(0,R+1))\Big) = 0.$$

Thus, since all the measures involved are Radon measures, for almost every $s \in (R, R+1)$, it follows that

$$\limsup_{k\to+\infty}\|R_{r_k}\|(\mathbf{B}(0,s)) = 0 \tag{2.3}$$

and

$$\limsup_{k\to+\infty}\mathbb{M}\left(\langle\tilde{T}_{r_k}, d, s\rangle\right) = 0. \tag{2.4}$$

Note that (2.4) follows directly from the formula of the slice and the fact that T_{r_k} converges to T_0 in the flat norm. We may use again the slice formula to get

$$T_{r_k}\llcorner\mathbf{B}(0,s) = T_0\llcorner\mathbf{B}(0,s) + R_{r_k}\llcorner\mathbf{B}(0,s) - \langle\tilde{T}_{r_k}, d, s\rangle + \partial(\tilde{T}_{r_k}\llcorner\mathbf{B}(0,s)). \tag{2.5}$$

The almost minimality condition gives

$$\|T_{r_k}\|(\mathbf{B}(0,s)) \leq (1 + C_0(rs)^{\alpha_0})\,\|T_{r_k} + \partial\tilde{T}_{r_k}\|\,(\mathbf{B}(0,s))\,.$$

Putting into account the latter inequality, the triangle inequality and (2.5), we obtain that

$$\|T_{r_k}\|(\mathbf{B}(0,s)) \leq (1 + C_0(rs)^{\alpha_0})\left(\|T_0\|(\mathbf{B}(0,s)) + \|R_{r_k}\|(\mathbf{B}(0,s)) + \mathbb{M}\left(\langle\tilde{T}_{r_k}, d, s\rangle\right) + 2\|\partial\tilde{T}_{r_k}\|(\mathbf{B}(0,s))\right).$$

Note that, by our construction, it follows that $\|\partial\tilde{T}_{r_k}\|(\mathbf{B}(0,s)) \to 0$ as $k \to +\infty$. Finally, by the lower semicontinuity of the mass, (2.3), (2.4) and the last equation passed through $\lim_{k\to+\infty}$, we conclude the proof of (i).

Proof of (ii): Fix $R \in (0, +\infty)$, by the lower semicontinuity of the mass, for all $\tilde{T} \in \mathbf{I}_{m+1}(\mathbf{B}(0,R))$ we have that

$$\begin{aligned}
\|T_0\|(\mathbf{B}(0,R)) &\leq \liminf_{k\to+\infty}\|T_{r_k}\|(\mathbf{B}(0,R)) \\
&\leq \liminf_{k\to+\infty}(1 + C_0(r_k R)^{\alpha_0})\,\|T_{r_k} + \partial\tilde{T}\|\,(\mathbf{B}(0,R)) \\
&\overset{\text{(i)}}{=} \|T_0 + \partial\tilde{T}\|\,(\mathbf{B}(0,R))\,,
\end{aligned}$$

for all $R > 0$.

Proof of (iii): From (i) and the almost monotonicity formula, we know that $\Theta^m(T, x)$ exists and, $\forall r > 0$, we have that

$$\begin{aligned} \|T_0\|(\mathbf{B}(0,r)) &= \lim_{k\to+\infty} \|T_{r_k}\|(\mathbf{B}(0,r)) \\ &= \lim_{k\to+\infty} \frac{\|T\|(\mathbf{B}(0,r_k r))}{r_k^m} \\ &= \lim_{k\to+\infty} \frac{\omega_m r^m \|T\|(\mathbf{B}(0,r_k r))}{\omega_m (r_k r)^m} \\ &= \Theta^m(T,x)\omega_m r^m. \end{aligned}$$

Proof of (iv): By following closely the argument given in [22, Theorem 3.1, Chapter 7] using (iii), we prove that T_0 is in fact a cone. Indeed, by [8, Theorem 3.2] applied to T_0 which is area minimizing, we know that

$$Q^\star \int_{T_p\Gamma\cap\mathbf{B}(p,\rho)} (x-p)\cdot \vec{n}(x) d\mathcal{H}^{m-1}(x) = 0,$$

since $\vec{n} \perp T_p\Gamma$. Using (ii), we obtain that $\vec{H}_T$ is zero a.e., so, by (iii), we obtain the constancy of the mass ratio and, then using again [8, Theorem 3.2], we get that

$$\int_{\mathbf{B}(p,r)\setminus\mathbf{B}(p,s)} \frac{\left|(x-p)^\perp\right|^2}{|x-p|^{m+2}} d\|T\|(x) = 0.$$

Then, if we fix a cone C such that $\partial C = \partial T_0$, notice that since $\partial T_0 = Q^\star [\![T_p\Gamma]\!]$ we may choose for instance C as a half subspace, we can apply [22, Lemma 2.33, Chapter 6] for $T_0 - C$ and conclude that it is a cone and thus $T_0 = T_0 - C + C$ is a cone. □

3 Stratification for (C_0, r_0, α_0)-almost area minimizing currents

3.1 Almgren-De Lellis-White's stratification

Definition 3.1 (Conelike functions)**.** *An upper semicontinuous function $g : \mathbb{R}^{m+n} \to \mathbb{R}_+$ is called* **conelike** *provided:*

(i) $g(\lambda x) = g(x)$ for all $\lambda > 0$ and for all $x \in \mathbb{R}^{m+n}$,

(ii) If $g(x) = g(0)$, then $g(x + \lambda v) = g(x + v)$ for all $\lambda > 0$ and $v \in \mathbb{R}^{m+n}$.

If g is conelike we also define the **spine of** *g as the set*

$$\mathrm{spine}(g) := \{x \in \mathbb{R}^{m+n} : g(x) = g(0)\}.$$

By (i) in the last definition and upper semicontinuity, we have that $g(z) \le g(0)$ for all z. Note that, by [26, Theorem 3.1], spine(g) is a vector subspace and

$$\mathrm{spine}(g) = \{x \in \mathbb{R}^{m+n} : g(x + v) = g(v),\ \forall v \in \mathbb{R}^{m+n}\}.$$

Fix T and Γ as in Assumption 2, and set the class of functions

$$\mathscr{G}(p) := \{g_{p,T_0} : T_0 \text{ is a tangent cone to } T \text{ at } p\},$$

where

$$f_T(x) := \begin{cases} \Theta^m(T,x), & x \notin \Gamma \\ 2\Theta^m(T,x)+1, & x \in \Gamma \end{cases}, \quad \text{and} \quad g_{p,T_0}(x) := \begin{cases} \Theta^m(T_0,x), & x \notin T_p\Gamma \\ 2\Theta^m(T_0,x)+1, & x \in T_p\Gamma \end{cases},$$

then f and each g_{p,T_0} are upper semicontinuous from Proprosition 2.4.

Definition 3.2 (Spine of a cone)**.** *Let $p \in \Gamma$, T_0 be an oriented tangent cone with $\partial T_0 = Q^\star [\![T_p\Gamma]\!]$, we define the* **spine of** T_0*, and denote it by* $\operatorname{spine}(T_0)$*, to be the set of vectors $v \in T_p\Gamma$ such that $(\tau_v)_\sharp T_0 = T_0$ where $\tau_v(w) = w + v$. Clearly the* $\operatorname{spine}(T_0)$ *is always a subspace of $T_p\Gamma$.*

We now provide an equivalence for the definition of spine of an oriented tangent cone which follows the ideas furnished by Almgren in his stratification process, [2, Theorem 2.26], and used in [26], [21], [7].

Lemma 3.3 (Spine as constant density set)**.** *Let T and Γ be as in Assumption 2 and T_0 be an oriented tangent cone to T at $p \in \Gamma$. Then* $\operatorname{spine}(T_0) = \{x \in T_p\Gamma : \Theta^m(T_0,x) = \Theta^m(T_0,0)\}$.

Proof. Take $x \in \operatorname{spine}(T_0)$, by the definition of spine, we have the third equality below

$$\Theta^m(T_0,x) = \lim_{r\to 0} \frac{\|T_0\|(\mathbf{B}(x,r))}{\omega_m r^m} = \lim_{r\to 0} \frac{\|(\tau_x)_\sharp T_0\|(\mathbf{B}(0,r))}{\omega_m r^m} = \lim_{r\to 0} \frac{\|T_0\|(\mathbf{B}(0,r))}{\omega_m r^m} = \Theta^m(T_0,0).$$

On the other hand, consider x such that $\Theta^m(T_0,x) = \Theta^m(T_0,0)$, we claim that $x \in \operatorname{spine}(T_0)$. To prove this claim, we apply the monotonicity formula [4, Equation 31] to the cone T_0, which is area minimizing and $\partial T_0 = Q^\star [\![T_p\Gamma]\!]$, to obtain, for $0 < s < r$,

$$\frac{\|T_0\|(\mathbf{B}(x,r))}{\omega_m r^m} - \frac{\|T_0\|(\mathbf{B}(x,s))}{\omega_m s^m} = \int_{\mathbf{B}(x,r)\setminus\mathbf{B}(x,s)} \frac{|(z-x)^\perp|^2}{|z-x|^{m+2}} \, \mathrm{d}\,\|T_0\|(z). \tag{3.1}$$

It is well known that when T_0 is a cone, the right hand side of the last equation is equal to 0 (for instance, see [22, Lemma 2.33, Chapter 6]), so letting $s \to 0$ in (3.1) we reach

$$\frac{\|T_0\|(\mathbf{B}(x,r))}{\omega_m r^m} = \Theta^m(T_0,x) = \Theta^m(T_0,0).$$

Therefore, we have that $\|(\tau_x)_\sharp T_0\|(\mathbf{B}(0,r)) = \|T_0\|(\mathbf{B}(x,r)) = \|T_0\|(\mathbf{B}(0,r))$ thus, by measure theory, we get that $\|T_0\| = \|(\tau_x)_\sharp T_0\|$ as measures, which in turn ensures that $T_0 = \pm(\tau_x)_\sharp T_0$. But we know that $\partial T_0 = Q^\star [\![T_p\Gamma]\!]$ and $\partial(\tau_x)_\sharp T_0 = (\tau_x)_\sharp \partial T_0 = Q^\star [\![T_p\Gamma]\!]$, then we conclude that $T_0 = (\tau_x)_\sharp T_0$ which shows that $x \in \operatorname{spine}(T_0)$. □

We would like to show that we are in position to apply [26, Theorem 3.2], to that end, we prove the following lemma.

Lemma 3.4. *For each $p \in \Gamma$ and each oriented tangent cone T_0 to T at p, g_{p,T_0} is conelike.*

Proof. Property (i) in Definition 3.1 is a direct consequence of the scaling invariance of T_0. To what concerns property (ii), if $g_{p,T_0}(x) = g_{p,T_0}(0)$, and $x \in T_p\Gamma$, we have that $x \in \mathrm{spine}(T_0)$ and thus

$$\begin{aligned}\Theta^m(T_0, x+v) &= \Theta^m((\tau_x)_\sharp T_0, v)\\ &= \Theta^m(T_0, v)\\ &= \Theta^m(T_0, \lambda v)\\ &= \Theta^m((\tau_x)_\sharp T_0, \lambda v)\\ &= \Theta^m(T_0, x+\lambda v),\end{aligned}$$

for any $\lambda > 0$ and $v \in \mathbb{R}^{m+n}$.

If $g_{p,T_0}(x) = g_{p,T_0}(0)$ and $x \notin T_p\Gamma$, then, by definition of g_{p,T_0}, we have $\Theta^m(T_0, x) = 2\Theta^m(T_0, 0) + 1$. Since T_0 is a cone we have that $\Theta^m(T_0, x) = \Theta^m(T_0, \lambda x)$, for every $\lambda > 0$. Then $\Theta^m(T_0, \lambda x) = 2\Theta^m(T_0, 0) + 1$, for every $\lambda > 0$. Taking the limsup and recalling Proposition 2.4 *(iii)* we get $\limsup_{\lambda \to 0^+} \Theta^m(T_0, \lambda x) = 2\Theta^m(T_0, 0) + 1 \leq 2\Theta^m(T_0, 0)$, which is a contradiction. □

Remark 3.5. Note that, a simple consequence of the Lemma 3.3 and the proof of Lemma 3.4 is that $\mathrm{spine}(g_{p,T_0}) = \mathrm{spine}(T_0)$.

Definition 3.6 (Stratum)**.** *Let $p \in \Gamma$ and T be a m-current with $\partial T = Q^\star [\![\Gamma]\!]$, we define the* **$j$-stratum of Γ with respect to** T *as the set*

$$\mathscr{P}_j(T, \Gamma) = \{p \in \Gamma : \dim(\mathrm{spine}(T_0)) \leq j, \text{ for all tangent cone } T_0 \text{ to } T \text{ at } p\}.$$

Now, we shall directly apply Proposition 2.5 to check the conditions (1) and (2) of [26, Theorem 3.2] which in turn furnishes

Theorem 3.7 (Stratification Theorem, Theorem 3.2 of [26])**.** *For T and Γ as in Assumption 2, let*

$$\Sigma_i := \{x : f_T(x) > 0 \text{ and } \sup\{\dim(\mathrm{spine}(g)) : g \in \mathscr{G}(x)\} \leq i\},$$

then the Hausdorff dimension of Σ_i is at most i and Σ_0 is at most countable. In particular, we have the same statements for the stratum $\mathscr{P}_i(T, \Gamma)$, i.e., the Hausdorff dimension of $\mathscr{P}_i(T, \Gamma)$ is at most i, $\mathscr{P}_0(T, \Gamma)$ is at most countable, and

$$\mathscr{P}_0(T, \Gamma) \subset \mathscr{P}_1(T, \Gamma) \subset \cdots \subset \mathscr{P}_{m-1}(T, \Gamma) = \Gamma.$$

3.2 Open books, flat cones, one-sided and two-sided points

The characterization of tangent cones is an important tool in the subsequent theory, in fact, when dealing with two dimensional area minimizing cones we have general structure results, see for instance [18, Lemma 3.1] or [9, Proposition 4.1]. If we consider arbitrary dimensions, there is no general structure theorem for area minimizing tangent cones, however, assuming that the density of the cone is constant along the boundary of the cone which is equivalent to assume that the spine has maximal dimension, as showed in Lemma 3.3, we can characterize tangent cones as in [5, Theorem 5.1], or [7, Lemma 3.17]. We enunciate and prove in Lemma 3.12 rigorous statement of the assertions just mentioned. To go further in our treatment we need the following definitions.

Definition 3.8 (Open book)**.** *Let* $T_0 \in \mathbf{I}_m^{\mathrm{loc}}(\mathbb{R}^{m+n})$ *be an oriented cone and* V *is an oriented* $(m-1)$*-dimensional linear subspace of* $\mathbb{R}^{m+n}$*. We say that* T_0 *is an* **open book with boundary** $[\![V]\!]$ **and multiplicity** $Q^\star$*, if* $\partial T_0 = Q^\star [\![V]\!]$ *and there exist* $Q_1, \dots, Q_N \in \mathbb{N} \setminus \{0\}$ *and* $\pi_1, \dots, \pi_N$ *distinct* m*-dimensional half-planes with* $\pi_i \neq -\pi_j$*, for any* $1 \leq i, j \leq N$*, such that*

(*i*) $\partial [\![\pi_i]\!] = [\![V]\!], \forall i \in \{1, \dots, N\}$,

(*ii*) $T_0 = \sum_{i=1}^N Q_i [\![\pi_i]\!]$ *with* $Q^\star = \sum_{i=1}^N Q_i$.

If exists $i \neq j$ *such that* $\pi_i \neq \pi_j$*, we say that* T_0 *is a* **genuine open book with boundary** $[\![V]\!]$ **and multiplicity** $Q^\star$*.*

Definition 3.9 (Flat cones)**.** *Let* $T_0 \in \mathbf{I}_m^{\mathrm{loc}}(\mathbb{R}^{m+n})$ *be an oriented cone and* V *is an oriented* $(m-1)$*-dimensional linear subspace of* $\mathbb{R}^{m+n}$*. We say that* T_0 *is a* **flat cone with boundary** $[\![V]\!]$ **and multiplicity** $Q^\star$*, if* $\partial T_0 = Q^\star [\![V]\!]$ *and there exist a* m*-dimensional closed plane* π*,* $Q^{\mathrm{int}} \in \mathbb{N}$ *and* $Q \in \mathbb{N} \setminus \{0\}, Q \geq Q^\star$*, such that*

(*i*) $\operatorname{spt} T_0 = \pi$ *is an* m*-dimensional subspace,*

(*ii*) $\partial [\![\pi^+]\!] = -\partial [\![\pi^-]\!] = [\![V]\!]$,

(*iii*) $T_0 = Q^{\mathrm{int}} [\![\pi]\!] + Q [\![\pi^+]\!] + (Q - Q^\star) [\![\pi^-]\!]$.

If $Q^{\mathrm{int}} = 0$*, we say that* T_0 *is a* **boundary flat cone with multiplicity** $Q^\star$*. If* $Q^{\mathrm{int}} = 0$ *and* $Q > Q^\star$*, we will call* T_0 *a* **two-sided boundary flat cone with multiplicity** $Q^\star$*. If* $Q^{\mathrm{int}} = 0$ *and* $Q = Q^\star$*, we will call* T_0 *a* **one-sided boundary flat cone with multiplicity** $Q^\star$*.*

Note that, T_0 is an open book which is not genuine if, and only if, T_0 is an one-sided boundary flat cone.

Definition 3.10. *Let* $T_0 \in \mathbf{I}_m^{\mathrm{loc}}(\mathbb{R}^{m+n})$ *be an oriented cone and* V *is an oriented* $(m-1)$*-dimensional linear subspace of* $\mathbb{R}^{m+n}$ *such that* $\partial T_0 = Q^\star [\![V]\!]$*. If* $p \in \operatorname{spt}(\partial T_0)$*, we say that*

(i) p *is a* **boundary flat point** *provided* T_0 *is a flat cone,*

(ii) p *is a* **one-sided boundary flat point** *provided* T_0 *is an open book which is non genuine,*

(iii) p *is a* **two-sided boundary flat point** *provided* T_0 *is a two-sided boundary flat cone.*

Lemma 3.11 (The set of one-sided points is open)**.** *Let* T, Γ *and* $p \in \Gamma$ *as in Assumption 2. If* $\Theta^m(T, p) < \frac{Q^\star+1}{2}$*, then there exists a neighbourhood* U *of* p *such that* $\Theta^m(T, q) < \frac{Q^\star+1}{2}$ *for every* $q \in U \cap \Gamma$*.*

Proof. It follows directly from the upper semicontinuity of the density function, see Proposition 2.4. □

Note that, if $m = 2$ and the tangent cone T_p is a two-sided boundary flat cone, Theorem 3.15 ensures that the least possible density on p is $\frac{Q^\star}{2} + 1$. The following lemma is a generalization of [7, Lemma 3.17] to the case of higher multiplicity with essentially the same proof.

Lemma 3.12. *Let* T, Γ *and* $p \in \Gamma$ *as in Assumption 2 with* $C_0 = 0$*. If* T_0 *is an oriented tangent cone to* T *at* p *with* $\dim(\operatorname{spine}(T_0)) = m - 1$*, then*

(i) *If* $\Theta^m(T_0, 0) = \frac{Q^\star}{2}$, T_0 *is an open book,*

(ii) *If* $\Theta^m(T_0, 0) > \frac{Q^\star}{2}$, T_0 *is a two-sided boundary flat cone.*

Remark 3.13. We also mention that the assumption that the spine has maximal dimension in Γ was assumed in a similar fashion (see Lemma 3.1) in [5, Theorem 5.1].

Proof. By the assumption, we have that $\mathrm{spine}(T_0) = T_p\Gamma$. By [2, Theorem 2.2 (3)], there exists an one-dimensional area minimizing current T_{01} in $(T_p\Gamma)^\perp$ such that $T_0 = [\![T_p\Gamma]\!] \times T_{01}$. This fact that Almgren proved is an application of [16, Theorem 5.4.8] and [16, Section 4.3.15], using that T_0 is an oriented cylinder with direction v for any $v \in T_p\Gamma = \mathrm{spine}(T_0)$. Thus, since $\partial T_0 = Q^\star [\![T_p\Gamma]\!]$, we obtain that $\partial T_{01} = (-1)^{m-1}Q^\star [\![0]\!]$, the fact that T_{01} is invariant under homotheties allows us to write, for some $Q \in \mathbb{N} \setminus \{0\}$,

$$(-1)^{m-1}T_{01} = \sum_{i=1}^{Q} [\![\ell_i^+]\!] + \sum_{j=1}^{Q-Q^\star} [\![\ell_i^-]\!]\,,\ T_{01} = \sum_{i=1}^{Q} \|\, [\![\ell_i^+]\!] \,\| + \sum_{j=1}^{Q-Q^\star} \|\, [\![\ell_i^-]\!] \,\|,$$

where ℓ_i^+, ℓ_j^- are all oriented half-lines such that $\partial [\![\ell_i^+]\!] = [\![0]\!]$ and $\partial \left[\!\!\left[\ell_j^-\right]\!\!\right] = -[\![0]\!]$. In particular, we have

$$\mathrm{spt}(T_0) \subset \left(\bigcup_{i=1}^{Q} T_p\Gamma + \ell_i^+ \right) \cup \left(\bigcup_{i=1}^{Q-Q^\star} T_p\Gamma + \ell_i^- \right). \tag{3.2}$$

Note that, if $Q > Q^\star$, $\partial([\![\ell_i^+]\!] + \left[\!\!\left[\ell_j^-\right]\!\!\right]) = 0$ and $[\![\ell_i^+]\!] + \left[\!\!\left[\ell_j^-\right]\!\!\right]$ is area minimizing for any choice of i and j which ensures that the support of $[\![\ell_i^+]\!] + \left[\!\!\left[\ell_j^-\right]\!\!\right]$ is a straight line ℓ_{ij}. Since the choice of i and j is arbitrary, then we have $\mathrm{spt}([\![\ell_i^+]\!] + \left[\!\!\left[\ell_j^-\right]\!\!\right]) \subset \ell$, where ℓ is a straight line which, by (3.2), concludes the proof of (ii). If $Q = Q^\star$, we have that T_{01} is a sum of lines which might be distinct, and this concludes the proof of (i). □

3.3 Two dimensional case

In Lemma 3.12, T_0 has dimension m and we assume that the dimension of the spine is maximal. Nevertheless, if $m = 2$, we can drop the hypothesis on the spine since we have a full characterization of tangent cones with boundary being a subspace as stated in the proposition below.

Proposition 3.14 (Proposition 4.1, [9])**.** *Let* T_0 *be a* 2*-dimensional area minimizing cone in* $\mathbb{R}^{2+n}$ *with* $\partial T_0 = Q^\star [\![\ell]\!]$ *for some positive integer* $Q^\star$ *and a straight line* ℓ *containing the origin. Then we can decompose* $T_0 = {T_0}^{\mathrm{int}} + {T_0}^{\flat}$ *into two area minimizing cones with supports intersecting only at the origin which satisfy*

(i) $\partial {T_0}^{\mathrm{int}} = 0$ *and thus* ${T_0}^{\mathrm{int}} = \sum_{i=1}^{N} Q_i [\![\pi_i]\!]$ *where* $Q_1, \ldots, Q_N$ *are positive integers and* $\pi_1, \ldots, \pi_N$ *are distinct oriented* 2*-dimensional planes such that* $\pi_i \cap \pi_j = \{0\}$ *for all* $i \neq j$,

(ii) ${T_0}^{\flat}$ *is either a two-sided boundary flat cone or an open book.*

Let us also recall two pivotal results in the theory which will be used in this work. We also denote dist_H for the Hausdorff distance between closed sets and we denote by $e(p,r)$ the usual **spherical excess of a current** T, namely

$$e(p,r) := \frac{\|T\|\left(\mathbf{B}(p,r)\right)}{\pi r^2} - \Theta(T,p),$$

we also define the Holder seminorm used to measure the regularity of Γ, for any open set U,

$$[\Gamma]_{0,\alpha,U} := \sup_{q\neq p\in\Gamma\cap U} \frac{|T_p\Gamma - T_q\Gamma|}{|p-q|^\alpha}.$$

We have the following decay properties.

Theorem 3.15 (Uniqueness of tangent cones and speed of convergence, Theorem 2.1, [9])**.** *Let T and Γ be as in Assumption 2 with $m=2$. Then there are positive constants ε_0, C and β with the following property. If $p\in\Gamma$ and $e(p,r)\le\varepsilon_0^2$ for some $r\le \mathrm{dist}\,(p,\partial\mathbf{B}(0,1))$, then there exists a unique tangent cone T_p to T at p which, for every $\rho\in(0,r]$, satisfies:*

$$|e(p,\rho)| \le C|e(p,r)|\left(\frac{\rho}{r}\right)^{2\beta} + C\left(C_0^2 + [\Gamma]^2_{0,\beta,\mathbf{B}(p,r)}\right)\left(\frac{\rho}{r}\right)^{2\beta},$$

$$\mathbf{d}_{\mathbf{B}(0,1)}\left(T_{p,\rho},T_p\right) \le C|e(p,r)|^{\frac12}\left(\frac{\rho}{r}\right)^{\beta} + C\left(C_0 + [\Gamma]_{0,\beta,\mathbf{B}(p,r)}\right)\left(\frac{\rho}{r}\right)^{\beta},$$

$$\mathrm{dist}_H\left(\mathrm{spt}\left(T_{p,\rho}\right)\cap\mathbf{B}(0,1)\,,\mathrm{spt}(T_p)\cap\mathbf{B}(0,1)\right) \le C|e(p,r)|^{\frac12}\left(\frac{\rho}{r}\right)^{\beta} + C\left(C_0 + [\Gamma]_{0,\beta,\mathbf{B}(p,r)}\right)\left(\frac{\rho}{r}\right)^{\beta}.$$

We also state the Holder continuity of the map that to each point $p\in\Gamma$ assigns its unique tangent cone T_p.

Lemma 3.16 (Holder continuity)**.** *Let T,p,r be as in Theorem 3.15 and $q\in\Gamma\cap\mathbf{B}(p,r)$. Then the functions $q\mapsto T_q$ is Holder continuous, i.e., it holds*

$$\mathbf{d}\left(T_q\llcorner\mathbf{B}(0,1)\,,T_p\llcorner\mathbf{B}(0,1)\right) \le C|q-p|^\beta, \quad \forall q\in\mathbf{B}(p,r)\,. \tag{3.3}$$

Proof. For the proof we refer the reader directly to [8, Equation 4.7] which can be readily adjusted to the almost area minimizing setting. □

Definition 3.17 (Two-sided collapsed points)**.** *Let T and Γ be as in Assumption 2. A point $p\in\Gamma$ will be called* **two-sided collapsed point of** T *if*

- *(i) there exists a tangent cone T_0 to T at p which is a two-sided boundary flat cone,*
- *(ii) there exists a neighbourhood U of p such that $\Theta(T,q)\ge\Theta(T,p)$ for every $q\in\Gamma\cap U$.*

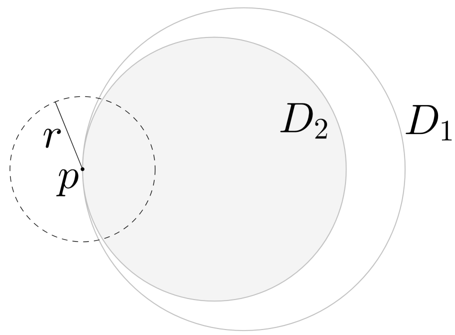


Figure 3: Condition (ii) essentially excludes points of tangential intersection of connected parts of $\operatorname{spt} T$, i.e., it forbid the existence of one-sided points arbitrarily close to two-sided points. For instance as in the picture p does not verify (ii), let $T = [\![D_1]\!] + [\![D_2]\!]$ with D_1 and D_2 being tangential circles at p then $\Theta(T,p) = 1 \geq \frac{1}{2} = \Theta(T,q)$ for all $q \neq p$ which belongs to the outer circumference.

Lemma 3.18 (The set of two-sided collapsed points is open)**.** *Let T and p as in Theorem 3.15. Assume that $p \in \Gamma$ is a two-sided collapsed point, then there is $\rho > 0$ such that $\Theta^2(T,q) = \Theta^2(T,p)$ for all $q \in \mathbf{B}(p,\rho) \cap \Gamma$. In particular, every such q is two-sided collapsed.*

Proof. Fix $Q \in \mathbb{N}, Q > Q^\star$ such that $\Theta^2(T,p) = Q - \frac{Q^\star}{2}$ and the unique tangent cone to T at p is $T_p = Q\,[\![\pi^+]\!] + (Q - Q^\star)\,[\![\pi^-]\!]$. If we choose $r > 0$ small enough we can assume that

$$\frac{\|T\|(\mathbf{B}(p,r))}{\pi r^2} \leq Q - \frac{Q^\star}{2} + \frac{1}{8}.$$

Now, we choose $s \in (0,r)$, in order to hold

$$\|T\|(\mathbf{B}(q,r-s)) \leq \|T\|(\mathbf{B}(p,r)) \leq \pi r^2\left(Q - \frac{Q^\star}{2} + \frac{1}{8}\right) \leq \pi(r-s)^2\left(Q - \frac{Q^\star}{2} + \frac{3}{16}\right),$$

for every $q \in \mathbf{B}(p,s) \cap \Gamma$. For every $\sigma \in (0, r-s)$, by the almost monotonocity formula, Proposition 2.3, we have that

$$\begin{aligned}\frac{\|T\|(\mathbf{B}(q,\sigma))}{\pi\sigma^2} &\leq e^{C_1((r-s)^{\beta_1} - \sigma^{\beta_1})}\frac{\|T\|(\mathbf{B}(q,r-s))}{\pi(r-s)^2} \\ &\leq e^{C_1(r-s)^{\beta_1}}\left(Q - \frac{Q^\star}{2} + \frac{3}{16}\right),\end{aligned} \tag{3.4}$$

for every $q \in \mathbf{B}(p,s) \cap \Gamma$. Then take $q \in \mathbf{B}(p,s) \cap \Gamma$ where this ball is chosen to be a subset of the neighbourhood U given by the definition of two-sided collapsed points. Hence, by Proposition 3.14 and Lemma 3.12, the tangent cone to T at q has to be of the form $T_q = Q'\,[\![\pi_q^+]\!] + (Q' - Q^\star)\,[\![\pi_q^-]\!]$, for some integer $Q' > Q^\star$, thus, letting $r \to 0$ in (3.4) we obtain

$$Q - \frac{Q^\star}{2} = \Theta^2(T,p) \leq \Theta^2(T,q) = Q' - \frac{Q^\star}{2} \overset{(3.4)}{\leq} Q - \frac{Q^\star}{2} + \frac{3}{16}.$$

□

The following theorem allows us to reduce the proof of Theorem 1.3 to the proof that any two-sided collapsed point is regular.

Theorem 3.19. *Let T and p as in Theorem 3.15 and assume that $C_0 = 0$. If $\mathrm{Reg}_b(T)$ is not dense in Γ, then there exists a two-sided collapsed singular point $p \in \Gamma$ with $\Theta^2(T,p) > \frac{Q^\star}{2}$.*

Remark 3.20. When we consider the setting of [8], i.e., when Γ belongs to a $C^{3,\alpha}$ convex barrier, $\alpha \in (0,1)$,, we have that two-sided points do not exists, in particular, $\mathrm{Reg}_b^2(T) = \emptyset$ and, by [8, Theorem 0.2], we know that $\mathrm{Reg}_b^1(T) = \Gamma$. In other words, the authors in [8] proved the full regularity of the current at the boundary.

