Nominal versus Real Board Independence:  
the Impact of Director Tenure

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ABSTRACT

In this paper, we investigate whether a regulation that mandates a greater proportion of outside directors on a corporate board results in a more independent board. Instead of taking the fraction of outside directors directly as the measure of board independence, we define it as the nominal independence level. The real independence level of the board is determined by both the nominal independence level and the length of the relationship between board members and the CEO. We assume that the real independence between a board member and the CEO decreases over time as long as the board member stays on the board. In our dynamic model, the board both monitors and advises the CEO, and the CEO decides whether to replace one of the directors in each period. The CEO’s tradeoff is between the possibly higher board expertise introduced by new directors versus the lower board real independence obtained by retaining the same directors. In our model, the higher the nominal independence level of the board, the more reluctant the CEO is in replacing existing directors. The resultant longer tenure of outside directors makes the CEO even less willing to replace them. Regulations that mandate higher nominal independence can have the unintended consequence that they lower both the real independence and the expertise of the board of directors in the long-run.

Key Words: Board Independence, Corporate Governance, Director Tenure
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1 Introduction

Corporate governance, especially the independence of the board of directors, attracted great attention after the corporate failures of Enron, WorldCom, Tyco, ImClone, and others. In response, regulations were enacted to protect investors by mandating requirements for board composition. In 2002, Congress passed the Sarbanes-Oxley Act, which requires the board’s audit committee to consist of a majority of independent members (also sometimes referred to as outside directors). The independent members are directors who have no affiliation with the company and are, therefore, supposedly less inclined to favor insiders when taking decisions and more inclined to protect shareholder interests. Both the New York Stock Exchange (NYSE) and NASDAQ amended their rules to require boards to have a majority of independent directors. The fraction of independent directors is often used as the measure of board independence (e.g., Linck et al., 2008; Faley et al., 2011; Ferreira et al., 2011).

Conceptually, independence is “the ability of an individual to maintain perspective or judgement that is unbiased by a relationship with others”(Larcker and Tayan, 2013). Unfortunately, in practice, it is often difficult to determine whether a director on the board is really independent. Both the NYSE and the NASDAQ have had to provide guidance on what “independent director” means. According to NYSE rule 303A.02(a)(i):

No director qualifies as independent unless the board of directors affirmatively determines that the director has no material relationship with the listed company (either directly or as a partner, shareholder or officer of an organization that has a relationship with the company).2

The NASDAQ has a similar rule. According to NASDAQ rule 5605(a)(2):

Independent Director means a person other than an Executive Officer or employee of the Company or any other individual having a relationship which, in the

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1See NYSE rule 303A.01 and NASDAQ rule 5605(b)(1).
2The NYSE rules are quoted from the NYSE listed company manual, which is available online at http://nysemanual.nyse.com/. The quotes are as of March 26 2014.
opinion of the Company’s board of directors, would interfere with the exercise of independent judgment in carrying out the responsibilities of a director.\textsuperscript{3}

Following the spirit of the rules by the securities exchanges, we refer to directors who care only about the shareholders’ interest when they newly join the board as outside directors. We define the fraction of outside directors on the board as the board’s “nominal independence.” Under the current rules, one reason “real independence” may differ from “nominal independence” is the duration of the relationship between outside directors and the CEO. As suggested by the Council of Institutional Investors (whose members consist of pension funds with more than $3 trillion), long tenure can affect a director’s “unbiased judgment,” and “[e]xtended tenure can lead an outside director to start to think more like an insider.” According to the EY Center for Board Matters (2015), “[s]ome investors believe that, after a certain point, a director’s ties to the company are deep and long-standing enough to potentially impair the director’s independence.” In other words, extended board tenure can create a culture of undue deference to management, which weakens the monitoring role of the board of directors. This can be a natural sociopsychological effect justified on behavioral grounds (Fichman and Levinthal, 1991) or a purely rational consequence of reputation concerns (in the sense that a director is reluctant to publicly overturn her previous judgements on management to avoid a reputation cost) (Corona and Randhawa, 2010).\textsuperscript{4} Meanwhile, increased firm-specific investments over time also make a director less willing to jeopardize her board seat by challenging management.\textsuperscript{5} Indeed, Huang (2013) documents evidence of high average tenure of outside board members being negatively associated with proxies for director oversight and company value.

Adopting tenure-related guidelines or restrictions for independent directors seems a

\textsuperscript{3}The NASDAQ rules are quoted from the NASDAQ stock market equity rules, which is available online at http://nasdaq.cchwallstreet.com/. The quotes are as of March 26 2014.

\textsuperscript{4}Similar arguments are frequently made in discussing auditor-client relationships. The claim is that the auditor becomes less independent from the client over time, which is used to argue in favor of auditor rotation.

\textsuperscript{5}We thank one of the anonymous referees for pointing this out.
natural response, as has been done in some economies (e.g., United Kingdom, France, and Hong Kong). Most countries adopt the “recommend and explain” approach. For example, under the UK Corporate Governance Code, a board should explain the reason for classifying a director with tenure longer than 9 years as independent in its annual disclosure. France is the only country with a mandatory regime under which directors who serve on the board for more than 12 years cannot be deemed independent. In the United States, there are no formal laws, rules, or regulations under which a long tenure would prevent a director from qualifying as independent. However, in September 2013, the Council of Institutional Investors revised its best-practices corporate governance policies to include tenure as a factor boards should consider when determining whether a director is independent. Also, one of the major proxy advisory firms, Institutional Shareholder Services (ISS), has added tenure to the checklist of factors used in its rating system QuickScore but still opposes a specific “narrow rule” for director tenure or term limits. One reason against a formal regulation on director tenure is that there is no ideal term limit which can be applied to all directors, because one size does not fit all. Due to a similar concern, Australia repealed a recent move toward a recommended director term limit.