Proof. Assume that $\mathrm{Sing}_b(T)$ has no empty interior, then we can define

$$C_i := \left\{ p \in \Gamma : \Theta^2(T,p) \geq i - \frac{1}{2} \right\} \cap \mathrm{int}(\mathrm{Sing}_b(T)).$$

By Proposition 2.4, the density restricted to the boundary is upper semicontinuous, then C_i is relatively closed in $\mathrm{int}(\mathrm{Sing}_b(T))$. Let D_i be the topological interior of C_i and E_i be the relatively open set $D_i \setminus C_{i+1}$ in $\mathrm{int}(\mathrm{Sing}_b(T))$. We fix $p \in \Gamma$ and the natural number i such that

$$i - \frac{1}{2} \leq \Theta^2(T,p) < i + \frac{1}{2}. \tag{3.5}$$

Assume that $p \notin \cup_{i \geq 1} E_i$, by the latter inequalities, we have $p \in C_i \setminus D_i$ which leads to

$$\mathrm{int}(\mathrm{Sing}_b(T)) \setminus \cup_i E_i \subset \cup_i C_i \setminus D_i.$$

Observe that $C_i \setminus D_i$ is relatively closed in $\mathrm{int}(\mathrm{Sing}_b(T))$ and then $\mathrm{int}(\mathrm{Sing}_b(T)) \setminus \cup_i E_i$ is the union of countably many closed subsets of $\mathrm{int}(\mathrm{Sing}_b(T))$ which guarantees, by the Baire Category Theorem, that $\cup_i E_i$ cannot be empty. So, there is $E_i \neq \emptyset$ relatively open in Γ, hence, in view of [8, Theorem 0.2], since E_i contains only singular points, any $p \in E_i$ satisfies $\Theta^2(T,p) \geq \frac{Q^\star + 1}{2}$ and, from Proposition 3.14 and Lemma 3.12, p is two-sided boundary flat point. Fix $p \in E_i$, we know that there exists $Q \in \mathbb{N}, Q > Q^*$ such that $\Theta^2(T,p) = Q - \frac{Q^\star}{2}$, thus

- if $\Theta^2(T,p) \in \mathbb{N}$, we get $\Theta^2(T,p) = i$. Now, assume by contradiction that there is $q \in E_i$ such that $\Theta^2(T,q) < \Theta^2(T,p)$, then, by (3.5), we necessarily have $i - \frac{1}{2} \leq \Theta^2(T,q) < i + \frac{1}{2}$ which ensures, by the classification of tangent cones, $\Theta^2(T,q) = Q - \frac{Q^\star + 1}{2}$. Since $\Theta^2(T,q) = Q' - \frac{Q^\star}{2}$ for some $Q' \in \mathbb{N}$, we obtain $Q' = Q - \frac{1}{2}$ which is a contradiction. We then conclude that p is a singular point which is also two-sided collapsed.
- if $\Theta^2(T,p) \notin \mathbb{N}$, then $\Theta^2(T,p) = i - \frac{1}{2}$ and, since E_i is relatively open, there is a relatively open, in Γ, neighborhood $U \subset E_i$ of p. By definition of E_i, for every $q \in U$, $\Theta^2(T,q) \geq i - \frac{1}{2} = \Theta^2(T,p)$ which ensures that p is two-sided collapsed.

□

We have now reduced our situation to prove the following theorem.

Theorem 3.21 (Two-sided collapsed points are regular)**.** *Let T and Γ be as in Assumption 2 with $C_0 = 0$. Then any two-sided collapsed point of T is a two-sided regular point of T.*

In fact, the rest of the paper is devoted to prove Theorem 3.21, we also mention that we prove this to the general setting of m-dimensional area minimizing currents with boundary multiplicity $Q^\star \geq 1$. However, our main result (Theorem 1.3) is stated for $2d$ area minimizing currents with boundary multiplicity $Q^\star \geq 1$ because we need to apply Theorem 3.19 which we proved only in this setting.

4 Approximations of currents by multi-valued collapsed Dirichlet minimizers

4.1 Definitions and regularity of collapsed Dirichlet minimizers

We refer the reader to [11] and [12] for standard definitions and notations about the theory of multiple valued functions. Throughout all this section we will consider an open set $\Omega \subset \mathbb{R}^m$ together with a $(m-1)$-submanifold γ of class $C^{3,\alpha}$ dividing Ω in two disjoint open sets Ω^+ and Ω^-.

Definition 4.1. *Let $\varphi \in W^{\frac{1}{2},2}(\gamma, \mathcal{A}_{Q^\star}(\mathbb{R}^n))$, $Q, Q^\star \in \mathbb{N}$, $Q \geq Q^\star \geq 1$. A* $(Q - \frac{Q^\star}{2})$**-valued function with interface** (γ, φ)*, consists of a pair (f^+, f^-) satisfying the following properties*

(i) $f^+ \in W^{1,2}(\Omega^+, \mathcal{A}_Q(\mathbb{R}^n))$, $f^- \in W^{1,2}(\Omega^-, \mathcal{A}_{Q-Q^\star}(\mathbb{R}^n))$,

(ii) $f^+|_\gamma = f^-|_\gamma + \varphi$.

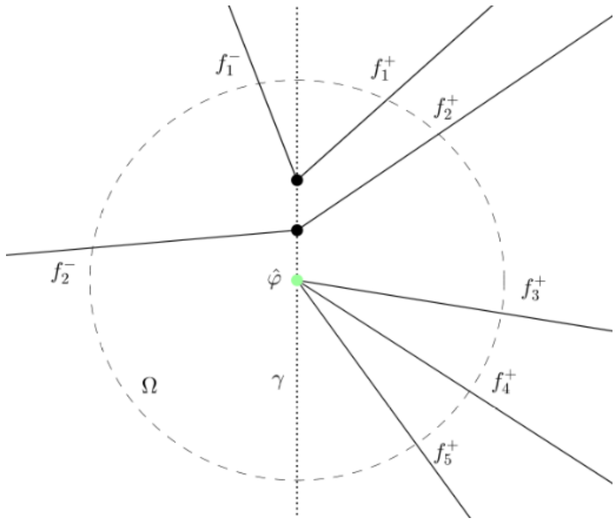
We define the **Dirichlet energy of** (f^+, f^-) *as* $\mathrm{Dir}(f^+, f^-, \Omega) := \mathrm{Dir}(f^+, \Omega^+) + \mathrm{Dir}(f^-, \Omega^-)$*. Such a pair will be called* **Dir-minimizing in** Ω*, if for all* $\left(Q - \frac{Q^\star}{2}\right)$*-valued function (g^+, g^-) with interface (γ, φ) which agrees with (f^+, f^-) outside of a compact set $K \subset\subset \Omega$ satisfies* $\mathrm{Dir}(f^+, f^-, \Omega) \leq \mathrm{Dir}(g^+, g^-, \Omega)$.

Remark 4.2. Note that when $Q^\star$ is an even number, we unfortunately overlap Definition 4.1 and Almgren's definition of Q-valued functions. However, since it will not cause any confusion in what follows, so we will allow this abuse of notation.

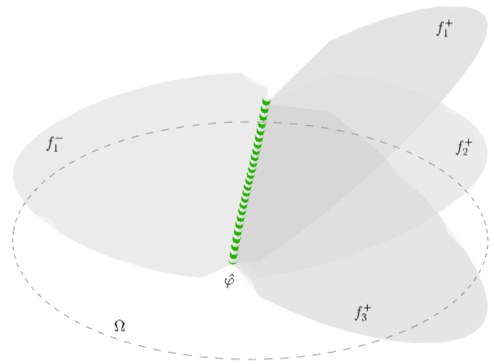
The interesting case to be treated here is when $Q > Q^\star > 1$. When $Q = Q^\star = 1$, the pair (f^+, f^-) consists of a single-valued function f^+ and its Dir-minimality is equivalent to the harmonicity of f^+. The case $Q > Q^\star = 1$ is studied in [7, Section 4]. The *one-sided* case, i.e. $Q = Q^\star$, has to be treated differently and it is done in dimension 2 in [8].

Definition 4.3. *Let (f^+, f^-) be a* $\left(Q - \frac{Q^\star}{2}\right)$*-valued function with interface (γ, φ) and $\varphi = Q^\star [\![\hat{\varphi}]\!]$ for a single valued function $\hat{\varphi}$. We say that (f^+, f^-)* **collapses at the interface***, if* $f^+|_\gamma = Q [\![\hat{\varphi}]\!]$.

Notice that, (f^+, f^-) satisfy the properties of the preceding definition, if and only if, $f^-|_\gamma = (Q - Q^\star) [\![\hat{\varphi}]\!]$.



(a) $n = 1, Q = 5, Q^\star = 3, f^+(x) = \sum_{i=1}^{5} [\![f_i^+(x)]\!]$ and $f^+(x) = \sum_{i=1}^{2} [\![f_i^-(x)]\!]$, so that the $(Q - \frac{Q^\star}{2})$-valued function (f^+, f^-) has interface $(\gamma, \hat{\varphi})$ where $\gamma = \{x = 0\}$ and $(x, \hat{\varphi}(x))$ is constantly equal to the green point.



(b) Assume $\Omega \subset \mathbb{R}$ and $n = 1, Q = 3, Q^\star = 2, f^+(x) = \sum_{i=1}^{3} [\![f_i^+(x)]\!]$ and $f^+(x) = [\![f_1^-(x)]\!]$, so that the $(Q - \frac{Q^\star}{2})$-valued function (f^+, f^-) collapses at the interface $(\gamma, \hat{\varphi})$ where $\gamma = \{x = 0\}$ and $\hat{\varphi}$ is represented by the green curve.

With these definitions settled, we aim to prove the harmonic regularity of collapsed $(Q - \frac{Q^\star}{2})$-valued maps along the same lines for $Q^\star = 1$ as it is done in [7, Theorem 4.5]. As we mentioned above, this part of the linear theory in our setting is true only when we consider $Q > Q^\star$ since we will construct some competitors in the arguments which need the existence of multi-valued functions defined in both sides of γ.

Theorem 4.4 (Regularity of collapsing $(Q - \frac{Q^\star}{2})$-Dir minimizers)**.** *Let $\varphi : \gamma \to \mathcal{A}_{Q^\star}(\mathbb{R}^n)$, where $\varphi = Q^\star [\![\hat{\varphi}]\!]$ for some $\hat{\varphi} \in C^{1,\alpha}(\gamma, \mathbb{R}^n)$, γ be a $(m-1)$-submanifold of class C^3 in $\mathbb{R}^m$, $Q > Q^\star \geq 1$, and (f^+, f^-) be a $\left(Q - \frac{Q^\star}{2}\right)$-valued Dir-minimizer with interface (γ, φ). If (f^+, f^-) collapses at the interface, then there is a single-valued harmonic function $h : \Omega \to \mathbb{R}^n$ such that $f^+ = Q [\![h|_{\Omega^+}]\!]$ and $f^- = (Q - Q^\star) [\![h|_{\Omega^-}]\!]$.*

If we do not assume that the pair (f^+, f^-) is collapsed and impose that γ is real analytic, we can obtain that the singular set of the pair is discrete when $Q^\star = 1$, see [19, Theorem 1.6]. Let us turn to the proof of Theorem 4.4, firstly we define the tangent function and then we characterize these tangent function.

Definition 4.5 (Tangent function)**.** *Let (f^+, f^-) be a $\left(Q - \frac{Q^\star}{2}\right)$-valued function with interface $(\gamma, Q^\star [\![0]\!])$. Fix $p \in \gamma$ and define a* **blowup of f at p at scale r** *as follows*

$$f^{\pm}_{p,r}(x) := \frac{f^{\pm}(p + rx)}{\sqrt{r^{2-m}\,\mathrm{Dir}(f^+, f^-, \mathrm{B}_r(p))}}, \forall r > 0,$$

where we assume that (f^+, f^-) is not identically $(Q [\![0]\!], (Q - Q^\star) [\![0]\!])$ in every ball $\mathrm{B}_r(0)$. For any sequence $r_k \to 0$, if the limit exists, we say that $g^{\pm} = \lim_{k \to +\infty} f^{\pm}_{p,r_k}$ is a **tangent function at p to f***.*

Lemma 4.6. *Let $Q > Q^\star$, (f^+, f^-) be a $\left(Q - \frac{Q^\star}{2}\right)$ Dir-minimizer which collapses at the interface $(\gamma, Q^\star [\![0]\!])$, where γ is a C^3 $(m-1)$-submanifold in $\mathbb{R}^m$, and fix $p \in \gamma$. Consider a tangent function*

$\left(h_0^+,h_0^-\right)$ to (f^+,f^-) at p and $\{e_1,\cdots,e_{m-1}\}$ a base of $T_p\gamma$. For each $i\in\{1,\cdots,m-1\}$, we define (h_i^+,h_i^-) to be a tangent to (h_{i-1}^+,h_{i-1}^-) at e_i. Then $(h^+,h^-):=(h_{m-1}^+,h_{m-1}^-)$ is given by $(Q\,[\![L]\!],(Q-Q^\star)\,[\![L]\!])$ where L is a nonzero linear function which vanishes on $T_p\gamma$.

Proof. Assume $T_p\gamma=\{x:x_m=0\}$. The consequences of [7, Lemma 4.29, Remark 4.31] readily holds in our higher multiplicity case, so we have the following properties:

(A) (h^+,h^-) is a $\left(Q-\frac{Q^\star}{2}\right)$ Dir-minimizer which collapses at the interface $(T_p\gamma,Q^\star\,[\![0]\!])$,

(B) (h^+,h^-) depends only on x_m namely there exist Q-valued function $\alpha^+:\mathbb{R}_+\to\mathcal{A}_Q(\mathbb{R}^n)$ and a $(Q-Q^\star)$-valued function $\alpha^-:\mathbb{R}_-\to\mathcal{A}_{Q-Q^\star}(\mathbb{R}^n)$ such that $h^\pm(x)=\alpha^\pm(x_m)$,

(C) (h^+,h^-) is an I-homogeneous function for some $I>0$, namely there is a Q-point P and a $(Q-Q^\star)$-point P' such that $\alpha^+(x_m)=x_m^IP$ and $\alpha^-(x_m)=(-x_m)^I P'$,

(D) $\mathrm{Dir}\left(h^+,\mathrm{B}_1(0)\right)+\mathrm{Dir}\left(h^-,\mathrm{B}_1(0)\right)=1$.

Since (h^+,h^-) is a Dir-minimizer and both h^+ and h^- are C^2, both h^+ and h^- are classical harmonic functions, therefore, since they depend only upon one variable, we necessarily have that $I=1$. So there are coefficients $\beta_1^+,\dots,\beta_Q^+$ and $\beta_1^-,\dots,\beta_{Q-Q^\star}^-$ such that

$$h^+(x)=\sum_{i=1}^{Q}[\![\beta_i^+x_m]\!]\,,\text{if } x_m>0,\quad\text{and}\quad h^-(x)=\sum_{i=1}^{Q-Q^\star}[\![\beta_i^-x_m]\!]\,,\text{if } x_m<0.$$

If $Q>Q^\star>1$, then we can assume that $\beta_{j_0}^+\neq\beta_{i_0}^-$. Now, we will construct a competitor of (h^+,h^-) with less Dir-energy which is the desired contradiction. Note that, in order to construct a competitor, we have to assure that it has the same interface of (h^+,h^-), i.e. it takes $[\![0]\!]$ at least $Q^\star$ times at $T_p\gamma=\{x_m=0\}$. For $x=(x',x_m)$, define

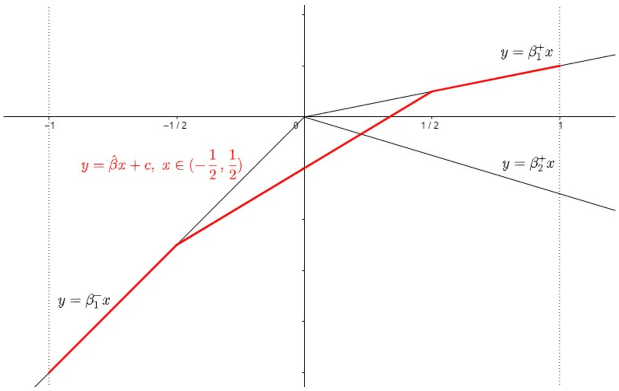
$$\hat h^+(x)=\begin{cases}\left[\!\!\left[\hat\beta x_m+c(|x'|)\right]\!\!\right]+\sum_{j=1,j\neq j_0}^{Q}\left[\!\!\left[\beta_j^+x_m\right]\!\!\right], & \text{if } x\in\overline{\mathrm{B}_{\frac12}^+(0)},\\ h^+(x), & \text{if } x\in\mathrm{B}_1^+(0)\setminus\overline{\mathrm{B}_{\frac12}^+(0)}.\end{cases}$$

$$\hat h^-(x)=\begin{cases}\left[\!\!\left[\hat\beta x_m+c(|x'|)\right]\!\!\right]+\sum_{i=1,i\neq i_0}^{Q-Q^\star}[\![\beta_i^+x_m]\!]\,, & \text{if } x\in\overline{\mathrm{B}_{\frac12}^-(0)},\\ h^-(x), & \text{if } x\in\mathrm{B}_1^-(0)\setminus\overline{\mathrm{B}_{\frac12}^-(0)}.\end{cases}$$

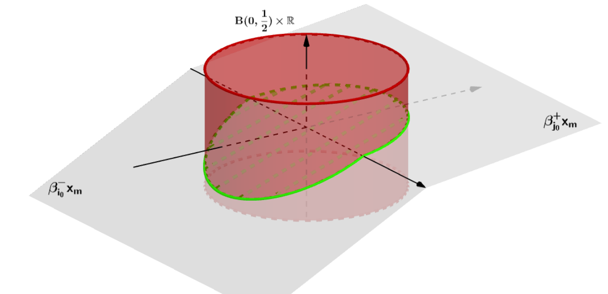
where $\hat\beta=\frac{\beta_{j_0}^++\beta_{i_0}^-}{2}$, $c(|x'|)=\bar\beta\sqrt{1/4-|x'|^2}$ and $\hat\beta=\frac{\beta_{j_0}^+-\beta_{i_0}^-}{2}$.
By direct computation, we have

$$\mathrm{Dir}(\hat h^+,\mathrm{B}_{1/2}^+(0))=|\mathrm{B}_{1/2}^+(0)|\left[\sum_{j=1,j\neq j_0}^{Q}|\beta_{j_0}^+|^2+|\hat\beta|^2\right]+\int_{\mathrm{B}^{m-1}(0,1/2)}\int_0^{\sqrt{\frac14-|x'|^2}}\frac{|\bar\beta|^2|x'|^2}{\frac14-|x'|^2}dx'dx_m,$$

the integral on the right hand side can be bounded by $|\bar\beta|^2|\mathrm{B}_{1/2}^+(0)|$ since $\bar\beta\neq0$ and the integrating function is radial. By the very same computation, we finally have that



(a) if $\Omega \subset \mathbb{R}$ and $n = 1, Q = 2, Q^\star = 1, h^+(x) = [\![\beta_1^+ x]\!] + [\![\beta_2^+ x]\!]$ and $h^-(x) = [\![\beta_1^- x]\!]$, we define the competitor $(\hat{h}^+, \hat{h}^-)$ as the red function which has the same interface of (h^+, h^-) and less Dir-energy.



(b) With $m = 2$ competitor $(\hat{h}^+, \hat{h}^-)$ represented by the green hypersurface is not linear inside the cylinder, but it also satisfies what we need, i.e. it has the same interface of (h^+, h^-) and it has less Dir-energy.

$$\begin{aligned}\mathrm{Dir}(\hat{h}^+, \hat{h}^-, \mathrm{B}_{1/2}(0)) &< |\mathrm{B}_{1/2}^+(0)| \Big[\sum_{j=1, j\neq j_0}^{Q} |\beta_j^+|^2 + \sum_{i=1, i\neq i_0}^{Q-Q^\star} |\beta_i^+|^2 + 2|\hat{\beta}|^2 + 2|\bar{\beta}|^2 \Big] \\ &= \mathrm{Dir}(h^+, h^-, \mathrm{B}_{1/2}(0)).\end{aligned}$$

By construction they have the same Dir-energy outside $\mathrm{B}_{1/2}(0)$, thus every β_j^+ has to coincide with β_i^- and we finish the proof of the lemma. □

Definition 4.7. *Let us denote* $\boldsymbol{\eta}(P) = \frac{1}{Q}\sum_{i=1}^{Q} P_i$ *the center of the* Q*-point* $P = \sum_{i=1}^{Q} [\![P_i]\!]$.

As a simple corollary of the above lemma we have:

Corollary 4.8. *Let* $Q > Q^\star$ *and assume* (f^+, f^-) *is a* $\left(Q - \frac{Q^\star}{2}\right)$ *Dir-minimizer which collapses at* $(\gamma, Q^\star [\![0]\!])$, *where* γ *is a* C^3 $(m-1)$*-submanifold in* $\mathbb{R}^m$. *If* $\boldsymbol{\eta} \circ f^- = \boldsymbol{\eta} \circ f^+ = 0$, *then* $f^+ = Q [\![0]\!]$ *and* $f^- = (Q - Q^\star) [\![0]\!]$.

Proof. If (f^+, f^-) is identically $(Q [\![0]\!], (Q - Q^\star) [\![0]\!])$ in a neighborhood U of a point $p \in \gamma$, then, by the interior regularity theory of Dir-minimizer (precisely, [10, Proposition 3.22]), (f^+, f^-) is identically $(Q [\![0]\!], (Q - Q^\star) [\![0]\!])$ in the connected component of the domain of (f^+, f^-) which contains p. Thus, if the corollary were false, then there would be a point $p \in \gamma$ such that $\mathrm{Dir}(f^+, \mathrm{B}_r^+(p)) + \mathrm{Dir}(f^-, \mathrm{B}_r^-(p)) > 0$ for every $r > 0$ such that $\mathrm{B}_r(p) \subset \Omega$. If we consider (h^+, h^-) as in Lemma 4.6, we conclude that $\boldsymbol{\eta} \circ h^+ = \boldsymbol{\eta} \circ h^- = 0$, since such property is inherited by each tangent map. But then the nonzero linear function L of the conclusion of Lemma 4.6 should be equal to $\boldsymbol{\eta} \circ h^+$ on $\{x_m > 0\}$ and $\boldsymbol{\eta} \circ h^-$ on $\{x_m \leq 0\}$. Hence L should vanish identically, contradicting Lemma 4.6. □

Before the proof of Theorem 4.4, we introduce the following notation which will be used throughout the paper.

$$f \oplus \zeta := \sum_i [\![f_i + \zeta]\!],$$

where f is a Q-valued function with a measurable selection of single-valued functions f_i and ζ is a single-valued functions both defined on the same domain.

Proof of Theorem 4.4. **The case $\hat{\varphi} \equiv 0$:** Firstly, using the regularity theory of classical elliptic equations, we obtain that the functions $\boldsymbol{\eta} \circ f^{\pm}$ are differentiable up to the boundary γ, i.e., belong to $C^1\left(\Omega^{\pm} \cup \gamma\right)$. Let ν be the unit normal to γ. We claim that

$$\partial_\nu \left(\boldsymbol{\eta} \circ f^+\right)(p) = \partial_\nu \left(\boldsymbol{\eta} \circ f^-\right)(p) \quad \text{for all } p \in \gamma \cap \Omega. \tag{4.1}$$

In fact, assume by contradiction that, at some point $p \in \gamma \cap \Omega$, we have $\partial_\nu \left(\boldsymbol{\eta} \circ f^+\right)(p) \neq \partial_\nu \left(\boldsymbol{\eta} \circ f^-\right)(p)$ and consider a tangent function (h^+, h^-) to (f^+, f^-) at p which is the limit of some rescaled sequence $\left(f^+_{p,\rho_k}, f^-_{p,\rho_k}\right)$, where we denote

$$f^{\pm}_{p,\rho_k}(x) := \frac{f^{\pm}(p + \rho_k x)}{\sqrt{\rho_k^{2-m} \operatorname{Dir}(f^+, f^-, \mathrm{B}_{\rho_k}(p))}}.$$

Observe that, since at least one among $\partial_\nu \left(\boldsymbol{\eta} \circ f^+\right)(p)$ and $\partial_\nu \left(\boldsymbol{\eta} \circ f^-\right)(p)$ differs from 0, we necessarily have

$$c_1 \rho_k^m \geq \operatorname{Dir}\left(\boldsymbol{\eta} \circ f^+, \boldsymbol{\eta} \circ f^-, \mathrm{B}_{\rho_k}(p)\right) \geq c_0 \rho_k^m,$$

for some constants $c_1 = c_1(f^+, f^-) > 0, c_0 = c_0(f^+, f^-) > 0$. Thus, by rescaling, we obtain

$$\begin{aligned}
c_1 \frac{\rho_k^m}{\operatorname{Dir}\left(f^+, f^-, \mathrm{B}_{\rho_k}(p)\right)} &\geq \frac{\operatorname{Dir}\left(\boldsymbol{\eta} \circ f^+, \boldsymbol{\eta} \circ f^-, \mathrm{B}_{\rho_k}(p)\right)}{\operatorname{Dir}\left(f^+, f^-, \mathrm{B}_{\rho_k}(p)\right)} \\
&= \frac{\operatorname{Dir}\left(\boldsymbol{\eta} \circ f^+_{p,\rho_k}, \boldsymbol{\eta} \circ f^-_{p,\rho_k}, \mathrm{B}_1(0)\right)}{\operatorname{Dir}\left(f^+_{p,\rho_k}, f^-_{p,\rho_k}, \mathrm{B}_1(0)\right)} \\
&= \operatorname{Dir}\left(\boldsymbol{\eta} \circ f^+_{p,\rho_k}, \boldsymbol{\eta} \circ f^-_{p,\rho_k}, \mathrm{B}_1(0)\right) \\
&\geq c_0 \frac{\rho_k^m}{\operatorname{Dir}\left(f^+, f^-, \mathrm{B}_{\rho_k}(p)\right)}.
\end{aligned} \tag{4.2}$$

Therefore, we have the following two alternatives:

(I) If $\limsup_k (\rho_k)^{-m} \operatorname{Dir}\left(f^+, f^-, \mathrm{B}_{\rho_k}(p)\right) = +\infty$, by (4.2), denoting by $\left(h_0^+, h_0^-\right)$ the tangent function to (f^+, f^-) at p, passing to the limit in (4.2), we have that $\operatorname{Dir}(\boldsymbol{\eta} \circ h_0^+, \boldsymbol{\eta} \circ h_0^-, \mathrm{B}_1(0)) = 0$ and then $\boldsymbol{\eta} \circ h_0^{\pm} \equiv 0$. By Corollary 4.8, $\left(h_0^+, h_0^-\right)$ should be trivial. But this is not possible, because the energy of a tangent function satisfies $\operatorname{Dir}\left(h_0^+, h_0^-, \mathrm{B}_1(0)\right) = 1$, see Lemma 4.6.

(II) If $\limsup_k (\rho_k)^{-m} \operatorname{Dir}\left(f^+, f^-, \mathrm{B}_{\rho_k}(p)\right) < +\infty$, by (4.2), we have $\operatorname{Dir}(\boldsymbol{\eta} \circ h_0^+, \boldsymbol{\eta} \circ h_0^-, \mathrm{B}_1(0)) > 0$ and thus $\boldsymbol{\eta} \circ h_0^+$ and $\boldsymbol{\eta} \circ h_0^-$ are distinct functions and at least one among them is a nontrivial function. Indeed, they are distinct follows from the fact that the blowup of a differentiable function coincides with its differential and we are assuming that $\partial_\nu \left(\boldsymbol{\eta} \circ f^+\right)(p) \neq \partial_\nu \left(\boldsymbol{\eta} \circ f^-\right)(p)$. Since case (I) never occurs, we can apply this argument iteratively until we reach the pair (h^+, h^-) of Lemma 4.6 and then conclude that $\boldsymbol{\eta} \circ h^+$ and $\boldsymbol{\eta} \circ h^-$ are two distinct linear functions with one of them being non trivial and this contradicts Lemma 4.6.

We have verified the validity of (4.1) and it is enough to conclude the proof, indeed, it implies that the function

$$\zeta := \begin{cases} \boldsymbol{\eta} \circ f^+ & \text{on } \Omega^+, \\ \boldsymbol{\eta} \circ f^- & \text{on } \Omega^-. \end{cases} \tag{4.3}$$

is an harmonic function defined on the entire Ω. Using the notation above, we set

$$\tilde{f}^+ := f^+ \oplus (-\zeta) \text{ and } \tilde{f}^- := f^- \oplus (-\zeta). \tag{4.4}$$

By [10, Lemma 3.23], it is easy to see that $\left(\tilde{f}^+, \tilde{f}^-\right)$ is a $\left(Q - \frac{Q^\star}{2}\right)$ Dir-minimizer which collapses at the interface $(\gamma, [\![0]\!])$ and that $\boldsymbol{\eta} \circ \tilde{f}^+ = \boldsymbol{\eta} \circ \tilde{f}^- = 0$. Thus we apply Corollary 4.8 and conclude that $\tilde{f}^+ = Q\,[\![0]\!]$ and $\tilde{f}^- = (Q - Q^\star)\,[\![0]\!]$, which complete the proof.

The general case: We fix ν as an unit normal to γ. As in the particular case $\hat{\varphi} \equiv 0$, we claim that $\partial_\nu (\boldsymbol{\eta} \circ f^+) = \partial_\nu (\boldsymbol{\eta} \circ f^-)$. With this claim, proceeding as in the former case, we can define ζ as in (4.3) and conclude that it is a harmonic function. We then define $\left(\tilde{f}^+, \tilde{f}^-\right)$ as in (4.4). To this pair, we can apply the former case and conclude the proof of the theorem. To prove the claim, assume by contradiction that, for some $p \in \gamma$, we have that $\partial_\nu(\boldsymbol{\eta} \circ f^+(p)) \neq \partial_\nu (\boldsymbol{\eta} \circ f^-)(p)$. Without loss of generality we can assume that $p = 0, \hat{\varphi}(0) = 0$ and $D\hat{\varphi}(0) = 0$. Since at least one among $Df^\pm(0)$ does not vanish, we must have

$$\mathrm{Dir}\left(f^+, f^-, \mathrm{B}_\rho(0)\right) \geq \mathrm{Dir}\left(\boldsymbol{\eta} \circ f^+, \boldsymbol{\eta} \circ f^-, \mathrm{B}_\rho(0)\right) \geq c_0 \rho^m, \tag{4.5}$$

for some positive constant c_0. It also means that there exist a constant $\eta > 0$ and a sequence $\rho_k \downarrow 0$ such that

$$\mathrm{Dir}\left(f^+, f^-, \mathrm{B}_{\rho_k}(0)\right) \geq \eta \left(\mathrm{Dir}\left(f^+, f^-, \mathrm{B}_{2\rho_k}(0)\right)\right),$$

otherwise we would contradict the lower bound (4.5). We see that $f^\pm_{0,\rho_k}$ have finite energy on $\mathrm{B}_2(0)$ and thus there is strong convergence of a subsequence to a $\left(Q - \frac{Q^\star}{2}\right)$ Dir-minimizer (h^+, h^-) with interface $(T_0\gamma, Q^\star\,[\![0]\!])$. The latter must then have Dirichlet energy 1 on $\mathrm{B}_1(0)$. We then have two possibilities:

(I) $\limsup_k (\rho_k)^{-m}\,\mathrm{Dir}\,(f^+, f^-, \mathrm{B}_{\rho_k}(0)) = +\infty$. Arguing as in (I) of the former case, this gives that $\boldsymbol{\eta} \circ h^+ = \boldsymbol{\eta} \circ h^- = 0$ and thus we conclude that $\left(h_0^+, h_0^-\right)$ is trivial, which is a contradiction,

(II) $\limsup_k (\rho_k)^{-m}\,\mathrm{Dir}\,(f^+, f^-, \mathrm{B}_{\rho_k}(0)) < +\infty$. Assuming in this case that $T_0\gamma = \{x_m = 0\}$, we conclude that $\left(h_0^+, h_0^-\right)$ is a $\left(Q - \frac{Q^\star}{2}\right)$ Dir-minimizer with flat interface $(T_0\gamma, Q^\star\,[\![0]\!])$, but also that $\boldsymbol{\eta} \circ h_0^\pm(x) = C_d \partial_\nu (\boldsymbol{\eta} \circ f^\pm)(0) x_m$ for some positive constant C_d that in general is not necessarily equal to one, because we use a normalization constant to have $\mathrm{Dir}(h_0^+, h_0^-, \mathrm{B}_1(0)) = 1$. By the particular case $\hat{\varphi} \equiv 0$, we thus conclude that $\partial_\nu (\boldsymbol{\eta} \circ f^+)(0) = \partial_\nu (\boldsymbol{\eta} \circ f^-)(0)$.

□

4.2 Harmonic approximations

In this chapter we aim to approximate the area minimizing current T by Q copies of an harmonic function in the right side and $Q-Q^\star$ copies of the same harmonic function in the left side. To this end, we will first approximate the current T by $(Q-\frac{Q^\star}{2})$-Lipschitz functions which, if we do not assume the minimality of T, will not be necessarily minimizers for the Dirichlet energy. Once we consider the minimality condition on T we will be able to upgrade our approximations using the regularity theorem, see Theorem 4.4, to furnish the desired harmonic approximations.

For any π, π_0 belonging to $\mathrm{G}_{m,m+n}$, where $\mathrm{G}_{k,l}$ denotes the set of k-dimensional subspaces of $\mathbb{R}^l$, we introduce, for any $p \in \mathbb{R}^{m+n}$ the notation $\mathrm{B}_r(p,\pi)$ for the disks $\mathbf{B}(p,r) \cap (p+\pi)$, if π is omitted, then we assume $\pi = \pi_0 = \mathbb{R}^m \times \{0\}$, and $\mathbf{C}(p,r,\pi)$ for the cylinders $\mathrm{B}_r(p,\pi) + \pi^\perp$, we also fix $\mathbf{C}(x,r) := \mathbf{C}(p,r,\pi_0)$.

Definition 4.9. *Let $\alpha \in]\,0,1]$ and integers $m \geq 2$, $Q^\star \geq 1$, and take Γ any $(m-1)$-rectifiable set. Let T be an m-dimensional integral current with boundary $\partial T = Q^\star [\![\Gamma]\!]$ and assume that $p \in \Gamma$. Then*

(i) We call the **cylindrical excess relative to the plane** π *the quantity*

$$\mathbf{E}(T, \mathbf{C}(p,r)\,, \pi) := \frac{1}{\omega_m r^m} \int_{\mathbf{C}(p,r)} \frac{|\vec{T}(x) - \vec{\pi}|^2}{2}\, d\|T\|(x),$$

and the **cylindrical excess** *the quantity*

$$\mathbf{E}(T, \mathbf{C}(p,r)) := \min\{\mathbf{E}(T, \mathbf{C}(p,r)\,, \pi) : \pi \subset \mathbb{R}^{m+n}\}.$$

(ii) We call the **spherical excess relative to the plane** π *the quantity*

$$\mathbf{E}(T, \mathbf{B}(p,r)\,, \pi) := \frac{1}{\omega_m r^m} \int_{\mathbf{B}(p,r)} \frac{|\vec{T}(x) - \vec{\pi}|^2}{2}\, d\|T\|(x),$$

and the **spherical excess** *the quantity*

$$\mathbf{E}(T, \mathbf{B}(p,r)) := \min\{\mathbf{E}(T, \mathbf{B}(p,r)\,, \pi) : \pi \subset \mathbb{R}^{m+n}\}.$$

(iii) We say that the **boundary spherical excess** *is the quantity*

$$\mathbf{E}^\flat(T, \mathbf{B}(p,r)) := \min\left\{\mathbf{E}(T, \mathbf{B}(p,r)\,, \pi) : \, T_p\Gamma \subset \pi \subset \mathbb{R}^{m+n}\right\}.$$

(iv) The **height of T in a set $G \subset \mathbb{R}^{m+n}$ with respect to a plane** π *is defined as*

$$\mathbf{h}(T, G, \pi) := \mathrm{diam}(\mathbf{p}_\pi^\perp(\mathrm{spt}(T) \cap G)) = \sup\left\{\left|\mathbf{p}_\pi^\perp(q-p)\right| : q, p \in \mathrm{spt}(T) \cap G\right\},$$

where $\mathbf{p}_\pi^\perp$ denotes the orthogonal projection onto $\pi^\perp$.

(v) If $\mathrm{spt}(T) \subset \mathbf{C}(p,r,\pi)$, we define the **excess measure with respect to $\mathbf{C}(p,r,\pi)$** *as the measure which to each $F \subset \mathrm{B}_r(p,\pi)$ gives*

$$\mathbf{e}_T(F) := \frac{1}{2} \int_{F+\pi^\perp} |\vec{T} - \vec{\pi}|^2 d\|T\|.$$

In this subsection we assume that $\pi_0 = \mathbb{R}^m \times \{0\}$ and we use the notation $\mathbf{p}$ and $\mathbf{p}^\perp$ for the orthogonal projections onto π_0 and $\pi_0^\perp$ respectively, whereas $\mathbf{p}_\pi$ and $\mathbf{p}_\pi^\perp$ will denote, respectively, the orthogonal projections onto the plane π and its orthogonal complement $\pi^\perp$. For the remaining part of this work, we will call **dimensional constants** those which depends only on $m, n, Q^\star$ and Q.

Assumption 3. *Let $\alpha \in (0,1]$ and integers $m \geq 2$, $Q^\star \geq 1$. Consider Γ a $C^{2,\alpha}$ oriented $(m-1)$-submanifold without boundary. Let T be an m-dimensional integral current in $\mathbf{B}(0,2)$ with boundary $\partial T \llcorner \mathbf{B}(0,2) = Q^\star [\![\Gamma \cap \mathbf{B}(0,2)]\!]$ and assume that $p \in \Gamma$. We also assume $T_p\Gamma = \mathbb{R}^{m-1} \times \{0\} \subset \pi_0, \psi_1 : \mathbb{R}^{m-1} \to \mathbb{R}, \psi : \gamma \subset \mathbb{R}^m \times \{0\} \to \mathbb{R}^n, \psi_2 : \mathbb{R}^{m-1} \to \mathbb{R}^{n+1}, \psi_2(x) = (\psi_1(x), \psi(x, \psi_1(x)))$ with $\mathrm{graph}(\psi_1) = \gamma$ and $\Gamma = \mathrm{graph}(\psi_2)$ satisfying the bounds $\|D\psi_2\|_{0,\mathrm{B}_2(0)} \leq c_0$ and $\mathbf{A} := \|A_\Gamma\|_{0,\mathbf{B}(0,2)} \leq c_0$, where A_Γ denotes the second fundamental form of Γ and c_0 is a positive small geometric constant. We assume that*

- *(i) $p \in \Gamma$ is a two-sided collpased point with $Q - \frac{Q^\star}{2} = \Theta^m(T,p)$, for some integers $Q > Q^\star \geq 1$,*
- *(ii) $\gamma = \mathbf{p}(\Gamma)$ divides $\mathrm{B}_{4r}(p) \subset \pi_0$ in two disjoint open sets Ω^+ and Ω^-,*
- *(iii) $\mathbf{p}_\sharp T = Q [\![\Omega^+]\!] + (Q - Q^\star) [\![\Omega^-]\!]$.*

Observe that thanks to (iii) we have the identities

$$
\begin{aligned}
\mathbf{E}(T, \mathbf{C}(p,4r)) &= \frac{1}{\omega_m (4r)^m} \left(\|T\| (\mathbf{C}(p,4r)) - \left(Q\left|\Omega^+\right| + (Q-Q^\star)\left|\Omega^-\right| \right) \right), \\
\mathbf{e}_T(F) &= \|T\| (F \times \mathbb{R}^n) - \left(Q\left|\Omega^+ \cap F\right| + (Q-Q^\star)\left|\Omega^- \cap F\right| \right).
\end{aligned}
\tag{4.6}
$$

Definition 4.10. *Given a current T in a cylinder $\mathbf{C}(p,4r,\pi)$, we introduce the noncentered maximal function of $\mathbf{e}_T$ as*

$$
\mathbf{me}_T(y) := \sup_{y \in \mathrm{B}_s(z,\pi) \subset \mathrm{B}_{4r}(p,\pi)} \frac{\mathbf{e}_T(\mathrm{B}_s(z,\pi))}{\omega_m s^m}.
$$

The following theorem allows us to approximate the current by a $(Q - \frac{Q^\star}{2})$-Lipschitz map which coincides with the current in a closed set K which is called the **good set**. Moreover, we prove that the **bad set**, i.e. $\mathrm{B}_{3r}(0) \setminus K$, has small measure. The tricky part of this theorem is to show that we can take such approximation collapsing at the interface. Notice that, no minimality condition are being assumed to prove this result.

Theorem 4.11 (Lusin type weak Lipschitz approximation)**.** *There are positive geometric constants $C = C(m,n,Q,Q^\star)$ and $c_0 = c_0(m,n,Q^\star,Q)$ with the following properties. Assume T satisfies Assumption 3 and $\mathbf{E}(T,\mathbf{C}(p,4r)) \leq c_0$. Then, for any $\delta_* \in (0,1)$, there are a closed set $K \subset \mathrm{B}_{3r}(\mathbf{p}(p))$ and a $\left(Q - \frac{Q^\star}{2}\right)$ valued function (u^+, u^-) on $\mathrm{B}_{3r}(\mathbf{p}(p))$ which collapses at the interface $(\gamma, Q^\star [\![\psi]\!])$ satisfying the following properties:*

$$\operatorname{Lip}\left(u^{\pm}\right) \leq C\left(\delta_*^{1/2} + r^{\frac{1}{2}}\mathbf{A}^{\frac{1}{2}}\right), \tag{4.7}$$

$$\operatorname{osc}\left(u^{\pm}\right) \leq C\mathbf{h}\left(T, \mathbf{C}(p,4r), \pi_0\right) + Cr\mathbf{E}\left(T, \mathbf{C}(p,4r)\right)^{1/2} + Cr^2\mathbf{A}, \tag{4.8}$$

$$K \subset \mathrm{B}_{3r}\left(\mathbf{p}(p)\right) \cap \left\{\mathbf{me}_T \leq \delta_*\right\}, \tag{4.9}$$

$$\mathbf{G}_{u^{\pm}} \llcorner \left[\left(K \cap \Omega^{\pm}\right) \times \mathbb{R}^n\right] = T \llcorner \left[\left(K \cap \Omega^{\pm}\right) \times \mathbb{R}^n\right], \tag{4.10}$$

$$\left|\mathrm{B}_s\left(\mathbf{p}(p)\right) \setminus K\right| \leq \frac{C}{\delta_*}\mathbf{e}_T\left(\left\{\mathbf{me}_T > \delta_*\right\} \cap \mathrm{B}_{s+r_1 r}\left(\mathbf{p}(p)\right)\right), \quad \forall s \leq \left(3 - r_1\right)r, \tag{4.11}$$

$$\frac{\left\|T - \mathbf{G}_{u^+} - \mathbf{G}_{u^-}\right\|\left(\mathbf{C}(p,3r)\right)}{r^m} \leq \frac{C}{\delta_*}\mathbf{E}\left(T, \mathbf{C}(p,4r)\right), \tag{4.12}$$

where $r_1 = c_0 \sqrt[m]{\frac{\mathbf{E}(T,\mathbf{C}(p,4r))}{\delta_*}}$.

Proof. The proof of this theorem is a straightforward adaptation of the corresponding statement considering multiplicity $Q^\star = 1$, c.f. [7, Theorem 5.5], we reiterate that it is not used any minimality condition to prove this weak approximation. □

From now on the approximation of Theorem 4.11 is called the $\delta_*^{\frac{1}{2}}$ **-approximation of T in $\mathbf{C}(p,3r)$**. If $E := \mathbf{E}\left(T, \mathbf{C}(p,4r)\right)$, actually in the sequel we will choose $\delta_*^{\frac{1}{2}}$ to be E^β for a small suitable constant β. In the following theorem, we will add the minimality condition on T to prove that, if E is taken sufficiently small, then (u^+, u^-) is close to a minimizer of the Dirichlet energy, i.e., a $\left(Q - \frac{Q^\star}{2}\right)$-Dir-minimizer, which collapses at its interface and thus, by Theorem 4.4, consists of a single harmonic sheet.

Theorem 4.12 (Harmonic approximation)**.** *For every $\eta_* > 0$ and every $\beta \in \left(0, \frac{1}{4m}\right)$ there exist constants $\varepsilon = \varepsilon(m, n, Q^\star, Q, \eta_*, \beta) > 0$ and $C = C(m, n, Q^\star, Q, \eta_*, \beta) > 0$ with the following property. Let T and Γ be as in Assumption 2 with $C_0 = 0$ and under the conditions of Theorem 4.11, $E \leq c_0$, let (u^+, u^-) be the E^β-approximation of T in $\mathrm{B}_{3r}\left(\mathbf{p}(p)\right)$ and let K be the good set satisfying all the properties (4.7)-(4.12). If $E \leq \varepsilon$ and $r\mathbf{A} \leq \varepsilon E^{\frac{1}{2}}$, then*

$$\mathbf{e}_T\left(\mathrm{B}_{5r/2}\left(\mathbf{p}(p)\right) \setminus K\right) \leq \eta_* E, \tag{4.13}$$

and

$$\operatorname{Dir}\left(u^+, u^-, \Omega \cap \mathrm{B}_{2r}\left(\mathbf{p}(p)\right) \setminus K\right) \leq C\eta_* E. \tag{4.14}$$

Moreover, there exists an harmonic function $h : \mathrm{B}_{2r}\left(\mathbf{p}(p)\right) \to \mathbb{R}^n$ such that $h|_{\{x_m = 0\}} \equiv 0$ and *satisfies the following inequalities:*

$$r^{-2}\int_{\mathrm{B}_{2r}(\mathbf{p}(p))\cap\Omega^+}\mathcal{G}\left(u^+,Q\,[\![h]\!]\right)^2+\int_{\mathrm{B}_{2r}(\mathbf{p}(p))\cap\Omega^+}\left(|Du^+|-\sqrt{Q}|Dh|\right)^2\leq\eta_*Er^m, \tag{4.15}$$

$$r^{-2}\int_{\mathrm{B}_{2r}(\mathbf{p}(p))\cap\Omega^-}\mathcal{G}\left(u^-,(Q-Q^\star)\,[\![h]\!]\right)^2+\int_{\mathrm{B}_{2r}(\mathbf{p}(p))\cap\Omega^-}\left(|Du^-|-\sqrt{Q-Q^\star}|Dh|\right)^2\leq\eta_*Er^m, \tag{4.16}$$

$$\int_{\mathrm{B}_{2r}(\mathbf{p}(p))\cap\Omega^\pm}\left|D\left(\boldsymbol{\eta}\circ u^\pm\right)-Dh\right|^2\leq\eta_*Er^m. \tag{4.17}$$

Proof. Without loss of generality we assume that $p=0$, $r=1$, and $\psi(0)=0$.

Proof of (4.13) **and** (4.14). Firstly we want to note that (4.14) is a consequence of (4.13). Indeed, since, $\delta_*=E^{2\beta}$, use first (4.9), (4.11) and (4.13) to estimate

$$|\mathrm{B}_2\,(0)\setminus K|\leq C\eta_*E^{1-2\beta}.$$

Since $\mathrm{Lip}\,(u^\pm)\leq CE^{2\beta}$, (4.14) follows easily. We now let fixed β,η_*, and we will argue by contradiction. Assuming that the statement is false, we obtain a sequence of area minimizing currents T_k and submanifolds Γ_k as in Assumption 3 satisfying the following properties for all $k\in\mathbb{N}$:

(i) The cylindrical excesses $E_k:=\mathbf{E}\left(T_k,\mathbf{C}(0,4)\,,\pi_0\right)$ satisfy $E_k\leq\frac{1}{k}$,

(ii) Γ_k is the graph of the entire function $\psi_{2k}:\mathbb{R}^{m-1}\to\mathbb{R}^{n+1}$ satisfying the bound

$$\|\psi_{2k}\|_{C^2(\mathrm{B}_8(0))}\leq C\mathbf{A}_k\leq\frac{C}{k}E_k^{1/2}, \tag{4.18}$$

(iii) The estimate (4.13) fails, i.e., for some positive $c_2>0$,

$$\mathbf{e}_{T_k}\left(\mathrm{B}_{5/2}\,(0)\setminus K_k\right)>\eta_*E_k=3c_2E_k. \tag{4.19}$$

The pair $\left(f_k^+,f_k^-\right)$ are $\left(Q-\frac{Q^\star}{2}\right)$-valued maps defined on $\mathrm{B}_3\,(0)$ which collapses at its interface $(\gamma_k,Q^\star\,[\![\psi_k]\!])$ denotes the E_k^β-Lipschitz approximations of the current T_k and K_k the corresponding good set. We denote by $B_{k,r}^\pm$ the domains of the functions $f_k^\pm$ intersected with the ball $\mathrm{B}_r\,(0)\subset\pi_0$. From the Taylor expansion of the area functional, arguing as in [11, Remark 5.5], since $E_k\downarrow 0$, we conclude the following inequalities for every $s\in[5/2,3]$:

$$\begin{aligned}\int_{K_k\cap B_{k,s}^+}\frac{|Df_k^+|^2}{2}+\int_{K_k\cap B_{k,s}^-}\frac{|Df_k^-|^2}{2}&\overset{Taylor}{\leq}\left(1+CE_k^{2\beta}\right)\mathbf{e}_{T_k}\left(K_k\cap\mathrm{B}_s\,(0)\right)\\&\overset{(4.19)}{<}\left(1+CE_k^{2\beta}\right)\left(\mathbf{e}_{T_k}\left(\mathrm{B}_s\,(0)\right)-3c_2E_k\right)\\&\leq\mathbf{e}_{T_k}\left(\mathrm{B}_s\,(0)\right)-2c_2E_k.\end{aligned} \tag{4.20}$$

The last inequality holds when E_k is sufficiently small, i.e., k large enough. The rest of the proof is devoted to show that (4.20) contradicts the minimizing property of T_k. We have

$$\begin{aligned}\operatorname{Dir}\left(f_k^+, f_k^-, B_{k,3}\right) &\leq \operatorname{Dir}\left(f_k^+, f_k^-, B_{k,3}\cap K_k\right) + \operatorname{Dir}\left(f_k^+, f_k^-, B_{k,3}\setminus K_k\right)\\ &\overset{(4.20)}{\leq} \mathbf{e}_{T_k}\left(\mathrm{B}_3\left(0\right)\right) - 2c_2E_k + \operatorname{Dir}\left(f_k^+, f_k^-, B_{k,3}\setminus K_k\right)\\ &\leq \mathbf{e}_{T_k}\left(\mathrm{B}_3\left(0\right)\right) - 2c_2E_k + \frac{C}{2}E_k^{1-2\beta+2\beta},\end{aligned} \tag{4.21}$$

where in the last inequality we use the fact that $\operatorname{Lip}\left(f_k^{\pm}\right) \leq CE_k^{\beta}$ and $|\mathrm{B}_3\left(0\right)\setminus K_k| \leq CE_k^{1-2\beta}$. Now we define $\left(g_k^+, g_k^-\right)$ as $g_k^{\pm} := E_k^{-\frac{1}{2}}f_k^{\pm}$ which also collapses at the interface $\left(\gamma_k, E_k^{-\frac{1}{2}}Q^\star\left[\!\left[\psi_k\right]\!\right]\right)$. Then if we take γ to be the plane $\{x_m = 0\}\subset\pi_0$, by (4.18), we obtain the following convergences

$$\gamma_k \xrightarrow{C^1} \gamma, \quad \psi_k \xrightarrow{C^1} 0. \tag{4.22}$$

We now want to do an argument based on the interpolation of the sequence $g_k^{\pm}$ using the Interpolation Lemma, c.f. [7, Lemma 4.9] (we mention that this theorems works line by line in higher multiplicity), but unfortunately they do not have the same interface. To overcome this difficulty, we do the following construction. For each k, we let Φ_k be a diffeomorphism which maps $\mathrm{B}_3\left(0\right)$ onto itself and $\gamma_k\cap\mathrm{B}_3\left(0\right)$ onto $\gamma\cap\mathrm{B}_3\left(0\right)$. By the C^1 convergences above for k large enough, it is not difficult to see that we can assume without loss of generality

$$\left\|\Phi_k - \mathrm{Id}\right\|_{C^1} \to 0, \quad \Phi_k\left(\partial\mathrm{B}_r\left(0\right)\right) = \partial\mathrm{B}_r\left(0\right), \quad \forall r\in[2,3].$$

Furthermore, we have that $\left\|\psi_k\circ\Phi_k^{-1}\right\|_{C^1(\gamma)} \to 0$. Now consider, for every $x = (x', x_m)\in\mathbb{R}^m$, then we define $\kappa_k\in C^1(\mathrm{B}_3\left(0\right))$ as follows $\kappa_k(x) = (\psi_k\circ\Phi_k^{-1})(x', 0) + x_m$ then we have that $\kappa_k := \psi_k\circ\Phi_k^{-1}$ on γ and $\|\kappa_k\|_{C^1(\mathrm{B}_3(0))}\to 0$. We fix a measurable selection $f_k^{\pm}(x) = \sum_i \left[\!\left[\left(f_k^{\pm}\right)_i(x)\right]\!\right]$, and we set

$$\hat{g}_k^{\pm} := \left(g_k^{\pm}\circ\Phi_k^{-1}\right)\oplus(-\kappa_k),$$

thus $\left(\hat{g}_k^+, \hat{g}_k^-\right)$ are $\left(Q - \frac{Q^\star}{2}\right)$-valued maps which collapses at the same interface $\left(\gamma, Q^\star\left[\!\left[0\right]\!\right]\right)$ and by straightforward computations

$$\operatorname{Dir}\left(\hat{g}_k^{\pm}, \Phi_k^{-1}(A)\cap B_{k,3}^{\pm}\right) = (1+o(1))\left(\operatorname{Dir}\left(g_k^+, A\cap B_{k,3}^{\pm}\right) + \operatorname{Dir}\left(g_k^-, A\cap B_{k,3}^{\pm}\right)\right) + o(1), \tag{4.23}$$

for all measurable $A\subset\mathrm{B}_3\left(0\right)$ and $o(1)$ is independent of the set A. From the very definition of $g_k^{\pm}$ and (4.21), we conclude that the Dirichlet energy of $\left(\hat{g}_k^+, \hat{g}_k^-\right)$ is uniformly bounded. From this bound and (4.22) we may apply the compactness theorem, see [7, Theorem 4.8], we can find a not relabeled subsequence and a $\left(Q - \frac{Q^\star}{2}\right)$-valued map $\left(g^+, g^-\right)$ with interface $\left(\gamma, Q^\star\left[\!\left[0\right]\!\right]\right)$ such that $\left\|\mathcal{G}\left(\hat{g}_k^{\pm}, g^{\pm}\right)\right\|_{L^2\left(\mathrm{B}_3^{\pm}(0)\right)} \to 0$ and

$$\operatorname{Dir}\left(g^+, g^-\right) \leq \liminf_{k\to\infty}\operatorname{Dir}\left(\hat{g}_k^+, \hat{g}_k^-\right) \overset{(4.23)}{=} \liminf_{k\to\infty}\operatorname{Dir}\left(g_k^+, g_k^-\right).$$

Moreover, up to extracting a subsequence, we can assume that $\left|D\hat{g}_k^{\pm}\right| \to G^{\pm}$ weakly in $L^2\left(\mathrm{B}_3\left(0\right)\right)$. Once can then easily check, see for instance the proof of [11, Proposition 4.3], that

$$\left|Dg^{\pm}\right| \leq G^{\pm}, \text{ a.e. in } \mathrm{B}_3\left(0\right).$$

In particular, since $E_k \to 0$ and it bounds the size of the bad set, we have $|\mathrm{B}_3(0) \setminus K_k| \to 0$, hence for every $s \in (2,3)$:

$$\begin{aligned}
\mathrm{Dir}\left(g^{\pm}, \mathrm{B}_s^{\pm}(0)\right) &\le \liminf_{k\to\infty} \int_{\mathrm{B}_s^{\pm}(0)\cap K_k} \left(G^{\pm}\right)^2 \\
&\le \liminf_{k\to\infty} \mathrm{Dir}\left(\hat{g}_k^{\pm}, \mathrm{B}_s^{\pm}(0) \cap K_k\right) \\
&\overset{(4.23)}{\le} \liminf_{k\to\infty} \mathrm{Dir}\left(g_k^{\pm}, \mathrm{B}_s^{\pm}(0) \cap \Phi_k^{-1}(K_k)\right) \\
&= \liminf_{k\to\infty} \mathrm{Dir}\left(g_k^{\pm}, \mathrm{B}_s^{\pm}(0) \cap K_k\right).
\end{aligned} \tag{4.24}$$

Let $\varepsilon > 0$ be a small parameter to be chosen later, we apply [7, Lemma 5.8] to $(g^+, g^-)|_{\mathrm{B}_3(0)}$ with such an ε to produce a $(Q - \frac{Q^\star}{2})$-Lipschitz multivalued function $(g_\varepsilon^+, g_\varepsilon^-)$ satisfying:

$$\int_{\mathrm{B}_3^{\pm}(0)} \mathcal{G}\left(g^{\pm}, g_\varepsilon^{\pm}\right)^2 + \int_{\mathrm{B}_3^{\pm}(0)} \left(\left|Dg^{\pm}\right| - \left|Dg_\varepsilon^{\pm}\right|\right)^2 + \int_{\mathrm{B}_3^{\pm}(0)} \left|D\left(\boldsymbol{\eta} \circ g^{\pm}\right) - D\left(\boldsymbol{\eta} \circ g_\varepsilon^{\pm}\right)\right|^2 \le \varepsilon, \tag{4.25}$$

$$\int_{\partial \mathrm{B}_3^{\pm}(0)} \mathcal{G}\left(g^{\pm}, g_\varepsilon^{\pm}\right)^2 + \int_{\partial \mathrm{B}_3^{\pm}(0)} \left(\left|Dg^{\pm}\right| - \left|Dg_\varepsilon^{\pm}\right|\right)^2 \le \varepsilon.$$

Additionally, we would like to interpolate without increasing too much Dirichlet energy in the transition region. To solve this problem, let us define the Radon measures

$$\mu_k(A) = \int_{A\cap \mathrm{B}_3^+(0)} \left|D\hat{g}_k^+\right|^2 + \int_{A\cap \mathrm{B}_3^-(0)} \left|D\hat{g}_k^-\right|^2, \quad A \subset \mathrm{B}_3(0).$$

It is easy to check using (4.21) that $\mu_k(A) \le C$ where C is independent of k and A. So, up to a subsequence, we can assume that $\mu_k \overset{*}{\rightharpoonup} \mu$ for some Radon measure μ. We now choose $r \in (5/2, 3)$ and a subsequence, not relabeled, such that

(A) $\mu\left(\partial \mathrm{B}_r(0)\right) = 0$,

(B) $\mathbf{M}\left(\left\langle T_k - \left(\mathbf{G}_{f_k^+} + \mathbf{G}_{f_k^-}\right), |\mathbf{p}|, r\right\rangle\right) \le C E_k^{1-2\beta}$, where the map $|\mathbf{p}|$ is given by $\pi_0 \times \pi_0^\perp \ni (x,y) \mapsto |x|$.

Indeed, by standard measure theory arguments, (A) is true for all but countably many radii while (B) can be obtained from the estimate (4.12) through the slicing theory for currents. In particular, by (A) and the properties of weak convergence of measures, we have

$$\begin{aligned}
&\limsup_{s\to r} \limsup_{k\to\infty} \left[\int_{\mathrm{B}_r^+(0)\setminus \mathrm{B}_s^+(0)} \left|D\hat{g}_k^+\right|^2 + \int_{A\cap \mathrm{B}_r^-(0)\setminus \mathrm{B}_s^-(0)} \left|D\hat{g}_k^-\right|^2\right] \\
&\qquad \le \limsup_{s\to r} \mu\left(\overline{\mathrm{B}_r(0)} \setminus \mathrm{B}_s(0)\right) = 0.
\end{aligned}$$

Hence, given $r \in (5/2, 3)$ satisfying (A) and (B) above, we can now choose $s \in (5/2, 3)$ such that

$$\limsup_{k\to\infty} \int_{\mathrm{B}_r^+(0)\setminus \mathrm{B}_s^+(0)} \left|D\hat{g}_k^+\right|^2 + \int_{\mathrm{B}_r^-(0)\setminus \mathrm{B}_s^-(0)} \left|D\hat{g}_k^-\right|^2 \le \frac{c_2}{3}. \tag{4.26}$$

Finally, as aforementioned, we interpolate the pairs $\left(\hat{g}_k^+, \hat{g}_k^-\right)$ and $\left(g_\varepsilon^+, g_\varepsilon^-\right)$ which we can do now because all of them have the same interface $(\gamma, Q^\star [\![0]\!])$ and we have control on their Dirichlet energy. We finally apply, for each k, the interpolation lemma to connect the functions $\left(\hat{g}_k^+, \hat{g}_k^-\right)$ and $\left(g_\varepsilon^+, g_\varepsilon^-\right)$ on the annulus $\mathrm{B}_r(0) \setminus \mathrm{B}_s(0)$. This gives sets $\overline{\mathrm{B}_s(0)} \subset V_{\lambda,\varepsilon}^k \subset W_{\lambda,\varepsilon}^k \subset \mathrm{B}_r(0)$ and a $\left(Q - \frac{Q^\star}{2}\right)$ valued interpolation map $\left(\zeta_{k,\varepsilon}^+, \zeta_{k,\varepsilon}^-\right)$ with

$$\begin{aligned}\int_{\left(W_{\lambda,\varepsilon}^k\right)^\pm \setminus V_{\lambda,\varepsilon}^k} \left|D\zeta_{k,\varepsilon}^\pm\right|^2 &\leq C\lambda \int_{\left(W_{\lambda,\varepsilon}^k\right)^\pm \setminus V_{\lambda,\varepsilon}^k} \left(\left|D\hat{g}_k^\pm\right|^2 + \left|Dg_\varepsilon^\pm\right|^2\right) + \frac{C}{\lambda} \int_{\left(W_{\lambda,\varepsilon}^k\right)^\pm \setminus V_{\lambda,\varepsilon}^k} \mathcal{G}\left(\hat{g}_k^\pm, g_\varepsilon^\pm\right)^2 \\ &\leq C\lambda \int_{\left(W_{\lambda,\varepsilon}^k\right)^\pm \setminus V_{\lambda,\varepsilon}^k} \left(\left|D\hat{g}_k^\pm\right|^2 + \left|Dg_\varepsilon^\pm\right|^2\right) + \frac{C}{\lambda} \int_{\left(W_{\lambda,\varepsilon}^k\right)^\pm \setminus V_{\lambda,\varepsilon}^k} \left(\mathcal{G}\left(\hat{g}_k^\pm, g^\pm\right)^2 + \mathcal{G}\left(g^\pm, g_\varepsilon^\pm\right)^2\right).\end{aligned}$$

Hence

$$\limsup_{\lambda \to 0} \limsup_{\varepsilon \to 0} \limsup_{k \to \infty} \int_{\left(W_{\lambda,\varepsilon}^k\right)^\pm \setminus V_{\lambda,\varepsilon}^k} \left|D\zeta_{k,\varepsilon}^\pm\right|^2 = 0.$$

Thus we can find $\lambda, \varepsilon > 0$ sufficiently small such that

$$\limsup_{k \to \infty} \int_{\left(W_{\lambda,\varepsilon}^k\right)^\pm \setminus V_{\lambda,\varepsilon}^k} \left|D\zeta_{k,\varepsilon}^\pm\right|^2 < \frac{c_2}{3}. \tag{4.27}$$

Moreover, up to further reduce ε, by (4.25) we can also assume that

$$\int_{\mathrm{B}_r(0)^\pm} \left|Dg_\varepsilon^\pm\right|^2 \leq \int_{\mathrm{B}_r(0)^\pm} \left|Dg^\pm\right|^2 + \frac{c_2}{6}. \tag{4.28}$$

Now that we have interpolated the functions without adding too much energy, we define the $(Q - \frac{Q^\star}{2})$-Lipschitz function on $\mathrm{B}_r(0)$ with interface $(\gamma, Q^\star [\![0]\!])$ by

$$\hat{h}_{k,\lambda,\varepsilon}^\pm := \begin{cases} \hat{g}_k^\pm & \text{on } \mathrm{B}_r(0) \setminus \left(W_{\lambda,\varepsilon}^k\right)^\pm, \\ \zeta_{k,\varepsilon}^\pm & \text{on } \left(W_{\lambda,\varepsilon}^k\right)^\pm \setminus V_{\lambda,\varepsilon}^k \quad, \\ g_\varepsilon^\pm & \text{on } \left(V_{\lambda,\varepsilon}^k\right)^\pm. \end{cases}$$

Let us then consider $\left(Q - \frac{Q^\star}{2}\right)$-valued map $\left(h_{k,\lambda,\varepsilon}^+, h_{k,\lambda,\varepsilon}^-\right)$ defined on $B_{k,3}^\pm$ with interface $(\gamma_k, Q^\star [\![\psi_k]\!])$ given by

$$h_{k,\lambda,\varepsilon}^\pm := \left(\hat{h}_{k,\lambda,\varepsilon}^\pm \circ \Phi_k\right) \oplus \left(\kappa_k \circ \Phi_k\right),$$

which satisfies

$$\begin{aligned}\liminf_{k\to\infty}\mathrm{Dir}\left(h^{+}_{k,\lambda,\varepsilon},h^{-}_{k,\lambda,\varepsilon},B_{k,r}\right) &= \liminf_{k\to\infty}\mathrm{Dir}\left(\hat h^{+}_{k,\lambda,\varepsilon},\hat h^{-}_{k,\lambda,\varepsilon},\mathrm{B}_r(0)\right)\\ &\le \mathrm{Dir}\left(g^{+}_{\varepsilon},g^{-}_{\varepsilon},\mathrm{B}_r(0)\right)\\ &+\limsup_{k\to\infty}\mathrm{Dir}\left(\zeta^{+}_{k,\varepsilon},\zeta^{-}_{k,\varepsilon},\left(W^{k}_{\lambda,\varepsilon}\right)\setminus V^{k}_{\lambda,\varepsilon}\right)\\ &+\limsup_{k\to\infty}\mathrm{Dir}\left(\hat g^{+}_{k},\hat g^{-}_{k},\mathrm{B}_r(0)\setminus\mathrm{B}_s(0)\right)\\ &\overset{(4.26),(4.27),(4.28)}{<}\mathrm{Dir}\left(g^{+},g^{-},\mathrm{B}_r(0)\right)+c_2\\ &\overset{(4.24)}{<}\liminf_{k\to\infty}\left(\mathrm{Dir}\left(g^{+}_{k},g^{-}_{k},\mathrm{B}_r(0)\cap K_k\right)\right)+2c_2.\end{aligned}\tag{4.29}$$

Let us consider the function $w^{\pm}_{k,\lambda,\varepsilon}(x):=E_k^{1/2}h^{\pm}_{k,\lambda,\varepsilon}(x)$. Observe that, by the constructions of $\hat g^{\pm}_k$, $w^{\pm}_{k,\lambda,\varepsilon}|_{\partial\mathrm{B}_r(0)}=f^{\pm}_k|_{\partial\mathrm{B}_r(0)}$ and $\mathrm{Lip}\left(w^{\pm}_{k,\lambda,\varepsilon}\right)\le CE_k^{\beta}$. We are now ready to construct a sequence of competitors one for each T_k which for large k will contradict the almost minimality of the sequence T_k. First of all, by the isoperimetric inequality, [16, Section 4.4.2], there is a current S_k such that

$$\partial S_{k,r}:=\left\langle T_k-\left(\mathbf{G}_{f^{+}_k}+\mathbf{G}_{f^{-}_k}\right),|\mathbf{p}|,r\right\rangle\quad\text{and}\quad\mathbf{M}\left(S_{k,r}\right)\le C\left(E_k^{1-2\beta}\right)^{\frac{m}{m-1}}\overset{(\beta<\frac{1}{2m})}{=}o\left(E_k\right).\tag{4.30}$$

Let $Z_{k,r}:=\mathbf{G}_{w^{+}_k}\llcorner\mathbf{C}(0,r)+\mathbf{G}_{w^{-}_k}\llcorner\mathbf{C}(0,r)+S_{k,r}$. Since $w^{\pm}_{k,\lambda,\varepsilon}|_{\partial\mathrm{B}_r(0)}=f^{\pm}_k|_{\partial\mathrm{B}_r(0)}$, we can see that the boundary of Z_k matches that of $T_k\llcorner\mathbf{C}(0,r)$, thus it is an admissible competitor and will furnishes the desired contradiction. To that end, first we compare the Dirichlet energies of $w^{+}_{k,\lambda,\varepsilon}$ and f^{+}_k. To begin with this comparison, we note that, up to a subsequence not relabeled, it holds

$$\begin{aligned}\mathrm{Dir}(w^{+}_{k,\lambda,\varepsilon},w^{-}_{k,\lambda,\varepsilon},B_{k,r}) &= E_k\mathrm{Dir}(h^{+}_{k,\lambda,\varepsilon},h^{-}_{k,\lambda,\varepsilon},B_{k,r})\\ &\overset{(4.29)}{<}E_k\mathrm{Dir}(g^{+}_k,g^{-}_k,\mathrm{B}_r(0)\cap K_k)+c_2E_k\\ &=\mathrm{Dir}(f^{+}_k,f^{-}_k,B_{k,r}\cap K_k)+2c_2E_k,\end{aligned}\tag{4.31}$$

for k large enough. In addition, the latter inequality combined with the second inequality in (4.20) implies for k large enough that

$$\mathrm{Dir}(w^{+}_{k,\lambda,\varepsilon},w^{-}_{k,\lambda,\varepsilon},B_{k,r})<\mathbf{e}_{T_k}(\mathrm{B}_r(0))-c_2E_k+o(E_k).\tag{4.32}$$

Finally, we estimate

$$\begin{aligned}\mathbf{M}(Z_k)-\mathbf{M}(T_k) &\le \mathbf{M}(\mathbf{G}_{w^{+}_k}\llcorner\mathbf{C}(0,r))+\mathbf{M}(\mathbf{G}_{w^{-}_k}\llcorner\mathbf{C}(0,r))+\mathbf{M}(S_k)-\mathbf{M}(T_k)\\ &\overset{Taylor}{\le} Q|B^{+}_{k,r}|+(Q-Q^{\star})|B^{-}_{k,r}|+\mathrm{Dir}(w^{+}_{k,\lambda,\varepsilon},w^{-}_{k,\lambda,\varepsilon},B_{k,r})+o(E_k)-\mathbf{M}(T_k)\\ &\le Q|B^{+}_{k,r}|+(Q-Q^{\star})|B^{-}_{k,r}|+\mathrm{Dir}(w^{+}_{k,\lambda,\varepsilon},w^{-}_{k,\lambda,\varepsilon},B_{k,r})+o(E_k)\\ &-Q|B^{+}_{k,r}|-(Q-Q^{\star})|B^{-}_{k,r}|-\mathbf{e}_{T_k}(\mathrm{B}_r(0))\\ &\overset{(4.32)}{<}-c_2E_k+o(E_k),\end{aligned}\tag{4.33}$$

the expression is negative when k is large enough. In particular Z_k is a competitor with less mass than T_k and this completes the proof of the first part of the theorem.

Proof of (4.15)**,** (4.16) **and** (4.17)**.** As in the first part, we argue by contradiction assuming (i), (ii), and (iii) becomes

(iii)' The E_k^β-Lipschitz approximations (f_k^+, f_k^-) fail to satisfy one among the estimates (4.15), (4.16) and (4.17) for any choice of the function κ.

We use the same notations of the previous step for $\psi_k, g_k^\pm, \Phi_k, \kappa_k, \hat{g}_k^\pm$ and $g^\pm$. Therefore, we now claim that

(A) The convergence of $\hat{g}_k^\pm$ to $g^\pm$ is strong in $W^{1,2}(\mathrm{B}_{5/2}(0))$, namely

$$\lim_{k\to\infty} \mathrm{Dir}(\hat{g}_k^+, \hat{g}_k^-, \mathrm{B}_{5/2}(0)) = \mathrm{Dir}(g^+, g^-, \mathrm{B}_{5/2}(0)) .$$

(B) (g^+, g^-) is a $(Q - \frac{Q^\star}{2})$-minimizer in $\mathrm{B}_{5/2}(0)$.

Recall that, by Theorem 4.11 and the construction, (g^+, g^-) collapses at the interface $(\gamma, Q^\star [\![0]\!])$, thus provided we assume that (A) and (B) are proved, from Theorem 4.4 we would then infer the existence of a classical harmonic function $\hat{h}$ on $\mathrm{B}_{5/2}(0)$ which vanishes identically on $\{x_m = 0\}$ such that $g^+ = Q\left[\!\left[\hat{h}\right]\!\right]$ and $g^- = (Q - Q^\star)\left[\!\left[\hat{h}\right]\!\right]$. If we set $h_k = E_k^{1/2}\hat{h}$, the following hold

$$\int_{B^+_{k,5/2}} \mathcal{G}(f_k^+, Q [\![h_k]\!])^2 + \int_{B^+_{k,5/2}} \Big(|Df_k^+| - \sqrt{Q}|Dh_k|\Big)^2 = \ o(E_k),$$
$$\int_{B^-_{k,5/2}} \mathcal{G}(f_k^-, (Q - Q^\star) [\![h_k]\!])^2 + \int_{B^-_{k,5/2}} \Big(|Df_k^-| - \sqrt{(Q - Q^\star)}|Dh_k|\Big)^2 = o(E_k),$$
$$\int_{B^\pm_{k,5/2}} \big|D(\boldsymbol{\eta} \circ f_k^\pm) - Dh_k\big|^2 = o(E_k).$$

But these estimates are incompatible with (iii)' above. Hence, at least one between (A) and (B) needs to fail. As in the previous section we will use this to contradict the minimality of T_k. Note that in both cases there exists a $(Q - \frac{Q^\star}{2})$-valued function $(\bar{g}^+, \bar{g}^-)$ with interface $(\gamma, Q^\star [\![0]\!])$, $\gamma = \{x_m = 0\}$, and a positive constant $c_3 > 0$, such that

$$\mathrm{Dir}(\bar{g}^+, \bar{g}^-, \mathrm{B}_s(0)) \leq \liminf_{k\to\infty} \mathrm{Dir}(\hat{g}^+, \hat{g}^-, \mathrm{B}_s(0)) - 2c_3 \tag{4.34}$$

for all $s \in (5/2, 3)$. Indeed this is true with $(\bar{g}^+, \bar{g}^-) = (g^+, g^-)$ if (A) fails, while if (B) fails we choose $(\bar{g}^+, \bar{g}^-)$ to be a $(Q - \frac{Q^\star}{2})$-minimizer with boundary data $g^\pm$ on $\partial\mathrm{B}_{5/2}(0)$ extended to be equal to $g^\pm$ on $\mathrm{B}_3(0) \setminus \mathrm{B}_{5/2}(0)$. We can now follow the exactly the same argument as in the previous step to find a radius $r \in (5/2, 3)$ and functions $\hat{h}_k^\pm$ such that

$$\mathbf{M}(\langle T_k - (\mathbf{G}_{f_k^+} + \mathbf{G}_{f_k^-}), |\mathbf{p}|, r\rangle) \leq C E_k^{1-2\beta}$$

and, arguing as we have done for (4.29),

$$\begin{aligned}\liminf_{k\to\infty} \mathrm{Dir}(h^+, h^-, B_{k,r}) &\le \mathrm{Dir}(\bar{g}^+, \bar{g}^-, \mathrm{B}_r(0)) + c_3 \\ &\le \liminf_{k\to\infty} \mathrm{Dir}(g^+, g^-, B_{k,r}) - c_3.\end{aligned} \tag{4.35}$$

Defining $w_k^\pm$ as above, we again observe that $w_k^\pm|_{\partial \mathrm{B}_r^\pm(0)} = f_k^\pm|_{\partial \mathrm{B}_r^\pm(0)}$. We then construct the same competitor currents to test the minimality of T_k. First we consider a current S_k supported in Σ_k such that

$$\partial S_k = \langle T_k - (\mathbf{G}_{f_k^+} + \mathbf{G}_{f_k^-}), |\mathbf{p}|, r\rangle \text{ and } \mathbf{M}(S_k) \le C(E_k^{1-2\beta})^{\frac{m}{m-1}} = o(E_k)\,, \tag{4.36}$$

where we again used $\beta < \frac{1}{4m}$. Then we define, as before, $Z_k := \mathbf{G}_{w_k^+} \llcorner \mathbf{C}(0,r) + \mathbf{G}_{w_k^-} \llcorner \mathbf{C}(0,r) + S_k$, for which the minimality condition guarantees

$$\mathbf{M}(Z_k) \ge \mathbf{M}(T_k \llcorner \mathbf{C}(0,r))\,.$$

Since we proved the first part of the theorem, we use it to show that

$$\mathbf{e}_{T_k}(\mathrm{B}_r(0)) = \mathrm{Dir}(f_k^+, \mathrm{B}_r^+(0)) + \mathrm{Dir}(f_k^-, \mathrm{B}_r^-(0)) + O(\eta_k E_k)\,.$$

Observe that now we can choose $\eta_k \to 0$ as $k \to \infty$. Arguing as in (4.31) and relying on (4.35) we also have

$$\mathrm{Dir}(w_k^+, w_k^-, \mathrm{B}_r(0)) \le \mathrm{Dir}(f_k^+, f_k^-, \mathrm{B}_r(0)) - c_3 E_k + o(E_k)\,.$$

Accordingly, the latter inequality combined with (4.20) implies

$$\mathrm{Dir}(w_k^+, w_k^-, B_{k,r}) < \mathbf{e}_{T_k}(\mathrm{B}_r(0)) - c_3 E_k.$$

As before, see (4.33), we complete the proof. □

5 Superlinear decays and Lusin type strong Lipschitz approximations

The approximation furnished in the last subsection, Theorem 4.11, have sublinear exponents bounding the size of the bad set among other important quantities. In the construction of the center manifold, in order to derive good properties of it we will need superlinear decays in some of the estimates of Theorem 4.11. To that end, we need to improve our approximation. In fact, we need an accurate height bound and harmonic approximations to achieve a satisfactory excess decay and therefore provide stronger Lipschitz approximations that will appear in Theorem 5.8, c.f. [7, Chapter 6], with the subtle exponents estimating the error of such approximations. So, as a first step we state the height bound of the current T.

5.1 Height bound

Lemma 5.1 (Height bound, Lemma 10.4, [8]). *Let T, $\mathbf{C}(p,4r)$, Γ and $\pi_0 := \mathbb{R}^m \times \{0\}$ be as in Assumption 3 with $C_0 = 0$. Then, there exist positive constants $\varepsilon_h = \varepsilon_h(Q, Q^\star, m, n)$ and $C_h = C_h(Q, Q^\star, m, n)$ such that, if $\mathbf{E}(T, \mathbf{C}(p,4r)) + \mathbf{A} \leq \varepsilon_h$, then*

$$\mathbf{h}(T, \mathbf{C}(p,2r), \pi_0) \leq C_h(r^{-1}\mathbf{E}(T, \mathbf{C}(p,4r)) + \mathbf{A})^{\frac{1}{2}} r^{\frac{3}{2}}.$$

5.2 Improved excess estimate

We will follow a very well known process that using the height bound provided by Lemma 5.1 allows to obtain our proof of the improved excess decay. To achieve this result we will firstly prove a milder decay in Lemma 5.2 and then iterate it to reach the needed excess superlinear estimate. To prove this milder statement which will be stated for the modified excess function introduced in Definition 4.9, we will reduce this milder decay of the current T in some steps until we can rely on a similar decay for harmonic functions, this reduction will be possible thanks to Theorem 4.12.

Lemma 5.2 (Milder excess decay). *Let T and Γ be as in Assumptions 2 with $C_0 = 0$, $p \in \Gamma \cap U$ is a two-sided collapsed point where U is the neighborhood in Definition 3.17. Then, for every $q \in U \cap \Gamma$ and $\varepsilon > 0$, there is an $\varepsilon_0 = \varepsilon_0(\varepsilon, Q, Q^\star, m, n) > 0$ (assume that $\varepsilon_h \leq \varepsilon_0^2$) and a $M_0 = M_0(\varepsilon, Q, Q^\star, m, n) > 0$ with the following property. We set $\theta(\sigma) := \max\{\mathbf{E}^\flat(T, \mathbf{B}(q,\sigma)), M_0\mathbf{A}^2\sigma^2\}$, and assume*

$$\mathbf{A}^2\sigma^2 + E := \|A_\Gamma\|^2\sigma^2 + \mathbf{E}^\flat(T, \mathbf{B}(q,4\sigma)) < \varepsilon_0, \tag{5.1}$$

$$\|T\|(\mathbf{B}(q,4\sigma)) \leq \left(\Theta^m(T,p) + \frac{1}{4}\right)\omega_m(4\sigma)^m. \tag{5.2}$$

Then we have

$$\theta(\sigma) \leq \max\{2^{-4+4\varepsilon}\theta(4\sigma), 2^{-2+2\varepsilon}\theta(2\sigma)\}. \tag{5.3}$$

This milder statement is not enough for our purposes, since the excess are considered in balls with the same center q. However, it facilitates a lot the proof of the improved excess decay which we enunciate below.

Theorem 5.3 (Improved excess decay and height bound). *Let T and Γ be as in Assumption 2 with $C_0 = 0$. If $p \in \Gamma \cap U$ is a two-sided collapsed point with density $\Theta(T,p) = Q - \frac{Q^\star}{2}$, U is a neighborhood of Definition 3.17, then there exists $r > 0$ such that $\mathbf{B}(p,r) \subset U$, for all $q \in \mathbf{B}(p,r) \cap U$ there exists a m-dimensional plane $\pi(q)$ which $T_q\Gamma \subset \pi(q)$, and for all $\varepsilon > 0$ there is a constant $C = C(m, n, Q^\star, Q, \varepsilon) > 0$ with*

$$\mathbf{E}^\flat(T, \mathbf{B}(q,\rho)) \leq \mathbf{E}^\flat(T, \mathbf{B}(q,\rho), \pi(q)) \leq C\left(\frac{\rho}{r}\right)^{2-2\varepsilon} \mathbf{E}^\flat(T, \mathbf{B}(p,2r)) + C\rho^{2-2\varepsilon}r^{2\varepsilon}\mathbf{A}^2, \tag{5.4}$$

for all $\rho \in (0,r)$. Moreover, if we take $\rho \in (0, \frac{r}{2\sqrt{2}})$, then

$$\mathbf{h}(T, \mathbf{B}(q,\rho), \pi(q)) \leq C(r^{-1}\mathbf{E}^\flat(T, \mathbf{B}(p,2r)) + \mathbf{A})^{\frac{1}{2}}\rho^{\frac{3}{2}}, \ \forall q \in \Gamma \cap \mathbf{B}(p,r). \tag{5.5}$$

Remark 5.4. We announce that, in Theorem 5.5, we prove that $\pi(q)$ is in fact the support of the unique tangent cone to T at q.

We begin with the proof of the Lemma 5.2 which will be used to prove the Theorem 5.3.

Proof of Lemma 5.2. Without loss of generality by scaling, translating and rotating, we can assume $\sigma = 1$, $q = 0$, $\mathbf{E}^\flat(T, \mathbf{B}(0,2)) = \mathbf{E}(T, \mathbf{B}(0,2)\,, \pi_0)$, where $\pi_0 = \mathbb{R}^m \times \{0\}$, and $T_0\Gamma = \mathbb{R}^{m-1} \times \{0\}$. We begin assuming

$$\mathbf{E}^\flat(T, \mathbf{B}(0,2)) \geq 2^{-m} M_0 \mathbf{A}^2 \quad \text{and} \quad \mathbf{E}^\flat(T, \mathbf{B}(0,2)) \geq 2^{-4-m} \mathbf{E}^\flat(T, \mathbf{B}(0,4)). \tag{5.6}$$

Indeed, note that

$$\theta(1) = \max\{M_0\mathbf{A}^2, \mathbf{E}^\flat(T, \mathbf{B}(0,1))\} \leq \max\{M_0\mathbf{A}^2, 2^m \mathbf{E}^\flat(T, \mathbf{B}(0,2))\}\,.$$

So, if the first inequality in (5.6) fails, by the latter inequality, we have

$$\theta(1) \leq M_0\mathbf{A}^2 = 2^{-2}(2^2 M_0 \mathbf{A}^2) \leq 2^{-2}\theta(2)\,,$$

whereas, if the second inequality in (5.6) fails, then

$$\theta(1) \leq \max\{M_0\mathbf{A}^2, 2^{-4}\mathbf{E}^\flat(T, \mathbf{B}(0,4))\} \leq 2^{-4}\theta(4)\,.$$

Hence in both cases the conclusion should hold. Reiterating, under assumption (5.6), we need to show the decay estimate:

$$\mathbf{E}^\flat(T, \mathbf{B}(0,1)) \leq 2^{2\varepsilon-2}\mathbf{E}^\flat(T, \mathbf{B}(0,2))\,. \tag{5.7}$$

Let us now fix a positive $\eta < 1$, to be chosen sufficiently small later, and consider the cylinder $U_\eta := \mathrm{B}_{4-\eta}(0, \pi_0) + \mathrm{B}_{\sqrt{\eta}}(0, \pi_0^\perp)$, which by abuse of notation we denote by $B_{4-\eta} \times B^n_{\sqrt{\eta}}$. If $\varepsilon_0 > 0$ is sufficiently small, we claim that

$$\mathrm{spt}(T) \cap \partial U_\eta \subset \partial B_{4-\eta} \times B^n_{\sqrt{\eta}} \tag{5.8}$$

$$\mathbf{B}(0, 4-\eta) \cap \mathrm{spt}(T) \subset U_\eta\,. \tag{5.9}$$

Otherwise, arguing by contradiction, we would have a sequence of currents T_k satisfying the assumptions of the theorem with $\varepsilon_0 = \frac{1}{k}$, but violating either (5.8) or (5.9). Then T_k would converge, in the sense of currents, to a current T_∞ that is area minimizing whose excess w.r.t. π_0 is identically zero so its support is contained in the plane π_0, $\partial T_\infty = Q^\star [\![T_q\Gamma]\!]$. Thus we are in position to apply the Constancy Lemma, [16, 4.1.17], to asserts

$$T_\infty := Q' [\![B_4^+]\!] + (Q' - Q^\star) [\![B_4^-]\!]\ ,$$

where $B_4^\pm = B_4(0, \pi_0) \cap \{\pm x_m > 0\}$ and $Q' \geq Q^\star$ is a positive integer. Since $\partial T_k = Q^\star [\![\mathbb{R}^{m-1} \times \{0\}]\!]$, $\forall k \in \mathbb{N}$, we can use the area-minimizing property to obtain an uniform bound on $\|T_k\|(B_4)$ and thus be in place to apply [22, Theorem 7.2, Chapter 6] which says that, since T_k converge to T_∞ in the sense of currents, the supports of T_k converge to either $\overline{B}_4$ in case $Q' > Q^\star$ or $\overline{B}_4^+$ otherwise, i.e. if $Q' = Q^\star$, in the Hausdorff sense in every compact subset of $\mathbf{B}(0,4)$. This is a contradiction because the following both inequalities hold

$$\mathrm{dist}_H(\overline{\mathbf{B}(0,4-\eta)\setminus U_\eta}, \mathbf{B}(0,4)) > 0 \text{ and } \mathrm{dist}_H(\partial U_\eta \setminus (\partial B_{4-\eta}\times B^n_{\sqrt{\eta}}), \mathbf{B}(0,4)) > 0.$$

We have therefore proved (5.8) and (5.9).

We now let T_0 be a tangent cone of T at 0 and $\rho_k \to 0^+$ a sequence such that $T_{0,\rho_k} \to T_0$. By a standard argument using the Constancy Lemma, we know that

$$\mathbf{p}_{\pi_0\sharp} T_0 = Q' [\![\pi_0]\!] + (Q' - Q^\star) [\![\pi_0]\!] , \tag{5.10}$$

for some natural number Q'. By the lower semicontinuity of the total variation and (5.2), we further notice that we necessarily have $\|\mathbf{p}_{\pi_0\sharp} T_0\|(\mathbf{B}(0,4)) \le (Q - \frac{2Q^\star - 1}{4})\omega_m 4^m$. Hence, by the monotonicity formula

$$\Theta^m(\mathbf{p}_{\pi_0\sharp} T_0, 0) \le Q - \frac{2Q^\star - 1}{4}. \tag{5.11}$$

On the other hand, the upper semicontinuity of the density w.r.t. the convergence of area-minimizing currents [22, Chapter 7, Section 3, Eq. (12)] and the fact that p is a two-sided collapsed point allow us to conclude

$$\Theta^m(\mathbf{p}_{\pi_0\sharp} T_0, 0) \ge \limsup \Theta^m(\mathbf{p}_{\pi_0\sharp} T_{0,\rho_k}, 0) = Q - \frac{Q^\star}{2}. \tag{5.12}$$

By equations (5.11), (5.12), and (5.10), we have

$$Q - \frac{2Q^\star - 1}{4} \ge \Theta^m(\mathbf{p}_{\pi_0\sharp} T_0, 0) = Q' - \frac{Q^\star}{2} \ge Q - \frac{Q^\star}{2},$$

since Q' is an integer it turns out that $Q' = Q$. By (5.9), we can straightforwardly check that

$$\mathrm{dist}_H(\mathrm{spt}(T), \mathrm{spt}(T_0))) < \eta,$$

which, provided η and ε_0 are small enough, leads to

$$\|T\|(\mathbf{B}(0,r)) = \|T_0\|(\mathbf{B}(0,r)) + O(\eta^{m-1}), \quad \forall r \in (1, 4 - \frac{\eta}{2}).$$

Thus, we can state the following property:

(A) the mass of T in the ball $\mathbf{B}(0,r)$ is $\left(Q - \frac{Q^\star}{2}\right)\omega_m r^m + O(\eta^{m-1})$, for any radius $1 \le r \le 4 - \frac{\eta}{2}$.

Next, let us define $S_\eta := T \llcorner U_\eta$. Observe that (5.8) and (5.9) imply:

(B) $\partial S_\eta \llcorner \mathbf{C}(0, 4-\eta) = Q^\star [\![\Gamma \cap \mathbf{C}(0,4-\eta)]\!]$;

(C) $T \llcorner \mathbf{B}(0,4-\eta) = S_\eta \llcorner \mathbf{B}(0,4-\eta)$.

Choose a plane $\overline{\pi}$ which minimizes the boundary excess, i.e., which contains $T_0\Gamma$ and $\mathbf{E}(T, \mathbf{B}(0,4), \overline{\pi}) = \mathbf{E}^\flat(T, \mathbf{B}(0,4))$. Let us observe that, since π_0 is the optimal plane for $\mathbf{E}^\flat(T, \mathbf{B}(0,2))$, we have

$$\begin{aligned} |\overline{\pi}-\pi_0|^2\|T\|(\mathbf{B}(0,2)) &= \int_{\mathbf{B}(0,2)} |\overline{\pi}-\pi_0|^2\, d\|T\| \\ &\le 2\int_{\mathbf{B}(0,2)} |\vec{T}-\pi_0|^2\, d\|T\| + 2\int_{\mathbf{B}(0,2)} |\vec{T}-\overline{\pi}|^2\, d\|T\| \\ &\le 2\cdot 2^m \omega_m \mathbf{E}^\flat(T, \mathbf{B}(0,2)) + 2\cdot 4^m \omega_m \mathbf{E}^\flat(T, \mathbf{B}(0,4)) \\ &\le C\mathbf{E}^\flat(T, \mathbf{B}(0,4)). \end{aligned} \tag{5.13}$$

Moreover,

$$\begin{aligned} \mathbf{E}(S_\eta, \mathbf{C}(0,4-\eta)) &\le \mathbf{E}(T, \mathbf{B}(0, 4-\eta/2), \pi_0) \\ &\overset{\text{triangular}}{\le} \frac{2}{\omega_m(4-\eta/2)^m}\mathbf{E}^\flat(T, \mathbf{B}(0,4)) \\ &+ \frac{2}{\omega_m(4-\frac{\eta}{2})^m}|\overline{\pi}-\pi_0|^2\|T\|(\mathbf{B}(0,4-\eta/2)) \\ &\overset{\text{(A)}}{\le} 2\mathbf{E}^\flat(T, \mathbf{B}(0,4)) + C|\overline{\pi}-\pi_0|^2\|T\|(\mathbf{B}(0,2)) \\ &\overset{(5.13)}{\le} C\mathbf{E}^\flat(T, \mathbf{B}(0,4)), \end{aligned} \tag{5.14}$$

Moreover, recalling that $\mathbf{p} : \mathbb{R}^{m+n} \to \pi_0$ is the orthogonal projection, by the Constancy Lemma, [16, 4.1.17],

(D) $\mathbf{p}_\sharp S_\eta = Q_{\mathbf{p}} [\![\Omega^+]\!] + (Q_{\mathbf{p}} - Q^\star)[\![\Omega^-]\!]$, where $Q_{\mathbf{p}}$ is a positive integer and $\Omega^\pm$ are the regions in which $\mathrm{B}_4(0,\pi_0)$ is divided by $\mathbf{p}(\Gamma)$; in particular

$$\partial [\![\Omega^+]\!] \llcorner \mathbf{C}(0,4-\eta) = -\partial [\![\Omega^-]\!] \llcorner \mathbf{C}(0,4-\eta) = \mathbf{p}_\sharp [\![\Gamma]\!] \llcorner \mathbf{C}(0,4-\eta)\,.$$

Since $S_\eta = T\llcorner U_\eta$ and $U_\eta \subset \mathbf{B}(0, 4-\eta/2)$, clearly

$$\|S_\eta\|(\mathbf{C}(0,4-\eta)) \le \|T\|(\mathbf{B}(0,4-\eta/2)). \tag{5.15}$$

Since projections do not increase mass, we obtain

$$\|S_\eta\|(\mathbf{C}(0,4-\eta)) \ge \|\mathbf{p}_\sharp S_\eta\|(\mathbf{C}(0,4-\eta)). \tag{5.16}$$

Assuming that the constant ε_0 in the assumption of the theorem is sufficiently small, we conclude that $\mathbf{p}_\sharp [\![\Gamma]\!] \llcorner \mathbf{C}(0,4-\eta)$ is close to $T_0\Gamma = \mathbb{R}^{m-1}\times\{0\}$. In particular, $|\Omega^\pm|$ is close to $|\mathrm{B}^\pm_{4-\eta}(0)|$ and thus $Q_{\mathbf{p}}|\Omega^+| + (Q_{\mathbf{p}} - Q^\star)|\Omega^-|$ is close to $(Q_{\mathbf{p}} - \frac{Q^\star}{2})\omega_m(4-\eta)^m$ too. Therefore, if ε_0 is smaller than a geometric constant, we infer from (5.16) that

$$\|S_\eta\|(\mathbf{C}(0,4-\eta)) \ge (Q_{\mathbf{p}} - \frac{2Q^\star+1}{4})\omega_m(4-\eta)^m.$$

In addition, by (A), a sufficiently small ε_0 imply

$$(Q_{\mathbf{p}} - \frac{2Q^\star+1}{4})\omega_m(4-\eta)^m \overset{(5.16)}{\leq} \|S_\eta\|(\mathbf{C}(0,4-\eta))$$
$$\overset{(5.15)}{\leq} \|T\|(\mathbf{B}(0,4-\eta/2))$$
$$\overset{(A)}{\leq} (Q - \frac{2Q^\star-1}{4})\omega_m(4-\frac{\eta}{2})^m,$$

we achieve that $Q_{\mathbf{p}} \leq Q$ provided η is chosen smaller than a geometric constant. On the other hand,

$$\|S_\eta\|(\mathbf{C}(0,4-\eta)) \leq Q_{\mathbf{p}}|\Omega^+| + (Q_{\mathbf{p}} - Q^\star)|\Omega^-| + \mathbf{E}(S_\eta, \mathbf{C}(0,4-\eta))\,.$$

Using (5.14) and the argument above, if ε_0 is sufficiently small we get $\|S_\eta\|(\mathbf{C}(0,4-\eta)) \leq (Q_{\mathbf{p}} - \frac{2Q^\star-1}{4})\omega_m(4-\eta)^m$. Recall that (C) ensures that $\|T\|(\mathbf{B}(0,4-\eta)) \leq \|S_\eta\|(\mathbf{C}(0,4-\eta))$, and, using (A), we also have $\|T\|(\mathbf{B}(0,4-\eta)) \geq (Q - \frac{2Q^\star+1}{4})(4-\eta)^m$. Thus necessarily $Q_{\mathbf{p}} \geq Q$, consequently, we have $Q_{\mathbf{p}} = Q$.

Next, since $T \llcorner \mathbf{B}(0,2) = S_\eta \llcorner \mathbf{B}(0,2)$, then

$$\mathbf{A}^2 \overset{(5.6)}{\leq} 2^m M_0^{-1}\mathbf{E}^\flat(T, \mathbf{B}(0,2))$$
$$\leq 2^m \left(\frac{4-\eta}{2}\right)^m M_0^{-1}\mathbf{E}(S_\eta, \mathbf{C}(0,4-\eta))$$
$$\overset{(5.14)}{\leq} CM_0^{-1}\mathbf{E}^\flat(T, \mathbf{B}(0,4))\,.$$

By the last inequality and $Q_{\mathbf{p}} = Q$, we finally proved that we are in position to apply Theorem 4.12 with $\beta = \frac{1}{5m}$ and a sufficiently small parameter η_* to be chosen later, provided ε_0 is sufficiently small and M_0 is sufficiently large.

Reduction to excess decay for graphs

From now on we let (u^+, u^-) and h be as in Theorem 4.12. In particular, recall that (u^+, u^-) is the E^β-approximation of Theorem 4.11 and h is a single-valued harmonic function. Moreover, denote by E the cylindrical excess $\mathbf{E}(S_\eta, \mathbf{C}(0,4-\eta))$ and record the estimates:

$$\mathbf{A}^2 \leq C_0 M_0^{-1} E \quad \text{and} \quad E \leq C_0 \mathbf{E}^\flat(T, \mathbf{B}(0,2)), \tag{5.17}$$

where C_0 is a geometric constant and the second inequality follows by combining (5.14) and (5.6). Next, define π to be the plane given by the graph of the linear function $x \mapsto (Dh(0)x, 0)$. Since, by the Schwarz reflection principle and the unique continuation for harmonic functions, we obtain that h is odd, and $h(x', 0) = 0$, so, we have that

$$\pi \supset T_0\Gamma = \mathbb{R}^{m-1} \times \{0\}.$$

Moreover, by elliptic estimates,

$$|\pi| \leq C|Dh(0)| \leq (C\mathrm{Dir}(h, \mathrm{B}_{\frac{5}{2}(4-\eta)}(0)))^{\frac{1}{2}} \overset{Thm. 4.12}{\leq} CE^{\frac{1}{2}}. \tag{5.18}$$

Fix $\overline{\eta}$ to be chosen later. The following inequality is a consequence of the reduction argument given in [7, Theorem 6.8] where the authors reduce the whole discussion to the analysis of a decay for classical harmonic functions using Theorem 4.12

$$\mathbf{E}(\mathbf{G}_{u^+}+\mathbf{G}_{u^-},\mathbf{C}(0,1)\,,\pi)\leq(2-\overline{\eta})^{-(2-\varepsilon)}\mathbf{E}(\mathbf{G}_{u^+}+\mathbf{G}_{u^-},\mathbf{C}(0,2-\overline{\eta}))+C\overline{\eta}E\,. \tag{5.19}$$

Now, we claim that this inequality allows us to conclude (5.7). First of all, by the Taylor expansion of the mass of a Lipschitz graph, [12, Corollary 3.3], and the bound on Dirichlet energy of $u^{\pm}$ on the bad set, we conclude

$$\begin{aligned}\mathbf{E}(\mathbf{G}_{u^+}+\mathbf{G}_{u^-},\mathbf{C}(0,2-\overline{\eta}))&\leq\mathbf{E}(S_\eta,\mathbf{C}(0,2-\overline{\eta}))+C\int_{\Omega^+\setminus K}|Du^+|^2+C\int_{\Omega^-\setminus K}|Du^-|^2\\&\overset{(4.14)}{\leq}\mathbf{E}(S_\eta,\mathbf{C}(0,2-\overline{\eta}))+C\eta_*E.\end{aligned} \tag{5.20}$$

In second place, we have

$$\begin{aligned}\mathbf{E}(T,\mathbf{B}(0,1)\,,\pi)&\leq\mathbf{E}(S_\eta,\mathbf{C}(0,1)\,,\pi)\\&\leq\mathbf{E}(\mathbf{G}_{u^+}+\mathbf{G}_{u^-},\mathbf{C}(0,1)\,,\pi)+2\mathbf{e}_T(\mathrm{B}_1(0)\setminus K)+2|\pi|^2|\mathrm{B}_1(0)\setminus K|\\&\overset{(4.13)}{\leq}\mathbf{E}(\mathbf{G}_{u^+}+\mathbf{G}_{u^-},\mathbf{C}(0,1)\,,\pi)+C\eta_*E+2|\pi|^2|\mathrm{B}_1(0)\setminus K|\\&\overset{(5.18),(5.19),(5.20)}{\leq}(2-\overline{\eta})^{2-\varepsilon}\mathbf{E}(S_\eta,\mathbf{C}(0,2-\overline{\eta}))+C\eta_*E+C\overline{\eta}E.\end{aligned} \tag{5.21}$$

Using the height bound in Theorem 5.1, for $\varepsilon<\varepsilon_0$ sufficiently small, we have

$$\mathbf{h}(T,\mathbf{C}(0,2-\overline{\eta})\,,\pi_0)\leq C_h\left(\frac{\mathbf{E}(T,\mathbf{C}(0,4-2\overline{\eta}))}{1-\frac{\overline{\eta}}{2}}+\mathbf{A}\right)^{\frac{1}{2}}(2-\overline{\eta})^{\frac{3}{2}},$$

and thus

$$\mathrm{spt}(T)\cap\mathbf{C}(0,2-\overline{\eta})\subset\mathbf{B}(0,2)\ . \tag{5.22}$$

Since $S_\eta\llcorner\mathbf{B}(0,2)=T\llcorner\mathbf{B}(0,2)$, we obtain that

$$\begin{aligned}\mathbf{E}^\flat(T,\mathbf{B}(0,1))&\leq\mathbf{E}(T,\mathbf{B}(0,1)\,,\pi)\\&\overset{(5.21),(5.22)}{\leq}(2-\overline{\eta})^{-(2-\varepsilon)}\left(\frac{2}{2-\overline{\eta}}\right)^m\mathbf{E}(T,\mathbf{B}(0,2)\,,\pi)+C\eta_*E+\overline{\eta}E\\&=(2-\overline{\eta})^{-(2-\varepsilon)}\left(\frac{2}{2-\overline{\eta}}\right)^m\mathbf{E}^\flat(T,\mathbf{B}(0,2))+C\eta_*E+\overline{\eta}E\\&\overset{(5.17)}{\leq}\left[(2-\overline{\eta})^{-(2-\varepsilon)}\left(\frac{2}{2-\overline{\eta}}\right)^m+C(\eta_*+\overline{\eta})\right]\mathbf{E}^\flat(T,\mathbf{B}(0,2)).\end{aligned}$$

Hence, since the constant C in the last inequality is independent of the parameters $\eta_*,\overline{\eta}$, choosing the latter sufficiently small, we conclude (5.7). □

Proof of Theorem 5.3. Firstly, we want to prove that the assumptions (5.1) and (5.2) of Lemma 5.2 are satisfied. To this end, we notice that, since p is a two-sided collapsed point, by Definition 3.17, for every $\delta > 0$ there exists $\bar\rho = \bar\rho(\delta)$ small such that

(i) $\mathbf{E}^\flat(T, \mathbf{B}(p, 2\sigma)) + 4\mathbf{A}\sigma^2 \le \delta$ for every $\sigma \le \bar\rho$;

(ii) $\Theta(T, q) \ge \Theta(T, p) = Q - \frac{Q^\star}{2}$ for all $q \in \Gamma \cap \mathbf{B}(p, 2\bar\rho)$.

Next, since $\Theta(T,p) = Q - \frac{Q^\star}{2}$, if the radius $\bar\rho$ is chosen small enough we can assume that

$$\|T\|(\mathbf{B}(p, 4\bar\rho)) \le \omega_m \left(Q - \frac{Q^\star}{2} + \frac{1}{8}\right)(4\bar\rho)^m .$$

By a simple comparison, for η sufficiently small, if $q \in \mathbf{B}(p,\eta) \cap \Gamma$ and $\bar\rho' = \bar\rho - \eta$, then

$$\begin{aligned} \|T\|(\mathbf{B}\big(p, 4\bar\rho'\big)) &\le \|T\|(\mathbf{B}(p, 4\bar\rho)) \\ &\le \omega_m \left(Q - \frac{Q^\star}{2} + \frac{1}{8}\right)(4\bar\rho)^m \\ &\le \omega_m \left(Q - \frac{Q^\star}{2} + \frac{3}{16}\right)(4\bar\rho')^m . \end{aligned}$$

Next, by the latter inequality and by the monotonicity formula

$$\begin{aligned} \sigma^{-m}\|T\|(\mathbf{B}(q,\sigma)) &\le e^{\mathbf{A}(4\bar\rho' - \sigma)}(4\bar\rho')^{-m}\|T\|(\mathbf{B}\big(q, 4\bar\rho'\big)) \\ &\le e^{\mathbf{A}(4\bar\rho' - \sigma)}\omega_m \left(Q - \frac{Q^\star}{2} + \frac{3}{16}\right) \\ &\le e^{4\mathbf{A}\bar\rho}\omega_m \left(Q - \frac{Q^\star}{2} + \frac{3}{16}\right), \end{aligned}$$

for all $\sigma \le 4\bar\rho'$. In particular, if $\bar\rho$ is chosen sufficiently small, by the last inequality we then conclude

$$\|T\|(\mathbf{B}(q,\sigma)) \le \omega_m \left(Q - \frac{Q^\star}{2} + \frac{1}{4}\right)\sigma^m, \qquad \forall q \in \mathbf{B}(p,\eta) \cap \Gamma \text{ and } \forall \sigma \le 4\bar\rho' . \tag{5.23}$$

So, the density of T at q is bounded above by (5.23) and below by (ii). Set now $r := \min\{\eta, \bar\rho'\}$. For all points q in $\mathbf{B}(p,r) \cap \Gamma$ we claim that

$$\mathbf{E}^\flat(T, \mathbf{B}(q,r)) \le 2^m \mathbf{E}^\flat(T, \mathbf{B}(p, 2r)) + C\mathbf{A}^2 r^2 \overset{(i)}{\le} C\delta. \tag{5.24}$$

Indeed let π be a plane for which $\mathbf{E}^\flat(T, \mathbf{B}(p,2r)) = \mathbf{E}(T, \mathbf{B}(p,2r), \pi)$. By the regularity of Γ, we find a plane π_q such that $|\pi - \pi_q| \le Cr\mathbf{A}$ and $T_q\Gamma \subset \pi_q$. Then we can estimate

$$\begin{aligned} \mathbf{E}^\flat(T, \mathbf{B}(q,r)) &\le \mathbf{E}(T, \mathbf{B}(q,r), \pi_q) \le C\mathbf{E}(T, \mathbf{B}(p,2r), \pi_q) \\ &\overset{\text{triangle inequality}}{\le} C\mathbf{E}^\flat(T, \mathbf{B}(p,2r)) + Cr^2\mathbf{A}^2 \overset{(i)}{\le} C\delta . \end{aligned} \tag{5.25}$$

We will now show that the conclusions of the theorem hold for this particular radius r which, without loss of generality, we assume to be $r = 1$ and we also assume $p = 0$. So, we have proved that we are under the assumptions of Lemma 5.2, in fact, (5.25) and (5.23) ensure the following properties for every $q \in \mathbf{B}(0,1) \cap \Gamma$

(A) $\mathbf{E}^\flat(T, \mathbf{B}(q,1)) + \mathbf{A}^2 \leq C\mathbf{E}^\flat(T, \mathbf{B}(0,2)) + C\mathbf{A}^2 \leq C\delta$,

(B) $\|T\|(\mathbf{B}(q,s)) \leq (Q - \frac{2Q^\star - 1}{4})\omega_m s^m$ for every $s \leq 1$.

We now fix any point $q \in \Gamma \cap \mathbf{B}(0,1)$ and define $\mathfrak{m}(s) := \mathbf{E}^\flat(T, \mathbf{B}(q,s))$. We claim that

$$\mathfrak{m}(s) \leq Cs^{2-2\varepsilon} \max\{\mathfrak{m}(\tfrac{1}{4}), \mathfrak{m}(\tfrac{1}{2})\} + Cs^2\mathbf{A}^2, \qquad \forall s \in (0, \frac{1}{2}). \tag{5.26}$$

In order to prove (5.26), we firstly prove for $s = 2^{-k-1}$ and for all $k \in \mathbb{N}$ that

$$\mathfrak{m}(2^{-k-1}) \leq C \max\{2^{(2\epsilon-2)k}\mathfrak{m}(\frac{1}{4}), 2^{(2\epsilon-2)k+2}\mathfrak{m}(\frac{1}{2})\} + C2^{-2k-4}\mathbf{A} \tag{5.27}$$

is valid and then we will show how to derive (5.26) from (5.27). The proof of (5.27) will be done by induction. Notice that inequality (5.27) is trivially true for $k = 0$, indeed,

$$\mathfrak{m}(\frac{1}{2}) \leq 2^2\mathfrak{m}(\frac{1}{2}) \leq \max\{\mathfrak{m}(\frac{1}{4}), 2^2\mathfrak{m}(\frac{1}{2})\}.$$

If the inequality is true for $k_0 \geq 0$, we want to show it for $k = k_0 + 1$. We set $\sigma = 2^{-k-2}$ and notice that, by inductive assumption, we conclude that

$$\begin{aligned}\mathfrak{m}(2^{-k-1}) \leq \mathfrak{m}(4\sigma) = \mathfrak{m}(2^{-k_0-1}) &\leq \max\{2^{(2\varepsilon-2)k_0}\mathfrak{m}(\frac{1}{4}), 2^{(2\varepsilon-2)k_0+2}\mathfrak{m}(\frac{1}{2})\} \\ &\leq \max\{\mathfrak{m}(\tfrac{1}{4}), \mathfrak{m}(\tfrac{1}{2})\} \leq \mathfrak{m}(1) \overset{\text{(A)}}{\leq} C\delta.\end{aligned} \tag{5.28}$$

Hence, provided we choose $\delta = \delta(m, n, Q^\star, Q) > 0$ small in (A) and consequently r is sufficiently small too, we are in position to apply Lemma 5.2 which assures that

$$\begin{aligned}\mathfrak{m}(2^{-(k+1)-1}) = \mathfrak{m}(\sigma) &\overset{(5.3)}{\leq} \max\{2^{-2+2\varepsilon}\theta(2\sigma), 2^{-4+4\varepsilon}\theta(4\sigma)\} \\ &\leq C2^{-4+4\varepsilon}\theta(4\sigma) \\ &= C2^{-4+4\varepsilon}\max\{\mathfrak{m}(4\sigma), M_0\mathbf{A}^2(4\sigma)^2\} \\ &\overset{(5.28)}{\leq} C\max\{2^{(2\varepsilon-2)(k_0+1)}\mathfrak{m}(\frac{1}{4}), 2^{(2\varepsilon-2)(k_0+1)+2}\mathfrak{m}(\frac{1}{2})\} \\ &+ M_0\mathbf{A}^2\sigma^2,\end{aligned}$$

where we recall that C and M_0 are both constants that depends on $m, n, Q^\star, Q$ and ε, which finishes our induction steps and proves (5.27). To prove (5.26), we take $s \in (0, \frac{1}{2})$ and $k_s \in \mathbb{N}$ such that $s \in (2^{-k_s-2}, 2^{-k_s-1})$, hence, by (5.27),

$$\mathfrak{m}(s) \le \mathfrak{m}(2^{-k_s-1}) \le C\max\{2^{(2\varepsilon-2)k_s}\mathfrak{m}(\frac{1}{4}), 2^{(2\varepsilon-2)k_s+2}\mathfrak{m}(\frac{1}{2})\} + C2^{-2k_s-4}\mathbf{A}^2,$$

taking into account in the last inequality that $s^{2-2\varepsilon} > 4^{2-2\varepsilon}\cdot 2^{(2\varepsilon-2)k_s}$, we finish the proof of (5.26). We then conclude, for $\rho \in (0, \frac{1}{2})$, that the following equation holds

$$\begin{aligned}\mathbf{E}(T, \mathbf{B}(q,\rho)) \le \mathbf{E}^\flat(T, \mathbf{B}(q,\rho)) &\overset{(5.26)}{\le} C\rho^{2-2\varepsilon}\max\{\mathfrak{m}(\frac{1}{2}), \mathfrak{m}(\frac{1}{4})\} + C\rho^2\mathbf{A}^2 \\ &\le C\rho^{2-2\varepsilon}\mathbf{E}^\flat(T, \mathbf{B}(q,1)) + C\rho^2\mathbf{A}^2 \\ &\overset{(A)}{\le} C\rho^{2-2\varepsilon}\mathbf{E}^\flat(T, \mathbf{B}(0,2)) + C\rho^{2-2\varepsilon}\mathbf{A}^2.\end{aligned} \tag{5.29}$$

Furthermore, by (A), the estimate is trivial for $\frac{1}{2} \le \rho < 1$. For $0 < t < s < 1$, define $\pi(q,s)$ and $\pi(q,t)$ to be the optimal planes for $\mathbf{E}^\flat(T, \mathbf{B}(q,t))$ and $\mathbf{E}^\flat(T, \mathbf{B}(q,s))$, respectively. So, (5.29) implies

$$\begin{aligned}|\pi(q,s) - \pi(q,t)|^2 &= \frac{1}{\|T\|(\mathbf{B}(q,s))}\int_{\mathbf{B}(q,s)} |\pi(q,t) - \pi(q,s)|^2 d\|T\| \\ &\overset{(5.23)}{\le} C\mathbf{E}(T, \mathbf{B}(q,s)\,, \pi(q,s)) + C\mathbf{E}(T, \mathbf{B}(q,t)), \pi(q,t)) \\ &\overset{(5.29)}{\le} Cs^{2-2\varepsilon}\mathbf{E}^\flat(T, \mathbf{B}(0,2)) + Cs^{2-2\varepsilon}\mathbf{A}^2.\end{aligned}$$

Letting t goes to 0 in the last equations and thanks to the compactness of $G_m(\mathbb{R}^{m+n})$, we obtain the existence of a limit $\pi(q)$ such that

$$|\pi(q) - \pi(q,\rho)|^2 \le C\rho^{2-2\varepsilon}\mathbf{E}^\flat(T, \mathbf{B}(0,2)) + C\rho^{2-2\varepsilon}\mathbf{A}^2 \quad, \forall\, \rho < 1\,. \tag{5.30}$$

Hence, for all $\rho \in (0,1)$, we conclude that

$$\begin{aligned}\mathbf{E}^\flat(T, \mathbf{B}(q,\rho)) &\le \mathbf{E}^\flat(T, \mathbf{B}(q,\rho)\,, \pi(q)) \\ &\overset{\text{triangular}}{\le} C\mathbf{E}^\flat(T, \mathbf{B}(q,\rho)\,, \pi(q,\rho)) + C|\pi(q,\rho) - \pi(q)|^2 \\ &= C\mathbf{E}^\flat(T, \mathbf{B}(q,\rho)) + C|\pi(q,\rho) - \pi(q)|^2 \\ &\overset{(5.29),(5.30)}{\le} Cs^{2-2\varepsilon}\mathbf{E}^\flat(T, \mathbf{B}(0,2)) + Cs^{2-2\varepsilon}\mathbf{A}^2.\end{aligned} \tag{5.31}$$

which concludes the proof of (5.4).

We now turn to (5.5), let

$$S_\rho = T\llcorner\Big(B_\rho(q,\pi(q)) \times B^n_\rho(q,\pi(q)^\perp)\Big)\,.$$

Hence, we immediately have $T\llcorner\mathbf{B}(q,\rho) = S_\rho\llcorner\mathbf{B}(q,\rho)$. Moreover, arguing as in (B), (C) and (D) in the proof of Lemma 5.2, we are under Assumption 3, thus we can apply the Height bound (Lemma 5.1) to obtain

$$\mathbf{h}(S_\rho, \mathbf{C}(q,\rho)\,, \pi(q)) \le C_h\left(\rho^{-1}\mathbf{E}(S_\rho, \mathbf{C}(q,2\rho)\,, \pi(q)) + \mathbf{A}\right)^{\frac{1}{2}}\rho^{\frac{3}{2}}, \quad \forall \rho \in (0, \frac{1}{2}). \tag{5.32}$$

As in (5.14), we obtain that

$$\mathbf{E}(S_\rho, \mathbf{C}(q,\rho)\,, \pi(q)) \le C\mathbf{E}^\flat(T, \mathbf{B}\Big(q, \sqrt{2}\rho\Big)\,, \pi(q)), \quad \forall \rho \in (0, \frac{1}{\sqrt{2}}). \tag{5.33}$$

We are ready to conclude (5.5) as follows, for every $\rho \in (0, \frac{1}{2\sqrt{2}})$,

$$\begin{aligned}\mathbf{h}(T, \mathbf{B}(q,\rho), \pi(q)) &= \mathbf{h}(S_\rho, \mathbf{B}(q,\rho), \pi(q)) \\ &\overset{(5.32)}{\leq} C_h \left(\rho^{-1}\mathbf{E}(T, \mathbf{C}(q, 2\rho), \pi(q)) + \mathbf{A}\right)^{\frac{1}{2}} \rho^{\frac{3}{2}} \\ &\overset{(5.33)}{\leq} C_h \left(\rho^{-1}\mathbf{E}(T, \mathbf{C}\Big(q, 2\sqrt{2}\rho\Big), \pi(q)) + \mathbf{A}\right)^{\frac{1}{2}} \rho^{\frac{3}{2}},\end{aligned} \tag{5.34}$$

it is sufficient to apply the improved excess decay, (5.4), to conclude the proof. □

5.3 Uniqueness of tangent cones at two-sided collapsed points

In the spirit of Theorem 3.15 and Lemma 3.16 which state the uniqueness of tangent cones and the Holder continuity of the map $q \mapsto T_q$ for (C_0, r_0, α_0)-almost area minimizing currents of dimension 2, we prove the uniqueness of tangent and the Holder continuity of the same map for area minimizing currents of arbitrary dimension m. We state below the analogous of [7, Theorem 6.3] when the boundary is taken with multiplicity $Q^\star$.

Theorem 5.5 (Uniqueness of tangent cones at two-sided collapsed points)**.** *Let T, p, U and r be as in Theorem 5.3. Then for all $q \in \mathbf{B}(p,r) \subset U$, we have that q is a two-sided collapsed point with $\Theta^m(T,q) = \Theta^m(T,p)$ and there is a unique tangent cone $T_q = Q [\![\pi(q)^+]\!] + (Q - Q^\star) [\![\pi(q)^-]\!]$ to T at q, where $\pi(q)$ is an m-dimensional plane. Moreover, for any $\varepsilon > 0$, there is $C = C(\varepsilon) > 0$, such that*

$$|\pi(q) - \pi(z)| \leq C \left(r^{\varepsilon - 1} \left(\mathbf{E}^\flat(T, \mathbf{B}(p, 2r)) \right)^{\frac{1}{2}} + \mathbf{A} r^\varepsilon \right) |z - q|^{1-\varepsilon}, \ \forall z \in \mathbf{B}(p,r). \tag{5.35}$$

Remark 5.6. Note that, as we have proved in Lemma 3.18 in dimension 2 using the characterization of the tangent cones in the $2d$ setting, Theorem 5.5 ensures, for arbitrary dimension m, that the set of two-sided collapsed points is relatively open in Γ. Furthermore, it also guarantees that the density is constant in $\mathbf{B}(p,r) \cap \Gamma$.

The following proof goes along the same lines of [7, Theorem 6.3] and we report it here for completeness's sake.

Proof. We start taking $\pi(q)$ as the plane given by the improved excess decay, see Theorem 5.3. Let us now prove that, if T_q is a tangent cone to T at q w.r.t. the sequence $\rho_k \to 0$, then its support is $\pi(q)$. By rescaling we have that

$$\mathbf{E}(T, \mathbf{B}(q, \rho_k), \pi(q)) \leq C\mathbf{E}(T_{q,\rho_k}, \mathbf{B}(0,2), \pi(q)).$$

The latter rescaling and the improved excess decay, i.e., (5.4), furnish

$$\mathbf{E}(T_{q,\rho_k}, \mathbf{B}(0,2), \pi(q)) \leq C \left(\frac{\rho_k}{r}\right)^{2-2\varepsilon} \mathbf{E}^\flat(T, \mathbf{B}(p, 2r)) + C\rho_k^{2-2\varepsilon} r^{2\varepsilon} \mathbf{A}^2, \ \ \forall \rho_k < r. \tag{5.36}$$

We now let $\rho_k \to 0$ in (5.36) to conclude that $\mathbf{E}(T_q, \mathbf{B}(0,2), \pi(q)) = 0$ and hence T_q is supported in $\pi(q)$. We conclude that the tangent cone is unique and, by a standard argument involving the Constancy Lemma as already used many times above, it takes the form

$$T_q = Q(q) \left[\!\left[\pi(q)^+\right]\!\right] + (Q(q) - Q^\star) \left[\!\left[\pi(q)^-\right]\!\right] ,$$

for some $Q(q) \in \mathbb{N}$, since the tangent cone is an integral current. By assumption we have that p is a two-sided collapsed point and $q \in U$, $Q(q) - \frac{Q^\star}{2} = \Theta(T, q) \geq \Theta^m(T, p)$. Moreover, by (5.2), we obtain $Q(q) - \frac{Q^\star}{2} \leq \Theta^m(T, p) + \frac{1}{4}$ and therefore $\Theta^m(T, q) = \Theta^m(T, p)$. Finally, in order to finish the proof of the theorem, for $0 < t < s < 1$, we let $\pi(q, s)$ and $\pi(q, t)$ such that $\mathbf{E}^\flat(T, \mathbf{B}(q, t)) = \mathbf{E}^\flat(T, \mathbf{B}(q, t)\,, \pi(q, t))$ and $\mathbf{E}^\flat(T, \mathbf{B}(q, s)) = \mathbf{E}^\flat(T, \mathbf{B}(q, s)\,, \pi(q, s))$. We now take $0 < t < \rho := |q - z| < r$ and note that

$$\begin{aligned}
|\pi(q,t) - \pi(z,t)|^2 &= \fint_{\mathbf{B}(q,t)\cap\mathbf{B}(z,t)} |\pi(q,t) - \pi(z,t)|^2 d\|T\| \\
&\leq \frac{C}{\omega_m t^m} \int_{\mathbf{B}(q,t)} |\vec{T} - \pi(q,t)|^2 + \frac{C}{\omega_m t^m} \int_{\mathbf{B}(z,t)} |\vec{T} - \pi(z,t)|^2 d\|T\| \\
&= C(\mathbf{E}^\flat(T, \mathbf{B}(q,t)) + \mathbf{E}^\flat(T, \mathbf{B}(z,t))) \\
&\leq C\left(\frac{\rho}{r}\right)^{2-2\varepsilon} \mathbf{E}^\flat(T, \mathbf{B}(p, 2r)) + C\rho^{2-2\varepsilon} r^{2\varepsilon} \mathbf{A}^2,
\end{aligned}$$

where, in the second line, we have used that $\|T\|(\mathbf{B}(p,t)) \geq ct^m$, a simple consequence of the monotonicity formula. Hence, the latter inequality gives

$$|\pi(q,t) - \pi(z,t)| \leq C\left(r^{-2+2\varepsilon}\mathbf{E}^\flat(T, \mathbf{B}(p, 2r)) + r^{2\varepsilon}\mathbf{A}^2\right)^{\frac{1}{2}} |q - z|^{1-\varepsilon},$$

we let t goes to 0 to conclude (5.35). □

We state an important corollary of Theorems 5.5 and 5.3 which will be used often in the remaining chapters and relates the height when we change the reference plane to an optimal plane to the excess in p instead of consider the tangent plane at q.

Corollary 5.7 (Height bound relative to tilted optimal planes)**.** *Let $\Gamma, T, p, q, \pi(q)$ and r be as in Theorem 5.5 and let π be an optimal plane for $\mathbf{E}^\flat(T, \mathbf{B}(p, 2r))$. If we denote by $\mathbf{p}_\pi, \mathbf{p}_\pi^\perp, \mathbf{p}_{\pi(q)}$ and $\mathbf{p}_{\pi(q)}^\perp$ respectively the orthogonal projections onto $\pi, \pi^\perp, \pi(q)$ and $\pi(q)^\perp$, then, for all $q \in \Gamma \cap \mathbf{B}(p, r)$, we have*

$$|\pi(q) - \pi| \leq C(\mathbf{E}^\flat(T, \mathbf{B}(p, 2r))^{\frac{1}{2}} + \mathbf{A}r), \tag{5.37}$$

$$\operatorname{spt}(T)\cap\mathbf{B}\left(q, \frac{r}{2}\right) \subset \left\{x \in \mathbb{R}^{m+n} : \left|\mathbf{p}_{\pi(q)}^\perp(x - q)\right| \leq C(r^{-1}\mathbf{E}^\flat(T, \mathbf{B}(p, 2r))^{\frac{1}{2}} + \mathbf{A})^{\frac{1}{2}} |x - q|^{\frac{3}{2}}\right\}, \tag{5.38}$$

$$\operatorname{spt}(T) \cap \mathbf{B}\left(q, \frac{r}{2}\right) \subset \left\{x \in \mathbb{R}^{m+n} \colon \left|\mathbf{p}_\pi^\perp(x - q)\right| \leq C(\mathbf{E}^\flat(T, \mathbf{B}(p, 2r))^{\frac{1}{2}} + \mathbf{A}r)^{\frac{1}{2}} |x - q|\right\}. \tag{5.39}$$

Proof. To prove (5.37), we proceed as follows

$$\begin{aligned}
|\pi - \pi(q)|^2 &\leq 2|\pi - \pi(p)|^2 + 2|\pi(p) - \pi(q)|^2 \\
&\overset{(5.35)}{\leq} 2|\pi - \pi(p)|^2 + C(\mathbf{E}^\flat(T, \mathbf{B}(p, 2r))^{\frac{1}{2}} + \mathbf{A}r)^2,
\end{aligned} \tag{5.40}$$

and

$$
\begin{aligned}
|\pi-\pi(p)|^2 &\le C\frac{1}{\|T\|(\mathbf{B}(p,2r))}\int_{\mathbf{B}(p,2r)}\left(|\pi-\vec{T}|^2+|\vec{T}-\pi(p)|^2\right)d\|T\| \\
&\overset{(*)}{\le} C\mathbf{E}^\flat(T,\mathbf{B}(p,2r))+C\frac{1}{\|T\|(\mathbf{B}(p,2r))}\int_{\mathbf{B}(p,2r)}|\vec{T}-\pi(p)|^2 d\|T\| \\
&\overset{(5.4),(*)}{\le} C(\mathbf{E}^\flat(T,\mathbf{B}(p,2r))^{\frac{1}{2}}+\mathbf{A}r)^2,
\end{aligned}
\tag{5.41}
$$

where in $(*)$ we have used the standard argument with the monotonicity formula to obtain a bound $\|T\|(\mathbf{B}(p,2r))\ge cr^m$. Therefore (5.40) and (5.41) prove (5.37). Note that (5.38) follows immediately from (5.5). We next observe that

$$
\left|\mathbf{p}_\pi^\perp-\mathbf{p}_{\pi(q)}{}^\perp\right|^2=\left|\mathbf{p}_\pi-\mathbf{p}_{\pi(q)}\right|^2\le C\left|\pi-\pi(q)\right|^2. \tag{5.42}
$$

Furthermore, for $x\in\mathbf{B}(q,r)\cap\operatorname{spt}(T)$, it follows

$$
\begin{aligned}
|\mathbf{p}_\pi^\perp(x-q)|^2 &\le C|x-q|^2|\mathbf{p}_\pi^\perp-\mathbf{p}_{\pi(q)}{}^\perp|^2+C|\mathbf{p}_{\pi(q)}{}^\perp(x-q)|^2 \\
&\le C|\pi-\pi(q)|^2|x-q|^2+C|\mathbf{p}_{\pi(q)}{}^\perp(x-q)|^2 \\
&\overset{(5.37)}{\le} C(\mathbf{E}(T,\mathbf{B}(p,2r))^{\frac{1}{2}}+\mathbf{A}r)^2|x-q|^2+C|\mathbf{p}_{\pi(q)}{}^\perp(x-q)|^2 \\
&\overset{(5.38)}{\le} C(\mathbf{E}(T,\mathbf{B}(p,2r))^{\frac{1}{2}}+\mathbf{A}r)^2|x-q|^2+C(r^{-1}\mathbf{E}(T,\mathbf{B}(p,2r))^{\frac{1}{2}}+\mathbf{A})\,|x-q|^{\frac{3}{2}} \\
&\le C(\mathbf{E}(T,\mathbf{B}(p,2r))^{\frac{1}{2}}+\mathbf{A}r)^{\frac{1}{2}}|x-q|,
\end{aligned}
$$

where in the last inequality the fact $|x-q|<r$ took place, thus the latter inequality proves (5.39).

□

5.4 Lusin type strong Lipschitz approximations

As we remarked at the beginning of this section, Theorem 4.11 provides an approximation which is not enough for our purposes. In the last subsection, we used the harmonic approximation, Theorem 4.12, to obtain the superlinear excess decay, Theorem 5.3, which will now be used to provide our desired approximation with faster decays and stronger estimates as it is precisely stated in Theorem 5.8.

Assumption 4. *Let T and Γ be as in Assumption 2 with $C_0=0$, π be a m-dimensional subspace, $\{e_i\}_{i=1}^m$ a basis of $\mathbb{R}^m$ and $q\in\Gamma$. We use the following notations $\pi'=\operatorname{span}(\mathbf{p}_\pi(e_1),\cdots,\mathbf{p}_\pi(e_{m-1}))$, $\psi_1:(q+\pi')\to(q+\operatorname{span}(\mathbf{p}_\pi(e_m)))$, $\psi:\gamma\subset(q+\pi)\to(q+\pi)^\perp$, $\psi_2:(q+\pi')\to(q+\operatorname{span}(\mathbf{p}_\pi(e_m)))\times(q+\pi)^\perp$, $\psi_2(x)=(\psi_1(x),\psi(x,\psi_1(x)))$ with $\operatorname{graph}(\psi_1)=\gamma,\Gamma=\operatorname{graph}(\psi_2)$, and ψ is of class $C^{3,\alpha}$. We assume that*

(i) In the excess decay, Theorem 5.3, $p=0\in\Gamma$ and $r=1$,

(ii) $\mathbf{E}^\flat(T,\mathbf{B}(0,2))+\mathbf{A}<\varepsilon_1$, where $\varepsilon_1=\varepsilon_1(m,n,Q^\star,Q)>0$ is a small constant.

We would like to point out that the approximation and the estimates in the following theorem hold for any point in a suitable ball of the fixed two-sided collapsed point $p = 0$, compare with Theorem 4.11 where the approximation and estimates are build in balls centered in the fixed two-sided collapsed point p.

Theorem 5.8 (Strong Lipschitz approximation)**.** *Let T, Γ, ψ and $\gamma = \mathbf{p}_\pi(\Gamma)$ be as in Assumption 4, $q \in \Gamma \cap \mathbf{B}(0,1)$, $r < \frac{1}{8}$ and π be a plane such that $T_q\Gamma \subset \pi$ and $\mathbf{E}(T, \mathbf{C}(q, 4r, \pi)) < \varepsilon_1$. Then there are a closed set $K \subset \mathrm{B}_r(q,\pi)$ and a $\left(Q - \frac{Q^\star}{2}\right)$-valued map (u^+, u^-) on $\mathrm{B}_r(q,\pi)$ which collapses at the interface $(\gamma, Q^\star [\![\psi]\!])$ satisfying the following estimates:*

$$\mathrm{Lip}(u^\pm) \leq C(\mathbf{E}(T, \mathbf{C}(q, 4r, \pi)) + \mathbf{A}^2 r^2)^\sigma \tag{5.43}$$

$$\mathrm{osc}(u^\pm) \leq C(\mathbf{E}(T, \mathbf{C}(q, 4r, \pi)) + \mathbf{A}^2 r^2)^{\frac{1}{2}} r \tag{5.44}$$

$$\mathbf{G}_{u^\pm} \llcorner [(K \cap \Omega^\pm) \times \pi^\perp] = T \llcorner [(K \cap \Omega^\pm) \times \mathbb{R}^n] \tag{5.45}$$

$$|\mathrm{B}_r(q,\pi) \setminus K| \leq C(\mathbf{E}(T, \mathbf{C}(q, 4r, \pi)) + \mathbf{A}^2 r^2)^{1+\sigma} r^m \tag{5.46}$$

$$\mathbf{e}_T(\mathrm{B}_r(q,\pi) \setminus K) \leq C(\mathbf{E}(T, \mathbf{C}(q, 4r, \pi)) + \mathbf{A}^2 r^2)^{1+\sigma} r^m \tag{5.47}$$

$$\int_{\mathrm{B}_r(q,\pi)\setminus K} |Du|^2 \leq C(\mathbf{E}(T, \mathbf{C}(q, 4r, \pi)) + \mathbf{A}^2 r^2)^{1+\sigma} r^m \tag{5.48}$$

$$\left| \mathbf{e}_T(F) - \frac{1}{2} \int_F |Du^\pm|^2 \right| \leq C(\mathbf{E}(T, \mathbf{C}(q, 4r, \pi)) + \mathbf{A}^2 r^2)^{1+\sigma} r^m, \quad \forall F \subset \Omega^\pm \text{ measurable,} \tag{5.49}$$

where $\Omega^\pm$ are the two regions in which $\mathrm{B}_r(q,\pi)$ is divided by γ, $C \geq 1$ and $\sigma \in]0, \frac{1}{4}[$ are two positive constants which depend on $m, n, Q^\star$ and Q.

Proof. Our strategy to prove this theorem is to go back to the interior estimates done in [11]. So we will divide the proof into two steps, in Step 1 we will prove further estimates provided by the interior case which are needed to conclude our estimates (5.43)-(5.49), and in Step 2 we will exhibit how to obtain the theorem from the interior case.

Step 1: If we assume that ε_1 is smaller than the constant of [11, Theorem 2.4] (also denoted by ε_1) and the cylinder $\mathbf{C}(x, 4\rho, \pi)$ does not intersect Γ and is contained in $\mathbf{C}(q, 4r, \pi)$. Then, [11, Theorem 2.4] provides a map $f : \mathrm{B}_\rho(x,\pi) \to \mathcal{A}_Q(\pi^\perp)$ (or $\mathcal{A}_{Q-Q^\star}(\pi^\perp)$) and a closed set $\bar{K} \subset \mathrm{B}_\rho(x,\pi)$ such that

$$\mathrm{Lip}(f) \leq C_{21} \mathbf{E}(T, \mathbf{C}(x, 4\rho))^\sigma, \tag{5.50}$$

$$\mathbf{G}_f \llcorner (\bar{K} \times \mathbb{R}^n) = T \llcorner (\bar{K} \times \mathbb{R}^n), \tag{5.51}$$

$$|\mathrm{B}_\rho(x,\pi) \setminus \bar{K}| \leq C \mathbf{E}(T, \mathbf{C}(x, 4\rho))^{1+\sigma} \rho^m, \tag{5.52}$$

$$\left| \|T\|(\mathbf{C}(x,\rho)) - Q\omega_m \rho^m - \frac{1}{2} \int_{\mathrm{B}_\rho(x,\pi)} |Df|^2 \right| \leq C \mathbf{E}(T, \mathbf{C}(x, 4\rho))^{1+\sigma} \rho^m, \tag{5.53}$$

In order to simplify our notation, we assume that $\pi = \mathbb{R}^m \times \{0\}$ and use the shorthand notation $\mathrm{B}_t(x)$ for $\mathrm{B}_t(x,\pi)$. It remains to prove the analogous of (5.47), (5.48) and (5.49) when q is replaced by the interior point x. Notice that (5.50) and (5.52) give

$$\int_{F\setminus\bar K}|Df|^2 \le C\mathbf{E}(T,\mathbf{C}(x,4\rho))^{2\sigma}\left|\mathrm{B}_\rho(x)\setminus \bar K\right| \le C\mathbf{E}(T,\mathbf{C}(x,4\rho))^{1+3\sigma}\rho^m,$$

for every $F\subset \mathrm{B}_\rho(x)$ measurable, hence we achieve (5.48) taking $F=\mathrm{B}_\rho(x)$. Next recall that either $\|T\|(\mathbf{C}(x,\rho)) - Q\omega_m\rho^m = \mathbf{e}_T(\mathrm{B}_\rho(x))$ or $\|T\|(\mathbf{C}(x,\rho)) - (Q-Q^\star)\omega_m\rho^m = \mathbf{e}_T(\mathrm{B}_\rho(x))$, hence (5.53) can be reformulated as

$$\left|\mathbf{e}_T(\mathrm{B}_\rho(x)) - \frac12\int_{\mathrm{B}_\rho(x)}|Df|^2\right| \le C\mathbf{E}(T,\mathbf{C}(x,4\rho))^{1+\sigma}\rho^m\,.$$

We also have that

$$\begin{aligned}\frac12\int_{\mathrm{B}_\rho(x)}|Df|^2 &\le \left(\mathbf{E}(T,\mathbf{C}(x,4\rho)) + C\mathbf{E}(T,\mathbf{C}(x,4\rho))^{1+\sigma}\right)\rho^m\\ &\le C\mathbf{E}(T,\mathbf{C}(x,4\rho))\rho^m.\end{aligned}\tag{5.54}$$

The Taylor expansion of the area functional, [12, Corollary 3.3], and (5.50) give

$$\left|\mathbf{e}_{\mathbf{G}_f}(F) - \frac12\int_F|Df|^2\right| \le C\,\mathrm{Lip}(f)^2\int_F|Df|^2 \le C\mathbf{E}(T,\mathbf{C}(x,4\rho))^{1+2\sigma}\rho^m,\tag{5.55}$$

for every $F\subset \mathrm{B}_\rho(x)$ measurable. Therefore, by (5.54) and (5.55), we obtain

$$\begin{aligned}\mathbf{e}_T(\mathrm{B}_\rho(x)\setminus\bar K) &= \mathbf{e}_T(\mathrm{B}_\rho(x)) - \mathbf{e}_{\mathbf{G}_f}(\mathrm{B}_\rho(x)\cap\bar K)\\ &\le \left|\mathbf{e}_T(\mathrm{B}_\rho(x)) - \frac12\int_{\mathrm{B}_\rho(x)}|Df|^2\right|\\ &\quad + \left|\frac12\int_{\mathrm{B}_\rho(x)\cap\bar K}|Df|^2 - \mathbf{e}_{\mathbf{G}_f}(\mathrm{B}_\rho(x)\cap\bar K)\right|\\ &\quad + \int_{\mathrm{B}_\rho(x)\setminus\bar K}|Df|^2\\ &\le C\mathbf{E}(T,\mathbf{C}(x,4\rho))^{1+\sigma}\rho^m,\end{aligned}$$

which is (5.47). Finally, (5.49) is implied by the last inequality, (5.48) and (5.55) as follows, for every $F\subset \mathrm{B}_\rho(x)$ measurable we have

$$\begin{aligned}\left|\mathbf{e}_T(F) - \frac12\int_F|Df|^2\right| &\le \left|\mathbf{e}_{\mathbf{G}_f}(F\cap\bar K) - \frac12\int_{F\cap\bar K}|Df|^2\right| + \mathbf{e}_T(F\setminus\bar K) + \frac12\int_{F\setminus\bar K}|Df|^2\\ &\le C\mathbf{E}(T,\mathbf{C}(x,4\rho))^{1+\sigma}\rho^m.\end{aligned}$$

Step 2: Without loss of generality we assume that $T_q\Gamma = \mathbb{R}^{m-1}\times\{0\}$, $\pi=\mathbb{R}^m\times\{0\}$. We then use $\mathbf{C}(q,s)$ in place of $\mathbf{C}(q,s,\pi)$, $\mathrm{B}_s(q)$ in place of $\mathrm{B}_s(q,\pi)$, and $\mathbf{p}$ to be the orthogonal projection onto π. By Assumption 4, we have

$$\partial T\llcorner\mathbf{C}(q,4r) = Q^\star[\![\Gamma\cap\mathbf{C}(q,4r)]\!] \qquad\text{and}\qquad \mathbf{p}_\sharp(\partial T\llcorner\mathbf{C}(q,4r)) = Q^\star[\![\gamma\cap\mathrm{B}_{4r}(\mathbf{p}(q))]\!]\,.$$

As in the previous sections, denote by Ω^+ and Ω^- the two connected components of $\mathrm{B}_{4r}(q) \setminus \gamma$, we have

$$\mathbf{p}_\sharp T \llcorner \mathbf{C}(q, 4r) = Q \left[\!\left[\Omega^+\right]\!\right] + (Q - Q^\star) \left[\!\left[\Omega^-\right]\!\right] . \tag{5.56}$$

We denote L_0 be the m-cube $q + [-r, r]^m$ and, for any natural number k, let $\mathcal{C}_k$ be a collection of m-cubes given by

$$\mathcal{C}_k = \left\{ L = q + r2^{-k}x + [-2^{-k}r, 2^{-k}r]^m : x \in \mathbb{Z}^m, k \in \mathbb{N}, L \subset L_0, L \cap \mathrm{B}_r(q) \neq \emptyset \right\}.$$

We take a number $N \in \mathbb{N}$ such that the $2^{4-N}\sqrt{m}r$-neighborhood of $\cup_{L \in \mathcal{C}_N} L$ is contained in $\mathbf{C}(q, 4r)$ and we will proceed with the construction of a Whitney decomposition of

$$\tilde{\Omega} := \bigcup_{L \in \mathcal{C}_N} L \setminus \gamma.$$

Here and in what follows we set

$$\mathrm{sep}\,(L, \gamma) := \min\{|x - y| : x \in \gamma, y \in L\}\,.$$

We firstly define the following sets of m-cubes $\mathcal{R}_N = \mathcal{C}_N$,

$$\mathcal{W}_N := \left\{ L \in \mathcal{R}_N : \mathrm{diam}(L) \leq \frac{1}{16}\,\mathrm{sep}(L, \gamma) \right\}.$$

If $L \in \mathcal{R}_N \setminus \mathcal{W}_N$, we subdivide L in 2^m m-subcubes of side $2^{-(N+1)}r$ and assign them to $\mathcal{R}_{N+1}$. We proceed inductively to define $\mathcal{W}_k$ and $\mathcal{R}_{k+1}$ for every $k \geq N$. Therefore, we obtain a Whitney decomposition $\mathcal{W} = \cup_{k \geq N} \mathcal{W}_k$ which is a collection of closed dyadic m-cubes such that

$$\mathrm{int}(L) \cap \mathrm{int}(H) = \emptyset, \text{ for all } L, H \in \mathcal{W}, \tag{5.57}$$

$$\Omega^+ \cup \Omega^- \subset \cup_{L \in \mathcal{W}} L, \tag{5.58}$$

$$\frac{15}{16}\frac{1}{32}\mathrm{sep}\,(L, \gamma) < \ \mathrm{diam}(L) \leq \frac{1}{16}\mathrm{sep}\,(L, \gamma), \ L \in \mathcal{W}. \tag{5.59}$$

Note that (5.57) follows readily from the construction. In regard to (5.58) we take $z \in \Omega^\pm$ then, we have two mutually exclusive cases that could appear, namely either there exists $L \in \mathcal{W}_N$ such that $z \in L$, or for every $L \in \mathcal{W}_N$ results $z \notin L$. In the first case we finish readily the proof. In the second case take L such that $L \in \mathcal{R}_N \setminus \mathcal{W}_N$ and $z \in L$. Then we may subdivide it and pass to the next generation and find a new cube $L' \in \mathcal{R}_{N+1}$ such that $z \in L'$. Now we may apply the same reasoning inductively and construct a sequence $L_{k,j_{k,z}}$, $k \geq N$ such that $z \in L_{k,j_{k,z}}$, $L_{k,j_{k,z}} \subseteq L_{k+1,j_{k+1,z}}$, and $L_{k,j_{k,z}} \notin \mathcal{W}$ for every $k \geq N$. If the sequence $(L_{k,j_{k,z}})_k$ is not constant for large values of k, then the diameters of $L_{k,j_{k,z}}$ goes to zero as k goes to infinity, and thus we obtain for sufficiently large k that

$$\mathrm{diam}(L_{k,j_{k,z}}) \leq \frac{1}{16}\,\mathrm{sep}(L_{N,j_{N,z}}, \gamma) \leq \frac{1}{16}\,\mathrm{sep}(L_{k,j_{k,z}}, \gamma),$$

since $L_{k,j_{k,z}} \subset L_{N,j_{N,z}}$ which ensures that $L_{k,j_{k,z}} \in \mathcal{W}$ and therefore (5.58). To prove (5.59), observe that $\mathrm{sep}(L, \gamma) \leq \mathrm{sep}(\tilde{L}, \gamma) + \mathrm{diam}(L)$ for every $L \in \mathcal{C}_k, \tilde{L} \in \mathcal{C}_{k-1}$ and $L \subseteq \tilde{L}$. By construction for each $L \in \mathcal{W}_k$ there exists $\tilde{L} \in \mathcal{R}_{k-1} \setminus \mathcal{W}_{k-1}$ such that $L \subseteq \tilde{L}$. Thus $\frac{15}{16}\,\mathrm{sep}(L, \gamma) \leq \mathrm{sep}(\tilde{L}, \gamma)$, $2\mathrm{diam}(L) = \mathrm{diam}(\tilde{L}) > \frac{1}{16}\,\mathrm{sep}(\tilde{L}, \gamma)$. So $\mathrm{diam}(L) > \frac{1}{32}\,\mathrm{sep}(\tilde{L}, \gamma) \geq \frac{15}{16}\frac{1}{32}\,\mathrm{sep}(L, \gamma)$.

Another important property of this family is that if a m-cube stops then its neighbours of next generation must also stop. Precisely, let $H \in \mathcal{W}_j, L \in \mathcal{C}_{j+1}$ and $H \cap L \neq \emptyset$, then

$$\operatorname{sep}(L,\gamma) \geq \operatorname{sep}(H,\gamma) - \operatorname{diam}(L) \overset{(5.59)}{\geq} 16\operatorname{diam}(H) - \operatorname{diam}(L) \geq 31\operatorname{diam}(L) \geq 16\operatorname{diam}(L). \quad (5.60)$$

The chain of inequalities above guarantees that $\operatorname{sep}(L,\gamma) \geq 16\operatorname{diam}(L)$ which is the very definition of the family $\mathcal{W}_{j+1}$, i.e., $L \in \mathcal{W}$.

We denote with c_L the center of the m-cube $L \in \mathcal{W}$ and set $r_L := 3\operatorname{diam}(L)$ so that

$$L \subset \mathrm{B}_{\frac{1}{4}r_L}(c_L)\,. \quad (5.61)$$

We claim that for each cube L the current T restricted to the cylinder $\mathbf{C}(c_L, 4r_L)$ satisfies the assumptions of [11, Theorem 2.4]. Firstly note that, by (5.59), we have $\mathbf{C}(c_L, 4r_L) \cap \Gamma = \emptyset$ and thus $\partial T \llcorner \mathbf{C}(c_L, 4r_L) = 0$, we also obtain by the choice of N that $\mathbf{B}(c_L, 6r_L) \subset \mathbf{B}(q, 4r)$. Moreover, either $\mathrm{B}_{4r_L}(c_L) \subset \Omega^+$ or $\mathrm{B}_{4r_L}(c_L) \subset \Omega^-$ and thus by (5.56) we have

$$\mathbf{p}_\sharp T \llcorner \mathbf{C}(c_L, 4r_L) = \begin{cases} Q\,[\![\mathrm{B}_{4r_L}(c_L)]\!]\,, & \text{if } c_L \in \Omega^+, \\ (Q - Q^\star)\,[\![\mathrm{B}_{4r_L}(c_L)]\!]\,, & \text{if } c_L \in \Omega^-. \end{cases}$$

It remains to prove that the excess is small enough. Towards this goal we will make use of the excess decay, Theorem 5.3. In fact, we start distinguishing the two cases $r_L = 2^{-N}r$ and $r_L < 2^{-N}r$. If $r_L = 2^{-N}r$, rescaling the excess and by the assumptions of the theorem, we easily obtain that

$$\mathbf{E}(T, \mathbf{C}(c_L, 4r_L)) \leq 2^{mN}\mathbf{E}(T, \mathbf{C}(q, 4r)) < \varepsilon_1.$$

Now, for each $L \in \mathcal{W}$ with $r_L < 2^{-N}r$, we let x_L be the orthogonal projection of c_L on γ and $q_L \in \Gamma$ be the point $(x_L, \psi(x_L))$. The first inequality of (5.59) implies that

$$\mathbf{C}(c_L, 4r_L) \subset \mathbf{C}(q_L, 15r_L)\,.$$

From our choice of N, taking ε_1 smaller if necessary, by (5.39), we have

$$\operatorname{spt}(T) \cap \mathbf{C}(q_L, 16r_L) \subset \mathbf{B}(q_L, 17r_L) \subset \mathbf{C}(q, 4r)\,. \quad (5.62)$$

So, we deduce that

$$\begin{aligned} \mathbf{E}(T, \mathbf{C}(c_L, 4r_L)) &\overset{(5.62)}{\leq} \mathbf{E}(T, \mathbf{B}(q_L, 17r_L), \pi) \\ &\overset{\text{triangular}}{\leq} C\mathbf{E}(T, \mathbf{B}(q_L, 17r_L), \pi(q_L)) + C|\pi - \pi(q_L)| \\ &\leq C\mathbf{E}(T, \mathbf{C}(q, 4r)) + \mathbf{A}^2 r^2 < \varepsilon_1, \end{aligned} \quad (5.63)$$

where in the last inequality we have used the excess decay, Theorem 5.3, and (5.37). So, provided ε_1 is chosen sufficiently small, we can apply Step 1 in every cylinder $\mathbf{C}(c_L, 4r_L)$ and obtain either a Q-valued or a $(Q - Q^\star)$-valued map f_L on each half ball $\mathrm{B}^+_{r_L}(c_L)$ or $\mathrm{B}^-_{r_L}(c_L)$ and a closed set $K_L \subset \mathrm{B}^\pm_{r_L}(c_L)$ such that

$$\operatorname{Lip}(f_L) \leq C\mathbf{E}(T, \mathbf{C}(c_L, 4r_L))^\sigma, \quad (5.64)$$

$$\mathbf{G}_{f_L} \llcorner (K_L \times \mathbb{R}^n) = T \llcorner (K_L \times \mathbb{R}^n), \tag{5.65}$$

$$|\mathrm{B}_{r_L}(c_L) \setminus K_L| \le C \mathbf{E}(T, \mathbf{C}(c_L, 4r_L))^{1+\sigma} r_L^m, \tag{5.66}$$

$$\mathbf{e}_T(\mathrm{B}_{r_L}(c_L) \setminus K_L) \le C \mathbf{E}(T, \mathbf{C}(c_L, 4r_L))^{1+\sigma} r_L^m, \tag{5.67}$$

$$\int_{\mathrm{B}_{r_L}(c_L) \setminus K_L} |Df_L|^2 \le C \mathbf{E}(T, \mathbf{C}(c_L, 4r_L))^{1+\sigma} r_L^m, \tag{5.68}$$

$$\left| \mathbf{e}_T(F) - \frac{1}{2} \int_F |Df_L|^2 \right| \le C \mathbf{E}(T, \mathbf{C}(c_L, 4r_L))^{1+\sigma} r_L^m, \quad \forall F \subset \mathrm{B}_{r_L}(c_L) \text{ measurable.} \tag{5.69}$$

Next, for each L we let $\mathcal{N}_{\ge}(L)$ be what we call the neighboring m-cubes in $\mathcal{W}$ with bigger radius, i.e.

$$\mathcal{N}_{\ge}(L) = \{H \in \mathcal{W} \colon H \cap L \neq \emptyset, r_H \ge r_L\}.$$

We use the good sets provided by the interior approximation to define

$$K_L' = K_L \cap \bigcap_{H \in \mathcal{N}_{\ge}(L)} K_H, \quad K^+ = \bigcup_{L \in \mathcal{W}, L \subset \Omega^+} K_L' \cap L, \quad K^- = \bigcup_{L \in \mathcal{W}, L \subset \Omega^-} K_L' \cap L.$$

Furthermore, we set up two functions defined on K^+ and K^-, respectively, given by $\tilde{u}^+(x) := f_L(x)$, for any $x \in L \cap K^+, L \in \mathcal{W}$, and $\tilde{u}^-(x) := f_L(x)$, for all $x \in L \cap K^-, L \in \mathcal{W}$, thanks to (5.65) these functions are well defined. Note that these functions are defined on each square in $\mathcal{W}$ which are away from the boundary γ, it means that we have to properly extend these functions in order to have a $(Q - \frac{Q^\star}{2})$-valued map which collapses at the interface. Indeed, we will refine this idea in the sequel.

Note that, if $H \in \mathcal{N}_{\ge}(L)$, we already know by (5.61) that $L \subset \mathrm{B}_{\frac{1}{4}r_L}(c_L)$ and then, since $H \cap L \neq \emptyset$, we deduce that $L \subset B_{r_H}(c_H)$. Hence

$$\begin{aligned} |L \setminus K_L'| &\le |L \setminus K_L| + \sum_{H \in \mathcal{N}_{\ge}(L)} |L \setminus K_H| \\ &\le |L \setminus K_L| + \sum_{H \in \mathcal{N}_{\ge}(L)} |\mathrm{B}_{r_H}(c_H) \setminus K_H| \\ &\overset{(5.66)}{\le} C \mathbf{E}(T, \mathbf{C}(q, 4r))^{1+\sigma} r_L^m < C \varepsilon_1 r_L^m, \end{aligned} \tag{5.70}$$

in the last inequality we also use that the cardinality of $\mathcal{N}_{\ge}(L)$ is bounded by a geometric constant C', which allow us to bound r_H by r_L. We now claim the validity of the following:

$$\mathrm{Lip}(\tilde{u}^\pm) \le C(\mathbf{E}(T, \mathbf{C}(q, 4r)) + \mathbf{A}^2 r^2)^\sigma, \tag{5.71}$$

$$\mathbf{G}_{\tilde{u}^\pm} \llcorner (K^\pm \times \mathbb{R}^n) = T \llcorner (K^\pm \times \mathbb{R}^n), \tag{5.72}$$

$$\mathbf{e}_T(L \setminus K_L') \le C(\mathbf{E}(T, \mathbf{C}(q, 4r)) + \mathbf{A}^2 r^2)^{1+\sigma} r_L^m, \tag{5.73}$$

$$\int_{L \setminus K_L'} |D\tilde{u}^\pm|^2 \le C(\mathbf{E}(T, \mathbf{C}(q, 4r)) + \mathbf{A}^2 r^2)^{1+\sigma} r_L^m. \tag{5.74}$$

Before the proof of this claim, we show how to prove (5.43)-(5.49) from it. Define the good set as $K = K^+ \cup K^-$ and notice that in view of these last inequalities, up to now, we have finished the proof of (5.43), (5.44), and (5.45). Observe that

$$\sum_{L\in\mathcal{W}} r_L^m \leq C r^m \,, \tag{5.75}$$

which furnishes (5.46), (5.47) and (5.48) by summing over $L \in \mathcal{W}$, respectively, (5.70), (5.73) and (5.74). Regarding (5.49), we proceed as follows, fix a measurable set $F \subset \Omega^\pm$ and observe that, for any m-cube L in the Whitney decomposition $\mathcal{W}$ of $\Omega^\pm$ we have that

$$\begin{aligned}
\left|\mathbf{e}_T(F\cap L) - \frac{1}{2}\int_{F\cap L} |D\tilde{u}^+|^2\right| &\overset{\text{triangular}}{\leq} \left|\mathbf{e}_T(F\cap L\cap K^\pm) - \frac{1}{2}\int_{F\cap L\cap K^\pm} |D\tilde{u}^+|^2\right| \\
&\quad + \mathbf{e}_T(L\setminus K^\pm) + \mathrm{Lip}(\tilde{u}^+)^2 |L\setminus K^\pm| \\
&\overset{(5.70),(5.71),(5.73)}{\leq} \left|\mathbf{e}_T(F\cap L\cap K^\pm) - \frac{1}{2}\int_{F\cap L\cap K^\pm} |Df_L|^2\right| \\
&\quad + C(\mathbf{E}(T,\mathbf{C}(q,4r)) + \mathbf{A}^2 r^2)^{1+\sigma} r_L^m \\
&\overset{(5.69)}{\leq} C(\mathbf{E}(T,\mathbf{C}(q,4r)) + \mathbf{A}^2 r^2)^{1+\sigma} r_L^m \,.
\end{aligned}$$

By (5.75), summing over $L \in \mathcal{W}$, we obtain (5.49). Now, we turn our attention to the proof of the claim.

We start with the proof of the Lipschitz bound in (5.71), we let $H, L \in \mathcal{W}$ with $\mathrm{diam}(H) \geq \mathrm{diam}(L)$ and $x \in H, y \in L$, hence

- If $H \cap L \neq \emptyset$, by the very definitions of $\tilde{u}$ and $K^\pm$, we know that $\tilde{u}^\pm = f_H$ on $K^\pm \cap H$. Take any $z \in H \cap L$, thus the Lipschitz bound on each m-cube, i.e., (5.64), ensures

$$\mathcal{G}(\tilde{u}^\pm(x), \tilde{u}^\pm(y)) \leq \mathcal{G}(\tilde{u}^\pm(x), \tilde{u}^\pm(z)) + \mathcal{G}(\tilde{u}^\pm(z), \tilde{u}^\pm(y)) \leq C\mathbf{E}(T,\mathbf{C}(q,4r))^\sigma |x-y|.$$

- If $H \cap L = \emptyset$, let $x_\gamma, y_\gamma \in \gamma$ such that $\mathrm{dist}(x,\gamma) = \mathrm{dist}(x,x_\gamma)$ and $\mathrm{dist}(y,\gamma) = \mathrm{dist}(y,y_\gamma)$. We directly obtain that

$$\begin{aligned}
\mathcal{G}(\tilde{u}^\pm(x), Q^\pm [\![\psi(x_\gamma)]\!]) &\leq C\mathbf{E}(T,\mathbf{C}(q,4r))^{\frac{1}{2}} |x - x_\gamma|, \\
\mathcal{G}(\tilde{u}^\pm(y), Q^\pm [\![\psi(y_\gamma)]\!]) &\leq C\mathbf{E}(T,\mathbf{C}(q,4r))^{\frac{1}{2}} |y - y_\gamma|,
\end{aligned} \tag{5.76}$$

where $Q^+ = Q$ and $Q^- = Q - Q^\star$. Indeed, both inequalities are due to the fact that $\mathrm{dist}(x,\gamma)$ and r_L are comparable, e.g., (5.59), and that we have the height bound

$$\mathrm{spt}(T) \cap \mathbf{B}(0,2) \subset \left\{ x \in \mathbb{R}^{m+n} : |\mathbf{p}_\pi^\perp(x)| \leq C\varepsilon_1^{\frac{1}{2}} |x| \right\},$$

as in Corollary 5.7, in the cylinder $\mathbf{C}(x_\gamma, 16 r_L)$. Note also that, by the regularity of Γ, we obtain $|\psi(x_\gamma) - \psi(y_\gamma)| \leq C\mathbf{A}^{1/2} |x_\gamma - y_\gamma|$. As a consequence, by (5.76), we can estimate

$$\begin{aligned}
\mathcal{G}(\tilde{u}^\pm(x), \tilde{u}^\pm(y)) &\leq \mathcal{G}(\tilde{u}^\pm(x), Q^\pm [\![\psi(x_\gamma)]\!]) + (Q^\pm)^{\frac{1}{2}} |\psi(x_\gamma) - \psi(y_\gamma)| \\
&\quad + \mathcal{G}(\tilde{u}^\pm(y), Q^\pm [\![\psi(y_\gamma)]\!]) \leq C\left(\mathbf{E}(T,\mathbf{C}(q,4r)) + \mathbf{A}^2 r^2\right)^\sigma |x-y|
\end{aligned}$$

where we have used that $\sigma \leq \frac{1}{4}$ and that

$$|x - x_\gamma| + |y_\gamma - y| = \operatorname{dist}(x,\gamma) + \operatorname{dist}(y,\gamma) \leq C(r_L + r_H) \leq C r_H \leq C\frac{3\sqrt{m}}{2}|x-y|.$$

To see the last inequality observe that when $H \cap L = \emptyset$ by (5.60) we have that $|x-y| \geq \frac{2r_H}{3\sqrt{m}}$.

In particular that we have also proved that $(\tilde{u}^+, \tilde{u}^-)$ has a Lipschitz extension to $(K^\pm \cup \gamma) \cap \mathrm{B}_r(q)$ which, by (5.76), collapses at the interface $(\gamma \cap \mathrm{B}_r(q), Q^\star [\![\psi]\!])$. We next extend $\tilde{u}^\pm$ to the whole $\Omega^\pm$, and denote by $u^\pm$, keeping the Lipschitz estimate (5.71) up to a multiplicative geometric constant, c.f., Kirszbraun Theorem. Finally, inequality (5.74) follows directly by the estimates on the bad set and the Lipschitz bound, i.e., (5.70) and (5.71). Concerning inequality (5.73), we obtain it by (5.69) and (5.71). To conclude the proof of the theorem, we notice that equation (5.72) follows from (5.65).

□

6 Center manifolds $\mathcal{M}^\pm$ with boundary Γ

In this section we work under the assumption that $0 \in \mathbb{R}^{m+n}$ is a two-sided collapsed point and that $T_0\Gamma = \mathbb{R}^{m-1} \times \{0\}$ and therefore, from Theorem 5.5, the tangent cone of T at 0 is $Q\,[\![\pi_0^+]\!] + (Q - Q^\star)\,[\![\pi_0^-]\!]$, where

$$\pi_0^\pm = \{x \in \mathbb{R}^{m+n} : \pm x_m > 0, x_{m+1} = \ldots = x_{n+m} = 0\}\,.$$

Following the notation that we have used up to know, we denote by γ the projection onto π_0 of Γ and, given any sufficiently small open set $\Omega \subset \pi_0$ in $\mathbb{R}^m$ which is contractible and contains 0, we denote by $\Omega^\pm$ the two regions in which Ω is divided by γ, i.e., the portions on the right and left of γ. In this section, we build two distinct m-dimensional submanifolds $\mathcal{M}^\pm$ of class C^3 which will be called, respectively, **left and right center manifolds**. Both center manifolds will have $\Gamma \cap \mathbf{C}(0, 3/2, \pi_0)$ as their boundary, when considered as submanifolds in the cylinder $\mathbf{C}(0, 3/2, \pi_0)$ and they will be $C^{3,\kappa}$ for a suitable positive κ up to the boundary. Additionally, at each point $p \in \Gamma \cap \mathbf{C}(0, 3/2, \pi_0)$ the tangent space to both manifolds will be the same which is the tangent cone to T at p denoted by $\pi(p)$ as in Theorem 5.5. In particular $\mathcal{M} = \mathcal{M}^+ \cup \mathcal{M}^-$ will be a $C^{1,1}$ submanifold in $\mathbf{C}(0, 3/2, \pi_0)$ without boundary.

Our aim in this section is to provide a new approximation of the current T, the way we will do this is building the center manifolds $\mathcal{M}^\pm$ which can be understood as an average of the sheets of the current T on each side of Γ, in the construction of the center manifold we will fabricate maps $\mathcal{N}^\pm$ which are defined in $\mathcal{M}^\pm$ and show that these maps $\mathcal{N}^\pm$ approximate the current in the sense of the Lipschitz approximations furnished in the previous chapters, e.g., Theorem 5.8. With respect to the final argument of this work to conclude the proof of Theorem 1.3 using Theorem 3.21, we desire to prove that $\mathcal{N}^\pm$ is identically zero and thus the current has to satisfies $T \llcorner \mathbf{C}(0, 3/2, \pi_0) = Q\,[\![\mathcal{M}^+]\!] + (Q - Q^\star)\,[\![\mathcal{M}^-]\!]$ which assures that 0 is a two-sided regular point of T. This strategy will be developed in the remaining part of this work where we begin with the construction of the center manifolds and the approximating maps, after that we use the theory of $(Q - \frac{Q^\star}{2})$-Dir minimizing maps, see Section 4, to obtain that $\mathcal{N}^\pm|_{\mathcal{M}^\pm} \equiv 0$.

6.1 Construction of the Whitney decomposition

Since the algorithm is the same for both sides of γ, it means that we can repeat the same frame to build both center manifolds. We can focus without loss of generality on the construction of $\mathcal{M}^+$. We start by describing a procedure which furnishes a suitable Whitney-type decomposition of $\mathrm{B}^+_{3/2}(0)$ with cubes whose sides are parallel to the coordinate axes and have sidelength $2\ell(L)$. The center of any such cube L will be denoted by c_L and its sidelength will be denoted by $2\ell(L)$. We start by introducing a family of dyadic cubes $L \subset \pi_0$ in the following way: for $j \geq N_0$, where N_0 is an integer whose choice will be specified below, we introduce the families

$$\mathscr{C}_j := \{L : L \text{ is a dyadic cube of side } \ell(L) = 2^{-j} \text{ and } \mathrm{B}^+_{3/2}(0) \cap L \neq \emptyset\},$$

For each L define a radius

$$r_L := M_0\sqrt{m}\ell(L),$$

with $M_0 \geq 1$ to be chosen later. We then subdivide $\mathscr{C} := \cup_j \mathscr{C}_j$ into, respectively, **boundary cubes** and **non-boundary cubes**

$$\mathscr{C}^\flat := \{L \in \mathscr{C} : \mathrm{dist}(c_L, \gamma) < 64 r_L\}, \quad \mathscr{C}^\flat_j = \mathscr{C}^\flat \cap \mathscr{C}_j,$$
$$\mathscr{C}^\natural := \{L \in \mathscr{C} : \mathrm{dist}(c_L, \gamma) \geq 64 r_L\}, \quad \mathscr{C}^\natural_j = \mathscr{C}^\natural \cap \mathscr{C}_j.$$

Observe that some boundary cubes can be completely contained in $\mathrm{B}^+_{3/2}(0)$. For this reason we prefer to use the term "non-boundary" rather than "interior" for the cubes in $\mathscr{C}^\natural$. Indeed in what follows, without mentioning it any further, we will often use the same convention for several other subfamilies of $\mathscr{C}$.

Definition 6.1. *If $H, L \in \mathscr{C}$ we say that:*

- *H is a* **descendant** *of L and L is an* **ancestor** *of H, if $H \subset L$;*
- *H is a* **child** *of L and L is the* **parent** *of H, if $H \subset L$ and $\ell(H) = \frac{1}{2}\ell(L)$;*
- *H and L are* **neighbors** *if $\frac{1}{2}\ell(L) \leq \ell(H) \leq \ell(L)$ and $H \cap L \neq \emptyset$.*

Note, in particular, the following elementary consequence of the subdivision of $\mathscr{C}$:

Lemma 6.2. *Let H be a boundary cube. Then any ancestor L and any neighbor L with $\ell(L) = 2\ell(H)$ is necessarily a boundary cube. In particular: the descendant of a non-boundary cube is a non-boundary cube.*

Proof. For the case of ancestors it suffices to prove that if L is the parent of a boundary cube H, then L is a boundary cube. Since the parent of H is a neighbor of H with $\ell(L) = 2\ell(H)$, we only need to show the second part of the statement of the lemma. The latter is a simple consequence of the following chain of inequalities:

$$\begin{aligned}\mathrm{dist}(c_L, \gamma) &\leq \mathrm{dist}(c_H, \gamma) + |c_H - c_L| = \mathrm{dist}(c_H, \gamma) + 3\sqrt{m}\ell(H) \\ &< 64 r_H + 3\frac{r_H}{M_0} \leq \left(64 + 3M_0^{-1}\right)\frac{r_L}{2} \leq \frac{67}{2} r_L < 64 r_L.\end{aligned}$$

□

Definition 6.3 (Satellite balls). *Following the notations above, we set:*

(i) *If* $L \in \mathscr{C}^\natural$*, then we define the* **non-boundary satellite ball** $\mathbf{B}_L = \mathbf{B}(p_L, 64r_L)$ *where* $p_L \in \mathrm{spt}(T)$ *such that* $\mathbf{p}_{\pi_0}(p_L) = c_L$*, such* p_L *is a priori not unique, and* π_L *is a plane which minimizes the excess in* $\mathbf{B}_L$*, namely* $\mathbf{E}(T, \mathbf{B}_L) = \mathbf{E}(T, \mathbf{B}_L, \pi_L)$*,*

(ii) *If* $L \in \mathscr{C}^\flat$*, then we define the* **boundary satellite ball** $\mathbf{B}_L^\flat = \mathbf{B}\big(p_L^\flat, 2^7 64 r_L\big)$ *where* $p_L^\flat$ *is such that* $|\mathbf{p}_{\pi_0}(p_L^\flat) - c_L| = \mathrm{dist}(c_L, \gamma)$*. Note that in this case the point* $p_L^\flat$ *is uniquely determined because* Γ *is regular and* $\mathbf{A}$ *is assumed to be sufficiently small. Likewise* π_L *is a plane which minimizes the excess* $\mathbf{E}^\flat$*, namely such that* $\mathbf{E}^\flat(T, \mathbf{B}_L^\flat) = \mathbf{E}(T, \mathbf{B}_L^\flat, \pi_L)$ *and* $T_{p_L^\flat}\Gamma \subset \pi_L$*.*

A simple corollary of Theorem 5.3 and Corollary 5.7 is the following lemma.

Lemma 6.4. *Let* T *and* Γ *be as in Assumption 2. Then there is a positive dimensional constant* $C(m,n)$ *such that, if the starting size of the Whitney decomposition satisfies* $2^{N_0} \geq C(m,n)M_0$*, then the satellite balls* $\mathbf{B}_L^\flat$ *and* $\mathbf{B}_L$ *are all contained in* $\mathbf{B}_2$*. Moreover, there exists* ε_1 *such that, for any choice of* $M_0, \alpha_{\mathbf{e}} > 0$ *and* $\alpha_{\mathbf{h}} < \frac{1}{2}$*, if*

$$\mathbf{E}^\flat(T, \mathbf{B}_2) + \|\psi\|^2_{C^{3,a_0}} < \varepsilon_1\,, \tag{6.1}$$

then for every cube $L \in \mathscr{C}^\flat$ *we have*

$$\mathbf{E}^\flat(T, \mathbf{B}_L^\flat) \leq C_0 \varepsilon_1 r_L^{2-2\alpha_{\mathbf{e}}}, \tag{6.2}$$

$$\mathbf{h}(T, \mathbf{B}_L^\flat, \pi_L) \leq C_0 \varepsilon_1^{1/4} r_L^{1+\alpha_{\mathbf{h}}}, \tag{6.3}$$

$$|\pi_L - \pi_0| \leq C_0 \varepsilon_1^{1/2}, \tag{6.4}$$

$$\Big|\pi_L - \pi(p_L^\flat)\Big| \leq C_0 \varepsilon_1^{1/2} r_L^{1-\alpha_{\mathbf{e}}}, \tag{6.5}$$

where, $\pi(p_L^\flat)$ *is the* m*-dimensional tangent plane supporting the tangent cone to* T *at* $p_L^\flat$ *and* C_0 *depends only upon* $\alpha_{\mathbf{e}}$*,* $\alpha_{\mathbf{h}}$*,* m *and* n*.*

Proof. Take $x \in \mathbf{B}_L^\natural$, using the height bound in (iii) of Assumption 4 we obtain that

$$|x| \leq 64 r_L + |p_L| \leq 64\sqrt{m} M_0 2^{-N_0} + |c_L| + C\varepsilon_0^{1/2}\,|p_L|\,,$$

recalling that $c_L \in \mathbf{B}(0, 3/2)$, possibly choosing ε_0 small enough and taking the constants $C(m,n)$, N_0 big enough, we certainly obtain that $\mathbf{B}_L^\natural \subset \mathbf{B}(0,2)$. The proof that $\mathbf{B}_L^\flat \subset \mathbf{B}(0,2)$ is analogous with the exception that we might multiply the dimensional constant by 2^7. Inequality (6.4) is a direct consequence of (5.37). In this proof we will use the improved excess decay, i.e., Theorem 5.3, with $q = p_L^\flat, p = 0, r = 1, \rho = 2^7 64 r_L$. To prove estimate (6.2), we do as follows

$$\mathbf{E}^\flat(T, \mathbf{B}_L^\flat) = \mathbf{E}^\flat(T, \mathbf{B}\Big(p_L^\flat, 2^7 64 r_L\Big)) \overset{(5.4)}{\leq} C(2^7 64 r_L)^{2-2\alpha_{\mathbf{e}}} \mathbf{E}^\flat(T, \mathbf{B}(0,2)) + C(2^7 64 r_L)^{2-2\alpha_{\mathbf{e}}} \mathbf{A}^2,$$

and thus (6.1) concludes the proof of (6.2). With the same argument we prove (6.3) using in this time the height bound given by the excess decay, i.e., (5.5). It remains to prove (6.5), by the monotonicity formula, and recalling that $\Theta(T, p_L^\flat) = Q - \frac{Q^\star}{2} \geq \frac{3}{2}$, we have

$$\|T\|(\mathbf{B}_L^\flat) \geq \omega_m (2^7 64 r_L)^m.$$

Therefore, by the improved excess decay, we obtain

$$\mathbf{E}(T,\mathbf{B}^\flat_L,\pi_L) \le \mathbf{E}(T,\mathbf{B}^\flat_L,\pi(p^\flat_L)) \overset{(5.4),(6.1)}{\le} C_0\varepsilon_1 r_L^{2-2\alpha_\mathbf{e}}\,.$$

Hence

$$|\pi(p^\flat_L)-\pi_L|^2 \le C_0\big(\mathbf{E}(T,\mathbf{B}^\flat_L,\pi_L)+\mathbf{E}(T,\mathbf{B}^\flat_L,\pi(p^\flat_L))\big) \le C_0\varepsilon_1 r_L^{2-2\alpha_\mathbf{e}}\,.$$

□

6.2 Stopping conditions of the Whitney decomposition

We will now start to refine our Whitney decomposition putting into account the properties of small excess and height bound of the current, in the sense of Lemma 6.4, to then obtain further stronger information about the current on each cube of the decomposition. To this end let $C_\mathbf{e}, C_\mathbf{h}$ be two large positive constants that will be fixed later. We take a cube $L \in \mathscr{C}_{N_0}$ and we **do not** subdivide it if either the excess "is too big" or the current "is too high", precisely if it belongs to one of the following sets:

(1) $\mathscr{W}^\mathbf{e}_{N_0} := \{L \in \mathscr{C}^\natural_{N_0} : \mathbf{E}(T,\mathbf{B}_L) > C_\mathbf{e}\varepsilon_1\ell(L)^{2-\alpha_\mathbf{e}}\}$;

(2) $\mathscr{W}^\mathbf{h}_{N_0} := \{L \in \mathscr{C}^\natural_{N_0} : \mathbf{h}(T,\mathbf{B}_L,\pi_L) > C_\mathbf{h}\varepsilon_1^{1/2m}\ell(L)^{1+\alpha_\mathbf{h}}\}$.

We then define

$$\mathscr{S}_{N_0} := \mathscr{C}_{N_0} \setminus \left(\mathscr{W}^\mathbf{e}_{N_0} \cup \mathscr{W}^\mathbf{h}_{N_0}\right)\,.$$

The cubes in $\mathscr{S}_{N_0}$ will be subdivided in their childs, it means that we are subdividing the cube whenever the current is well behaved in it. In what follows we aim to show that the current is well behaved in the whole ball, it means that we will ensure that $\mathscr{W}_{N_0} := \mathscr{W}^\mathbf{e}_{N_0} \cup \mathscr{W}^\mathbf{h}_{N_0} = \emptyset$, and therefore $\mathscr{C}_{N_0} = \mathscr{S}_{N_0}$, by choosing $C_\mathbf{e}$ and $C_\mathbf{h}$ large enough, depending only upon $\alpha_\mathbf{h}, \alpha_\mathbf{e}, M_0$ and N_0, see Proposition 6.6 below.

We next describe the refining procedure assuming inductively that for a certain step $j \ge N_0+1$ we have defined the families $\mathscr{W}_{j-1}$ and $\mathscr{S}_{j-1}$. In particular we consider all the cubes L in $\mathscr{C}_j$ which are contained in some element of $\mathscr{S}_{j-1}$. Among them we select and set aside in the classes $\mathscr{W}_j := \mathscr{W}^\mathbf{e}_j \cup \mathscr{W}^\mathbf{h}_j \cup \mathscr{W}^\mathbf{n}_j$ those cubes where the following stopping criteria are met:

(1) $\mathscr{W}^\mathbf{e}_j := \{L \text{ child of } K \in \mathscr{S}^\natural_{j-1} : \mathbf{E}(T,\mathbf{B}_L) > C_\mathbf{e}\varepsilon_1\ell(L)^{2-2\alpha_\mathbf{e}}\}$,

(2) $\mathscr{W}^\mathbf{h}_j := \{L \text{ child of } K \in \mathscr{S}^\natural_{j-1} : L \notin \mathscr{W}^\mathbf{e}_j \text{ and } \mathbf{h}(T,\mathbf{B}_L,\pi_L) > C_\mathbf{h}\varepsilon_1^{1/2m}\ell(L)^{1+\alpha_\mathbf{h}}\}$,

(3) $\mathscr{W}^\mathbf{n}_j := \{L \text{ child of } K \in \mathscr{S}_{j-1} : L \notin \mathscr{W}^\mathbf{e}_j \cup \mathscr{W}^\mathbf{h}_j \text{ but } \exists L' \in \mathscr{W}_{j-1} \text{ with } L \cap L' \ne \emptyset\}$.

We keep refining the decomposition in the set

$$\mathscr{S}_j := \{L \in \mathscr{C}_j \text{ child of } K \in \mathscr{S}_{j-1}\} \setminus \mathscr{W}_j\,.$$

Observe that it might happen that the child of a cube in $\mathscr{S}_{j-1}$ does not intersect $\mathrm{B}^+_{3/2}(0)$: in that case, according to our definition, the cube does not belong to $\mathscr{S}_j$ neither to $\mathscr{W}_j$: it is simply discarded. As already mentioned, we use the notation $\mathscr{S}_j^\flat$ and $\mathscr{S}_j^\natural$ respectively for $\mathscr{S}_j \cap \mathscr{C}^\flat$ and $\mathscr{S}_j \cap \mathscr{C}^\natural$. Furthermore we set

$$\begin{aligned}
\mathscr{W} &:= \bigcup_{j\geq N_0} \mathscr{W}_j,\\
\mathscr{S} &:= \bigcup_{j\geq N_0} \mathscr{S}_j,\\
\mathbf{S}^+ &:= \bigcap_{j\geq N_0} \Big(\bigcup_{L\in\mathscr{S}_j} L\Big) = \mathrm{B}^+_{3/2}(0) \setminus \bigcup_{H\in\mathscr{W}} H\,.
\end{aligned}$$

Note, in particular, that the refinement of boundary cubes can *never* be stopped because of the conditions (1) and (2), as we state in the following.

Lemma 6.5. *$\mathscr{C}_j^\flat \cap \mathscr{W}_i = \emptyset$ for every $i, j \geq N_0$ and in particular $\gamma \cap \mathrm{B}^+_{3/2}(0) \subset \mathbf{S}^+$. Thus boundary cubes always belong to $\mathscr{S}$.*

Proof. Assume there is a boundary cube in $\mathscr{W}_i$, then let L be a boundary cube in $\mathscr{W}_i$ with largest side length. The latter must then belong to $\mathscr{W}_i^{\mathbf{n}}$ because Lemma 6.4 excludes the possibility of L to belong to either $\mathscr{W}_j^{\mathbf{e}}$ or $\mathscr{W}_j^{\mathbf{h}}$. However, by definition of the family, this would imply the existence of a neighbor $L' \in \mathscr{W}_i$ with $\ell(L') = 2\ell(L)$. By Lemma 6.2, L' would be a boundary cube in $\mathscr{W}$, contradicting the maximality of L. □

Clearly, descendants of boundary cubes might become non-boundary cubes and so their refining cubes can be stopped. We finally set $\mathscr{W}_j := \mathscr{W}_j^{\mathbf{e}} \cup \mathscr{W}_j^{\mathbf{h}} \cup \mathscr{W}_j^{\mathbf{n}}$. From now on we specify a set of assumptions on the various choices of the constants involved in the construction.

Assumption 5. *T and Γ are as in Assumptions 2 and we also assume that*

(a) $\alpha_{\mathbf{h}}$ is smaller than $\frac{1}{2m}$ and $\alpha_{\mathbf{e}}$ is positive but small, depending only on $\alpha_{\mathbf{h}}$,

(b) M_0 is larger than a suitable constant, depending only upon $\alpha_{\mathbf{e}}$,

(c) $2^{N_0} \geq C(m, n, M_0)$, in particular it satisfies the condition of Lemma 6.4,

(d) $C_{\mathbf{e}}$ is sufficiently large depending upon $\alpha_{\mathbf{e}}$, $\alpha_{\mathbf{h}}$, M_0 and N_0,

(e) $C_{\mathbf{h}}$ is sufficiently large depending upon $\alpha_{\mathbf{e}}, \alpha_{\mathbf{h}}, M_0, N_0$ and $C_{\mathbf{e}}$,

(f) (6.1) holds with an ε_1 sufficiently small depending upon all the other parameters.

Finally, there is an exponent $\alpha_{\mathbf{L}}$, which depends only on $m, n, Q^\star$ and Q and which is independent of all the other parameters, in terms of which several important estimates in Theorem 6.19 will be stated.

We are ensuring that there is a nonempty set of parameters satisfying all the requirements, since the parameters are chosen following a precise hierarchy. The hierarchy is consistent with that of [13], the reader could compare Assumption 5 with [13, Assumption 1.9].

6.3 Tilting optimal planes and L-interpolating functions

In this section, we will define the interpolating functions which will give rise to the function $\boldsymbol{\varphi}^+$ whose graphs will define the center manifold $\mathcal{M}^+$. In order to begin with the construction of these objects, we shall notice that an important fact is that up to now, in the construction of the decomposition, we have nice local information about the current, i.e., inside each square of the decomposition we can do a good analysis, however, we do not know how to work among those cubes, i.e., how to compare quantities as the excess and height between two different cubes of the decomposition. To that end, we enunciate the following crucial result which is the analogous of [7, Proposition 8.24] which is stated for $Q^\star = 1$, we mention that the proof of this proposition readily works for currents with boundary multiplicity equal to $Q^\star \geq 1$.

Proposition 6.6 (Tilting and optimal planes, Proposition 8.24, [7]). *Under the Assumptions 4 and 5, we have $\mathscr{W}_{N_0} = \emptyset$. Then the following estimates hold for any couple of neighbors $H, L \in \mathscr{S} \cup \mathscr{W}$ and for every $H, L \in \mathscr{S} \cup \mathscr{W}$ with H descendant of L:*

(a) denoting by π_H, π_L the optimal planes for the excess in $\mathbf{B}_H$ and $\mathbf{B}_L$, respectively, we have

$$|\pi_H - \pi_L| \leq \bar{C}\varepsilon_1^{1/2}\ell(L)^{1-\alpha_{\mathbf{e}}}, \qquad |\pi_H - \pi_0| \leq \bar{C}\varepsilon_1^{1/2},$$

(b)$^\natural$ $\mathbf{h}(T, \mathbf{C}_{48r_H}(p_H, \pi_0)) \leq C\varepsilon_1^{1/2m}\ell(H)$ *and* $\mathrm{spt}(T) \cap \mathbf{C}_{48r_H}(p_H, \pi_0) \subset \mathbf{B}_H$ *if* $H \in \mathscr{C}^\natural$,

(b)$^\flat$ $\mathbf{h}(T, \mathbf{C}_{2^7 48r_H}(p_H^\flat, \pi_0)) \leq C\varepsilon_1^{1/4}\ell(H)$ *and* $\mathrm{spt}(T) \cap \mathbf{C}_{2^7 48r_H}(p_H^\flat, \pi_0) \subset \mathbf{B}_H^\flat$ *if* $H \in \mathscr{C}^\flat$,

(c)$^\natural$ $\mathbf{h}(T, \mathbf{C}_{36r_L}(p_L, \pi_H)) \leq C\varepsilon_1^{1/2m}\ell(L)^{1+\alpha_{\mathbf{h}}}$ *and* $\mathrm{spt}(T) \cap \mathbf{C}_{36r_L}(p_L, \pi_H) \subset \mathbf{B}_L$ *if* $H, L \in \mathscr{C}^\natural$,

(c)$^\flat$ $\mathbf{h}(T, \mathbf{C}_{2^7 36r_L}(p_L^\flat, \pi_H)) \leq C\varepsilon_1^{1/4}\ell(L)^{1+\alpha_{\mathbf{h}}}$ *and* $\mathrm{spt}(T) \cap \mathbf{C}_{2^7 36r_L}(p_L^\flat, \pi_H)) \subset \mathbf{B}_L^\flat$ *if* $L \in \mathscr{C}^\flat$,

where $\bar{C} = \bar{C}(\alpha_{\mathbf{e}}, \alpha_{\mathbf{h}}, M_0, N_0, C_{\mathbf{e}}) > 0$ *and* $C = C(\alpha_{\mathbf{e}}, \alpha_{\mathbf{h}}, M_0, N_0, C_{\mathbf{e}}, C_{\mathbf{h}}) > 0$.

We now state the following results which is the analogous of [7, Proposition 8.7] will allow us to locally approximate the current by $(Q - \frac{Q^\star}{2})$-Lipschitz maps in the sense of Theorem 5.8. We also noticed that the proof given in [7, Proposition 8.7] for $Q^\star = 1$ readily works for currents with boundary multiplicity equal to $Q^\star \geq 1$.

Proposition 6.7 (Proposition 8.7, [7]). *Under the Assumptions 4 and 5 the following holds for every couple of neighbors $H, L \in \mathscr{S} \cup \mathscr{W}$ and any $H, L \in \mathscr{S} \cup \mathscr{W}$ with H descendant of L:*

$$\mathrm{spt}(T) \cap \mathbf{C}_{36r_L}(p_L, \pi_H) \subset \mathbf{B}_L \qquad \text{when } L \in \mathscr{C}^\natural,$$
$$\mathrm{spt}(T) \cap \mathbf{C}_{2^7 36r_L}(p_L^\flat, \pi_H) \subset \mathbf{B}_L^\flat \qquad \text{when } L \in \mathscr{C}^\flat,$$

and the current T satisfies the assumptions of [11, Theorem 2.4] in the cylinder $\mathbf{C}_{36r_L}(p_L, \pi_H)$ (resp. of Theorem 5.8 in the cylinder $\mathbf{C}_{2^7 36r_L}(p_L^\flat, \pi_H)$).

We will now construct the "interpolating functions" g_L for each cube L. To begin with the construction of this interpolation, we approximate the current T by $(Q - \frac{Q^\star}{2})$-Lipschitz functions (Proposition 6.7) that will determine the boundary condition of an elliptic system which comes from the linearization of the mean curvature condition for minimal surfaces. The solution of this elliptic system will be further represented by a function g_L defined in the tilted ball contained in

π_0 taking values in $\pi_0^\perp$, i.e., we changed to our new reference coordinate system. Since the construction will be local, i.e., in each cube, over the set $\mathrm{B}^+_{3/2}(0) \setminus \mathbf{S}^+$ we will patch every g_L together with a partition of the unity to obtain the function $\boldsymbol{\varphi}^+$, whose graph will be the center manifold, defined in the whole ball. So we need to define $\boldsymbol{\varphi}^+$ over $\mathbf{S}^+$ as well, to that end we introduce all the machinery needed for all cubes in $\mathscr{S} \cup \mathscr{W}$.

Definition 6.8 (π_L-approximations). *Under the Assumptions 4 and 5, we set*

(i) If $L \in \mathscr{S}_j^\flat$ for some j, take the Lipschitz approximation (f_L^+, f_L^-) in the cylinder $\mathbf{C}\big(p_L^\flat, 2^7 9 r_L, \pi_L\big)$ given by Proposition 6.7, we call (f_L^+, f_L^-) **a π_L-approximation of T in the cylinder $\mathbf{C}\big(p_L^\flat, 2^7 9 r_L, \pi_L\big)$**.

(ii) If $L \in \mathscr{S}_j^\natural \cup \mathscr{W}_j$ for some j, we take the Lipschitz approximation f_L in the cylinder $\mathbf{C}(p_L, 9r_L, \pi_L)$ given by Proposition 6.7, we call f_L **a π_L-approximation of T in the cylinder $\mathbf{C}(p_L, 9r_L, \pi_L)$**.

Definition 6.9 (L-tilded harmonic interpolations). *Under the Assumptions 4 and 5, we define*

(i) if L is a nonboundary cube, let $h_L : \mathrm{B}_{5r_L}(p_L, \pi_L) \to \mathbb{R}$ to be an harmonic function with boundary condition $h_L|_{\partial \mathrm{B}_{5r_L}(p_L,\pi_L)} = \boldsymbol{\eta} \circ f_L|_{\partial \mathrm{B}_{5r_L}(p_L,\pi_L)}$,

(ii) if L is a boundary cube, let $h_L : \mathrm{B}_{2^7 5 r_L}(p_L^\flat, \pi_L) \to \mathbb{R}$ to be an harmonic function with boundary condition $h_L|_{\partial \mathrm{B}_{2^7 5 r_L}(p_L^\flat,\pi_L)} = \boldsymbol{\eta} \circ f_L^+|_{\partial \mathrm{B}_{2^7 5 r_L}(p_L^\flat,\pi_L)}$.

We call h_L **L-tilted harmonic interpolating function**.

We now are ready to define the final function, g_L, on our "reference coordinate system", i.e., the domain of g_L is contained in π_0 and g_L takes values in $\pi_0^\perp$, with the property that its graph coincides with a suitable portion of the graph of h_L. The function g_L is furnished by [13, Lemma B.1] which we state below.

Proposition 6.10 (L-interpolating functions). *Under the assumptions of Proposition 6.7, we have*

(i) If L is a boundary cube, the function h_L is Lipschitz on $\mathrm{B}^\pm_{2^7 \cdot 9 r_L/2}(p_L^\flat, \pi_L)$ and we can define a function $g_L : \mathrm{B}^+_{2^7 4 r_L}(p_L^\flat, \pi_0) \to \pi_0^\perp$ such that $\mathbf{G}_{g_L} = \mathbf{G}_{h_L} \llcorner \mathrm{B}^+_{2^7 4 r_L}(p_L^\flat, \pi_0) \times \mathbb{R}^n$,

(ii) If L is a non-boundary cube, the function h_L is Lipschitz on $\mathrm{B}_{9r_L/2}(p_L, \pi_L)$ and we can define a function $g_L : \mathrm{B}_{4r_L}(p_L, \pi_0) \to \pi_0^\perp$ such that $\mathbf{G}_{g_L} = \mathbf{G}_{h_L} \llcorner \mathbf{C}_{4r_L}(p_L, \pi_0)\big)$.

The functions g_L is called **L-interpolating function**.

6.4 Glueing L-interpolations

We now define another set of cubes, the Whitney cubes at the step j, which will be similar to what we have constructed until now but we are including all the ancestors with respect to step j in the same family as follows

$$\mathscr{P}_j := \mathscr{S}_j \cup \bigcup_{i=N_0+1}^{j} \mathscr{W}_i.$$

Note that $\mathscr{P}_j$ is a "Whitney family of dyadic cubes" in the sense that if $K, L \in \mathscr{P}_j$ and $K \cap L \neq \emptyset$, then $\frac{1}{2}\ell(L) \leq \ell(K) \leq 2\ell(L)$. We fix a mollifier function ϑ satisfying

$$\vartheta \in C_c^\infty\left(\left[-\frac{17}{16}, \frac{17}{16}\right]^m, [0,1]\right), \ \vartheta|_{[-1,1]^m} \equiv 1, \text{ and, for each cube } L, \ \tilde{\vartheta}_L(y) := \vartheta\left(\frac{y - c(L)}{\ell(L)}\right).$$

We thus set a partition of unity of $\mathrm{B}^+_{3/2}(0)$ defined as

$$\vartheta_L : \mathrm{B}^+_{3/2}(0) \to \mathbb{R}, \ \vartheta_L(y) := \frac{\tilde{\vartheta}_L(y)}{\sum_{H \in \mathscr{P}_j} \tilde{\vartheta}_H(y)}.$$

Definition 6.11 (Glued interpolation at the step j)**.** *We set $\varphi_j := \sum_{L \in \mathscr{P}_j} \vartheta_L g_L$, this maps φ_j are called* **glued interpolation at the step j.**

6.5 Existence of a $C^{3,\kappa}$-center manifold

We are now ready to state the main theorem regarding the construction of the center manifolds needed in this paper, i.e., Theorem 6.12, which ensures that $(\varphi_j)_j$ is a sequence that converges to a $C^{3,\kappa}$ map, $\kappa > 0$, whose graph will be called the center manifold. This limit map has good properties as the smallness of the $C^{3,\kappa}$ norm, which is bounded by ε_1. After the main theorem, we will start the construction of the normal approximation in the sense of Theorem 5.8 but now the approximations will be defined on the center manifold and will take values on the normal bundle of the center manifold. The normal approximations will enjoy some good estimates relying in the estimates of Theorem 5.8 and the estimate obtained in the construction of our Whitney decomposition. The main theorem of this section is stated below and is the version adapted to our setting of Theorem 8.13 of [7].

Theorem 6.12 (Theorem 8.13 ,[7])**.** *Under Assumptions 4 and 5, there is a $\kappa := \kappa(\alpha_{\mathbf{e}}, \alpha_{\mathbf{h}}) > 0$, such that*

(i) $\varphi_j \in C^{3,\kappa}$, moreover $\|\varphi_j\|_{3,\kappa,\mathrm{B}^+_{3/2}(0)} \leq C\varepsilon_1^{1/2}$, for some $C := C(\alpha_{\mathbf{e}}, \alpha_{\mathbf{h}}, M_0, C_{\mathbf{e}}, C_{\mathbf{h}}) > 0$,

(ii) If $i \leq j$, $L \in \mathscr{W}_{i-1}$ and H is a cube concentric to L with $\ell(H) = \frac{9}{8}\ell(L)$, then $\varphi_j = \varphi_i$ on H,

(iii) φ_j converges in C^3 to a map $\varphi^+ : \mathrm{B}^+_{3/2}(0) \to \mathbb{R}^n$, whose graph is a $C^{3,\kappa}$ submanifold $\mathcal{M}^+$, which will be called the **right center manifold***;*

(iv) $\varphi^+ = \psi$ on $\gamma \cap B_{3/2}$, i.e., $\partial\mathcal{M}^+ \cap \mathbf{C}(0, 3/2) = \Gamma \cap \mathbf{C}(0, 3/2)$;

(v) For any $q \in \partial\mathcal{M}^+ \cap \mathbf{C}(0, 3/2)$, we have $T_q\mathcal{M}^+ = \pi(q)$ where $\pi(q)$ is the support of the unique tangent cone to T at q.

We will omit the proof of 6.12 because goes mutatis mutandis along the same lines of [7, Theorem 8.13] and even if this last theorem is stated just when $Q^\star = 1$, a careful inspection of its proof will reveal that it also holds when $Q^\star \geq 1$.

Remark 6.13. *The construction of $\mathcal{M}^+$ made in Theorem 6.12 is based on the decomposition of $\mathrm{B}^+_{3/2}(0)$. Under Assumption 5, the same construction can be made for $\mathrm{B}^-_{3/2}(0)$ and gives a $C^{3,\kappa}$ map $\varphi^- : \mathrm{B}^-_{3/2}(0) \to \mathbb{R}^n$ which agrees with ψ on $\gamma \cap B_{3/2}$. The graph of φ^- is a $C^{3,\kappa}$*

submanifold $\mathcal{M}^-$, which will be called the **left center manifold**. *Its boundary in the cylinder* $\mathbf{C}(0,3/2)$*, namely* $\mathbf{C}(0,3/2)\cap\partial\mathcal{M}^-$*, coincides, in a set-theoretical sense, with* $\mathbf{C}(0,3/2)\cap\partial\mathcal{M}^+$*, but it has opposite orientation, and moreover its tangent plane* $T_q\mathcal{M}^-$ *coincides with* $\pi(q)$ *for every point* $q\in\mathbf{C}(0,3/2)\cap\partial\mathcal{M}^-$*.*

In particular, the union $\mathcal{M}:=\mathcal{M}^+\cup\mathcal{M}^-$ of the two submanifolds is a $C^{1,1}$ submanifold in $\mathbf{C}(0,3/2)$ without boundary in $\mathbf{C}(0,3/2)$, which will be called the **center manifold**. Moreover, we will often state properties of the center manifold related to cubes L in one of the collections $\mathscr{W}_j$ described above. Therefore, we will denote by $\mathscr{W}^+$ the union of all $\mathscr{W}_j$ and by $\mathscr{W}^-$ the union of the corresponding classes of cubes which lead to the left center manifold $\mathcal{M}^-$. We emphasize again that so far we can only conclude the $C^{1,1}$ regularity of $\mathcal{M}$, because we do not know that the traces of the second derivatives of φ^+ and φ^- coincide on γ.

Definition 6.14. *Let us define the graph parametrization map of* $\mathcal{M}^+$ *as* $\mathbf{\Phi}^+(x):=(x,\varphi^+(x))$*. We will call* **right contact set** *the subset* $\mathbf{K}^+:=\mathbf{\Phi}^+(\mathbf{S}^+)$*. For every cube* $L\in\mathscr{W}^+$ *we associate a* **Whitney region** $\mathcal{L}$ **on** $\mathcal{M}^+$ *as follows:*

- $\mathcal{L}:=\mathbf{\Phi}^+(H\cap \mathrm{B}_1(0))$ *where* H *is the cube concentric to* L *such that* $\ell(H)=\frac{17}{16}\ell(L)$*.*

Analogously we define the map $\mathbf{\Phi}^-$*, the* **left contact set** $\mathbf{K}^-$ *and the* **Whitney regions on the left center manifold** $\mathcal{M}^-$*.*

To keep the text flow, we postpone the proof of the Theorem 6.12 to the last part of this section.

6.6 The $\mathcal{M}$-Lipschitz approximations defined on the center manifolds

Since the two portions $\mathcal{M}^-$ and $\mathcal{M}^+$ are $C^{3,\kappa}$ and they join with $C^{1,1}$ regularity along Γ, in a sufficiently small normal neighborhood of $\mathcal{M}$ there is a well defined orthogonal projection $\mathbf{p}$ onto $\mathcal{M}$. The thickness of the tubular neighborhood is inversely proportional to the norm of the second derivatives of $\varphi^\pm$ and hence, for ε_1 sufficiently small, we can assume that the thickness is 2 which leads to the next assumption.

Assumption 6. *Under Assumptions 4 and 5. We let* $\mathcal{M}:=\mathcal{M}^+\cup\mathcal{M}^-$ *and* ε_1 *be sufficiently small so that, if*

$$\mathbf{U}:=\{q\in\mathbb{R}^{m+n}:\exists! q'=\mathbf{p}(q)\in\mathcal{M}\ s.t.\ |q-q'|<1 \text{ and } q-q'\in T^\perp_{q'}\mathcal{M}\}\,,$$

where $T^\perp_{q'}\mathcal{M}:=(T_{q'}\mathcal{M})^\perp$*, then the map* $\mathbf{p}$ *extends to a Lipschitz map to the closure* $\overline{\mathbf{U}}$ *which is* $C^{2,\kappa}$ *on* $\mathbf{U}\setminus\mathbf{p}^{-1}(\Gamma)$ *and*

$$\mathbf{p}^{-1}(q')=q'+\overline{\mathrm{B}_1(0,(T_{q'}\mathcal{M})^\perp)} \text{ for all } q'\in\overline{\mathcal{M}}.$$

As highlighted before, we construct the center manifold $\mathcal{M}$ and also a function defined on $\mathcal{M}$ that approximates, with the desired superlinear exponents for the error, the current T (in the sense of Theorem 5.8). This approximation will take values on the normal bundle of $\mathcal{M}$, we define precisely this type of approximations. But before that, we shall also define the space of Q-tuples on a manifold analogously to what is done for Euclidean spaces in the Introduction of [10].

Definition 6.15. *Let M be an m-dimensional manifold, and, for each $P \in M$, we denote $[\![P]\!]$ the current with support equal to P, i.e., the current associated with the Dirac measure concentrated in P. Then we define the space of unordered Q-tuples in M, for any $Q \in \mathbb{N}, Q \geq 1$, as follows*

$$\mathcal{A}_Q(M) := \left\{ \sum_{i=1}^{Q} [\![P_i]\!] : P_i \in M \text{ for every } i \in \{1, \dots, Q\} \right\}.$$

Definition 6.16. *Let $\mathcal{M}$ be the center manifold as in Theorem 6.12 without loss of generality we can assume that we are under Assumption 6, $Q^+ = Q$ and $Q^- = Q - Q^\star$. We say that $(\mathcal{K}, F^+, F^-)$ is an* **$\mathcal{M}$-normal approximation of** *T, if*

(i) there exist Lipschitz functions $\mathcal{N}^+ : \mathcal{M}^+ \cap \mathbf{C}(0,1) \to \mathcal{A}_Q(T^\perp \mathcal{M}^+)$, $\mathcal{N}^+(x) \in \mathcal{A}_Q(T_x^\perp \mathcal{M}^+)$ and $\mathcal{N}^- : \mathcal{M}^- \cap \mathbf{C}(0,1) \to \mathcal{A}_{Q-Q^\star}(T^\perp \mathcal{M}^-)$, $\mathcal{N}^-(x) \in \mathcal{A}_{Q-Q^\star}(T_x^\perp \mathcal{M}^-)$, where $T^\perp \mathcal{M}^\pm := \sqcup_{x \in \mathcal{M}^\pm} T_x^\perp \mathcal{M}^\pm$ denotes the normal bundle of $\mathcal{M}^\pm$ and is seen as a subset of $\mathbb{R}^{m+n}$,

$$\begin{array}{rccl} \mathcal{N}^\pm : & \mathcal{M}^\pm \cap \mathbf{C}(0,1) & \to & \mathcal{A}_{Q^\pm}(T^\perp \mathcal{M}^\pm) \\ & x & \mapsto & \mathcal{N}^\pm(x) := \sum_{i=1}^{Q^\pm} [\![\mathcal{N}_i^\pm(x)]\!] \,, \end{array}$$

where $(\mathcal{N}_i^\pm : \mathcal{M}^+ \cap \mathbf{C}(0,1) \to T^\perp \mathcal{M})_{i \in \{1,\dots,Q^\pm\}}$ are measurable sections of the normal bundle, i.e., each $\mathcal{N}_i^\pm$ is a classical 1-valued measurable function satisfying $\mathcal{N}_i^\pm(x) \in T_x^\perp \mathcal{M}^\pm$. We then define the Lipschitz function given by

$$\begin{array}{rccl} F^\pm : & \mathcal{M}^\pm \cap \mathbf{C}(0,1) & \to & \mathcal{A}_{Q^\pm}(T^\perp \mathcal{M}^\pm) \\ & x & \mapsto & (\mathcal{N}^\pm \oplus \mathrm{id})(x). \end{array}$$

(ii) $\mathcal{K} \subset \mathcal{M}$ is closed and $\mathbf{T}_{F^\pm} \llcorner \mathbf{p}^{-1}(\mathcal{K} \cap \mathcal{M}^\pm) = T \llcorner \mathbf{p}^{-1}(\mathcal{K} \cap \mathcal{M}^\pm)$, where $\mathbf{T}_{F^\pm} := (F^\pm)_\sharp [\![\mathcal{M}]\!]$, according to [12, Definition 1.3],

(iii) $\mathbf{K}^+ \cup \mathbf{K}^- \subset \mathcal{K}$, $\mathcal{N}^\pm|_{\mathcal{K}} \equiv 0$, and then $F^+(x) = Q [\![x]\!]$ on $\mathcal{K}^+$ and $F^-(x) = (Q - Q^\star) [\![x]\!]$ on $\mathcal{K}^-$.

Observe that the pairs (F^+, F^-) and $(\mathcal{N}^+, \mathcal{N}^-)$ are intuitively $(Q - \frac{Q^\star}{2})$-valued maps in the spirit of Definition 4.1. Although this is very intuitive, these functions are defined on manifolds, so, we make the precise definition of it as follows.

Definition 6.17. *Firstly, we let Z be an m-dimensional manifold and Υ be a $(m-1)$-submanifold of the m-manifold M which splits M into the two connected components M^+ and M^-. Let $\Phi \in W^{\frac{1}{2},2}(\Upsilon, \mathcal{A}_{Q^\star}(Z))$, $Q, Q^\star \in \mathbb{N}$, $Q \geq Q^\star \geq 1$. A* **$(Q - \frac{Q^\star}{2})$-valued function with interface** *(Υ, Φ), consists of a pair (F^+, F^-) satisfying the following properties*

(i) $F^+ \in W^{1,2}(M^+, \mathcal{A}_Q(Z))$, $F^- \in W^{1,2}(M^-, \mathcal{A}_{Q-Q^\star}(Z))$,

(ii) $F^+|_\Upsilon = F^-|_\Upsilon + \Phi$.

We define the **Dirichlet energy of** *(F^+, F^-) as $\mathrm{Dir}(F^+, F^-, M) := \mathrm{Dir}(F^+, M^+) + \mathrm{Dir}(F^-, M^-)$. Such a pair will be called* **Dir-minimizing in** *M, if for all $\left(Q - \frac{Q^\star}{2}\right)$-valued function (G^+, G^-) with interface (Υ, Φ) which agrees with (F^+, F^-) outside of a compact set $K \subset\subset M$ satisfies $\mathrm{Dir}(F^+, F^-, M) \leq \mathrm{Dir}(G^+, G^-, M)$.*

Definition 6.18. *Let* (F^+, F^-) *be a* $\left(Q - \frac{Q^\star}{2}\right)$*-valued function with interface* (Υ, Φ) *and* $\Phi = Q^\star \left[\!\left[\hat{\Phi}\right]\!\right]$ *for the single valued function* $\hat{\Phi} \in W^{\frac{1}{2},2}(\Upsilon, Z)$*. We say that* (F^+, F^-) **collapses at the interface***, if* $F^+|_\Upsilon = Q \left[\!\left[\hat{\Phi}\right]\!\right]$*.*

The following theorem ensures the existence of an $\mathcal{M}$-normal approximation suitable for our purposes, i.e., with the desired exponents at the bound on the Lipschitz constant, the Dirichlet energy and the size of the complement set of $\mathcal{K}$. It is the same as [7, Theorem 8.19] but of course adapted to our context where $Q^\star$ is taken any arbitrary positive integer. We omit its proof because goes precisely as in the proof of [7, Theorem 8.19].

Theorem 6.19 (Local behaviour of the $\mathcal{M}$-normal approximation on Whitney regions. Theorem 8.19 of [7])**.** *Under Assumption 6 there is a constant* $\alpha_{\mathbf{L}} := \alpha_{\mathbf{L}}(m, n, Q^\star, Q) > 0$ *such that there exists an* $\mathcal{M}$*-normal approximation* $(\mathcal{K}, F^+, F^-)$ *satisfying the following estimates on any Whitney region* $\mathcal{L} \subset \mathcal{M}$ *associated to a cube* $L \in \mathscr{W}^+ \cup \mathscr{W}^-$*:*

$$\mathrm{Lip}(\mathcal{N}^\pm|_\mathcal{L}) \leq C\varepsilon_1^{\alpha_{\mathbf{L}}} \ell(L)^{\alpha_{\mathbf{L}}}, \tag{6.6}$$

$$\|\mathcal{N}^\pm|_\mathcal{L}\|_0 \leq C\varepsilon_1^{1/2m} \ell(L)^{1+\alpha_{\mathbf{h}}}, \tag{6.7}$$

$$\mathcal{H}^m(\mathcal{L} \setminus \mathcal{K}) + \|\mathbf{T}_F - T\|(\mathbf{p}^{-1}(\mathcal{L})) \leq C\varepsilon_1^{1+\alpha_{\mathbf{L}}} \ell(L)^{m+2+\alpha_{\mathbf{L}}}, \tag{6.8}$$

$$\int_\mathcal{L} |D\mathcal{N}^\pm|^2 \leq C\varepsilon_1 \ell(L)^{m+2-2\alpha_{\mathbf{e}}}, \tag{6.9}$$

for a constant $C = C(\alpha_{\mathbf{e}}, \alpha_{\mathbf{h}}, M_0, N_0, C_{\mathbf{e}}, C_{\mathbf{h}}) > 0$*. Moreover, for any* $a > 0$ *and any Borel* $\mathcal{V} \subset \mathcal{L}$*,*

$$\begin{aligned} \int_\mathcal{V} |\eta \circ \mathcal{N}^\pm| \mathrm{d}\mathcal{H}^m \leq C\varepsilon_1 &\left(\ell(L)^{m+3+\alpha_{\mathbf{h}}/3} + a\ell(L)^{2+\alpha_{\mathbf{L}}/2} \mathcal{H}^m(\mathcal{V})\right) \\ &+ \frac{C}{a} \int_\mathcal{V} \mathcal{G}(\mathcal{N}^\pm, Q\left[\!\left[\eta \circ \mathcal{N}^\pm\right]\!\right])^{2+\alpha_{\mathbf{L}}} \mathrm{d}\mathcal{H}^m. \end{aligned} \tag{6.10}$$

7 Blowup argument by the frequency function

In this section we finish the proof of the main theorem of this work, i.e., Theorem 1.3, which is a consequence of Theorem 3.21, for $m = 2$, as noticed in Subsection 3.3. We use the frequency function in the center manifold $\mathcal{M}$, c.f. [7, Chapter 9], originally introduced by Almgren and more recently totally reformulated and adapted to the boundary case by De Lellis and collaborators, this motivates us to call it the Almgren-De Lellis' frequency function. In Theorem 7.3, we show that the m-dimensional area minimizing current has to satisfies at most one of two conditions, where the first one essentially says that 0 is a two-sided regular point of T and the second one is an estimate with the Almgren-De Lellis' frequency function. Although, there are two alternatives in Theorem 7.3, we use a blowup argument in Theorem 7.4 to show that the second alternative never occurs, thus implying that the only possible situation is 0 being a two-sided regular boundary point.

7.1 Almgren-De Lellis' frequency function in $\mathcal{M}$

In order to define our main quantities, we start with the following lemma that shows that exists a good perturbation function of the distance function on the center manifold .

Lemma 7.1 (Lemma 9.2 ,[7])**.** *There exist positive continuous functions $d^{\pm}: \mathcal{M}^{\pm} \to \mathbb{R}_+$ which belong to $C^2(\mathcal{M}^{\pm} \setminus \{0\})$ and satisfies the following properties*

(a) $d^{\pm}(x) = \mathrm{dist}_{\mathcal{M}^{\pm}}(x,0) + O(\mathrm{dist}_{\mathcal{M}^{\pm}}(x,0)^2) = |x| + O(|x|^2)$,

(b) $|\nabla d^{\pm}(x)| = 1 + O(d^{\pm})$, *where ∇ is the gradient on the manifold $\mathcal{M}$,*

(c) $\frac{1}{2}\nabla^2(d^{\pm})^2(x) = g + O(d^{\pm})$, *where ∇^2 denotes the covariant Hessian on $\mathcal{M}$ (which we regard as a $(0,2)$ tensor) and g is the induced metric on $\mathcal{M}$ as a submanifold of $\mathbb{R}^{m+n}$,*

(d) $\nabla d^{\pm}(p) \in T_p\Gamma$ *for all $p \in \Gamma$, i.e.*

$$\nabla d^{\pm} \cdot \vec{n}^{\pm} = 0 \ \text{on } \Gamma, \tag{7.1}$$

where $\vec{n}^{\pm}$ denotes the outer unit normal to $\mathcal{M}^{\pm}$ inside $\mathcal{M}$.

In particular this implies

$$\nabla^2 d^{\pm}(x) = \frac{1}{d}\Big(g - \nabla d^{\pm}(x) \otimes \nabla d^{\pm}(x)\Big) + O(1) \tag{7.2}$$

and

$$\Delta\, d^{\pm} = \frac{m-1}{d^{\pm}} + O(1) \tag{7.3}$$

where Δ denotes the Laplace-Beltrami operator on $\mathcal{M}$, namely the trace of the Hessian ∇^2. Moreover:

(S) All the constants estimating the $O(\cdot)$ error terms in the above estimates can be made smaller than any given $\eta > 0$, provided the parameter ε_1 in Assumption 5 is chosen appropriately small (depending on η).

Let us define three functions that we be used in the definition of the frequency function as follows

$$\phi(t) := \begin{cases} 1, & \text{for } 0 \le t \le \frac{1}{2}, \\ 2(1-t), & \text{for } \frac{1}{2} \le t \le 1, \\ 0, & \text{for } t \ge 1, \end{cases} \tag{7.4}$$

$$\mathrm{D}_{\phi,d^{\pm}}(\mathcal{N}^{\pm}, r) := \int_{\mathcal{M}^{\pm}} \phi\left(\frac{d^{\pm}(x)}{r}\right) |D\mathcal{N}^{\pm}|^2(x)\, \mathrm{d}\,\mathrm{Vol}^{\pm}, \tag{7.5}$$

$$\mathrm{H}_{\phi,d^{\pm}}(\mathcal{N}^{\pm}, r) := -\int_{\mathcal{M}^{\pm}} \phi'\left(\frac{d^{\pm}(x)}{r}\right) |\nabla d^{\pm}(x)|^2 \frac{|\mathcal{N}^{\pm}(x)|^2}{d^{\pm}(x)}\, \mathrm{d}\,\mathrm{Vol}^{\pm}, \tag{7.6}$$

where $\mathrm{Vol}^{\pm}$ denotes the standard volume form on $\mathcal{M}^{\pm}$.

Definition 7.2. *The* **Almgren-De Lellis' frequency function** *is defined as the ratio*

$$\mathrm{I}_{\phi,d^{\pm}}(\mathcal{N}^{\pm}, r) := \frac{r\mathrm{D}_{\phi,d^{\pm}}(\mathcal{N}^{\pm}, r)}{\mathrm{H}_{\phi,d^{\pm}}(\mathcal{N}\pm, r)}.$$

We also set the notation

$$\mathcal{C}^{\pm} := \left\{ y \in \mathbf{B}(0,1) : \mathbf{p}(y) \in \mathcal{M}^{\pm} \text{ and } |y - \mathbf{p}(y)| \leq \operatorname{dist}(y,\Gamma)^{3/2} \right\}$$

for the horned neighborhoods of $\mathcal{M}^{\pm}$ in which T is supported, compare with Corollary 5.7 and item (v) of Theorem 6.12.

Theorem 7.3 (Theorem 9.3, [7])**.** *Let T and Γ be as in Assumption 6, $Q^+ = Q, Q^- = Q - Q^\star$ and consider ϕ and $d^{\pm}$ as above. Then only one of the following alternatives holds*

(a) $T \llcorner \mathcal{C}^{\pm} = Q^{\pm} [\![\mathcal{M}^{\pm}]\!]$ *in a neighborhood of* 0,

(b) $\lim_{r\to 0} \mathrm{I}_{\phi,d^{\pm}}(\mathcal{N}^{\pm}, r) > 0$.

7.2 Blowup argument

Letting $Q^+ := Q$ and $Q^- := Q - Q^\star$, we define the multivalued maps with domain and codomain in Euclidean spaces,

$$N^{\pm}(x) = \sum_{i=1}^{Q^{\pm}} [\![(N^i)^{\pm}(x)]\!] \,,$$

such selections $\{N^i\}_{i=1}^{Q^{\pm}}$ are given by the formulas

$$\begin{array}{rccl} (N^i)^{\pm} : & \mathrm{B}_1^{\pm}(0) \subset \mathbb{R}^m & \to & \mathbb{R}^n \\ & x & \mapsto & \mathbf{p}_{\{0\}\times\mathbb{R}^n}\big((N^i)^{\pm}(x, \boldsymbol{\varphi}^{\pm}(x))\big). \end{array}$$

Observe that the pair (N^+, N^-) is a $\left(Q - \frac{Q^\star}{2}\right)$-valued function with interface $(\gamma, Q^\star [\![0]\!])$. We now set the following notation for the Dirichlet energy

$$\mathrm{Dir}(r) := \frac{1}{2}\int_{\mathrm{B}_1^+(0)} |DN^+|^2 + \frac{1}{2}\int_{\mathrm{B}_1^-(0)} |DN^-|^2 := \mathrm{Dir}^+(r) + \mathrm{Dir}^-(r),$$

and the corresponding rescaling of $N^{\pm}$

$$N_r^{\pm}(x) := \sum_i \left[\!\!\left[r^{m/2-1}\,\mathrm{Dir}^{\pm}(r)^{-1/2}(N^i)^{\pm}(rx) \right]\!\!\right].$$

Finally, we can state the key result to give our final contradiction argument.

Theorem 7.4. *Let T and Γ be as in Assumption 6. If it holds*

$$\lim_{r\to 0} \mathrm{I}_{\phi,d^{\pm}}(N^{\pm}, r) > 0, \tag{7.7}$$

for at least one of the regions $\mathcal{C}^{\pm}$, then there exists a sequence $\rho_k \to 0$ as $k \to +\infty$ such that the sequence of pairs $(N^+_{\rho_k}, N^-_{\rho_k})$ would converge in $\mathrm{B}_1(0)$ locally strongly in L^2 to a $\left(Q - \frac{Q^\star}{2}\right)$ Dir-minimizer (N_0^+, N_0^-) where $N_0^+ : \mathrm{B}_1^+(0) \to \mathcal{A}_Q(\mathbb{R}^n)$ and $N_0^- : \mathrm{B}_1^-(0) \to \mathcal{A}_{Q-Q^\star}(\mathbb{R}^n)$, it holds that

$$\lim_{k\to\infty}\left(\int_{\mathrm{B}_R^+(0)} |DN^+_{\rho_k}|^2 + \int_{\mathrm{B}_R^-(0)} |DN^-_{\rho_k}|^2\right) = \int_{\mathrm{B}_R^+(0)} |DN_0^+|^2 + \int_{\mathrm{B}_R^-(0)} |DN_0^-|^2, \forall R \in (0,1), \tag{7.8}$$

(N_0^+, N_0^-) collapses at the interface $(T_0\gamma, Q^\star [\![0]\!])$, we have the following properties

(*i*) (N_0^+, N_0^-) *is nontrivial and in particular* $\mathrm{Dir}(N_0^+, N_0^-, \mathrm{B}_1(0)) = 1$;

(*ii*) $\boldsymbol{\eta} \circ N_0^\pm \equiv 0$.

As it is explained in Subsection 3.3, Theorem 1.3 for currents of dimension 2 follows from Theorem 3.21 which we are now able to prove for area minimizing currents of dimension $m \geq 2$, codimension $n \geq 2$ and multiplicity $Q^\star \geq 1$.

Theorem (Theorem 3.21)**.** *Let T and Γ be as in Assumption 2 with $C_0 = 0$. Then any two-sided collapsed point of T is a two-sided regular point of T.*

Proof. Now, since we are under Assumption 6, we can apply Theorem 7.3 and we show that (b) of Theorem 7.3 never occurs, then we are always in the case (a) of Theorem 7.3 which ensures that 0 is a boundary two-sided regular point of T. With this aim in mind observe that by the harmonic regularity of $\left(Q - \frac{Q^\star}{2}\right)$-Dir minimizers which collapse at the interface, i.e., Theorem 4.4, we have that $N_0^\pm = Q^\pm [\![h]\!]$ for some classical 1-valued harmonic function $h : \mathrm{B}_1(0) \to \mathbb{R}^n$, hence we necessarily have

$$N_0^+ = Q\, [\![\boldsymbol{\eta} \circ N_0^+]\!] \quad \text{and} \quad N_0^- = (Q - Q^\star)\, [\![\boldsymbol{\eta} \circ N_0^-]\!] \,.$$

By (ii) of Theorem 7.4, we have that $h \equiv 0$, but this is contradiction with $\mathrm{Dir}(N_0^+, N_0^-, \mathrm{B}_1(0)) = 1$. Thus (b) of Theorem 7.3 never occurs. This last fact surely completes the proof of the theorem. □

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