In our model, the “real independence” level of the board of directors is determined by both the nominal independence level and the length of relationship between outside directors and the CEO. The real independence between an inside director and the CEO is zero, while the real independence between an outside director and the CEO decreases over time if the board member stays on the board. The real independence of the board is determined by the average real independence of the board members. The question we address is: does a regulation that mandates higher nominal independence result in higher or lower real independence?

Board tenure varies across different firms due to the variation in the board succession plans. One important reason for replacing an existing board member and hiring a new director is to bring new expertise into the existing board to deal with changes in the
business environment. In this paper, we assume the firm demands the expertise of new outside directors to adapt to such changes in the business environment (e.g., transforming into new product strategies or expansion to new markets). Therefore, replacing existing outside directors with new outside directors brings new expertise to the board as well as increases the “real independence” of the board.

Previous research has recognized the influence of incumbent CEOs on director selection (e.g., Hermelin and Weisbach, 1998; Shivdasani and Yermack, 1999; Withers et al., 2012). For simplicity, in our main model, we assume that the incumbent CEO chooses whether or not to replace existing outside directors. There is a conflict between the CEO’s and shareholders’ preferences over projects, as in Adams and Ferreira (2007). By choosing no replacement and keeping the same board of directors, the CEO can take advantage of the decreased real independence which leads to a smaller conflict over project choice between the board and the CEO and thereby to less monitoring of the CEO by the board. Lower monitoring enables the CEO to have a higher chance of retaining control over project choice.

The board of directors not only monitors the CEO but also provides managerial advice to the CEO (Adams and Ferreira, 2007; Baldenius et al., 2014; Linck et al., 2008). The CEO, hoping to obtain better advice from the board of directors, has incentives to replace existing outside directors with new directors who might provide better advice. The CEO faces a trade-off between lower monitoring and possible better advising when deciding whether to replace existing outside directors with new directors.

To make the distinction between nominal and real independence meaningful while focusing on director tenure, a multi-period model is needed. A three period model suffices to make our point. To illustrate the intuition for our results, consider two boards that are both formed at the beginning of Period 0 (which is a dummy initialization period since

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In discussing director succession, the Stuart Spencer Board Index 2014 (Stuart, 2014) states “Globalization and advancements in technology are giving rise to a slew of competitive and business threats and opportunities. At a time when growth and innovation are top priorities for most organizations, companies are transforming themselves through new product strategies, different product mixes, and expansion into new markets and geographies. In an ideal world, outside directors with relevant experience can serve as valuable advisers to the board and management about the company’s market, geographic and product directions.”
there is no replacement decision at the start of Period 0) and are otherwise the same except that the number of outsiders on the first board is determined by Regulation $R$ and the number of outsiders on the second board is determined by Regulation $R'$, where $R'$ requires a greater proportion of outsiders than $R$. Since there is no replacement decision in Period 0, the second board has both higher nominal and higher real independence at the start of the first period. From the perspective of the CEO, higher real board independence not only increases monitoring but also makes the project chosen by the board less desirable. Consequently, the marginal cost of higher real board independence is increasing in real independence. However, the marginal benefit of higher expertise is constant. (This assumption is relaxed later in the paper.) Since the second board has higher real independence at the start of the first period, the CEO is less willing to replace outside directors. As a result of increased retention of directors in the second board, the two boards will be closer in their real independence at the start of the second period. In the second period, a new force emerges that makes the CEO of the second board less willing to replace outside directors. When the CEO of the first board has replaced a larger fraction of directors than the CEO of second board in the first period, the longer tenure of the directors on the second board results in every replacement causing a larger increase in real independence. This new force driven by tenure makes the CEO of the second board less willing to replace outside directors than the first board, even if the second board has the same (or lower) real independence than the first board has. Hence, the second board can actually end up with a lower real independence level. That is, an unintended consequence of regulation that requires greater nominal board independence is that both real independence and advising expertise may be reduced in the long-run, because of the effect of regulation on board turnover.

Consistent with our results, outside board member turnover has trended down over the past decades according to the 2013 Spencer Stuart U.S. Board Index.\textsuperscript{7} Our results may

\textsuperscript{7}According to the 2013 Stuart Board Index (p.6), the boards of S&P 500 companies elected 339 new independent board members this past proxy season, down 11 percent from five years ago and 14 percent
also help explain why many studies (see Bhagat and Black, 1999 and the survey by Adams et al., 2010) fail to find an association between performance and the fraction of independent directors on the board, as the fraction of independent directors is not necessarily a good measure of the independence level of the board.

We have designed our model to highlight conditions under which mandating (by regulation) higher nominal independence leads to lower real independence. The analysis focuses on interior solutions, which is somewhat restrictive. When the comparison is between two boards with similar proportions of outsiders, the sufficient conditions for an interior solution are not particularly demanding. However, when the difference in the proportions of outside directors is large, the solution is interior (for both boards) only for a narrow set of parameters. In the latter case, our result should be interpreted as an existence result. In Section 7, we present extensions of our model in which mandated higher nominal independence results in either lower or higher real independence. For example, if the contribution in expertise of new directors is sufficiently decreasing in the number of new directors hired, higher nominal independence will result in higher real independence. Also, if the board of directors instead of the CEO makes the replacement decision and real independence decreases slowly enough, higher nominal independence again results in higher real independence.

The paper proceeds as follows. Section 2 provides a review of related literature. Section 3 presents the model setup and the sequence of events. Section 4 states the board’s optimal monitoring problem and solves it. Section 5 presents the CEO’s trade-off in deciding on board turnover in each period and analyzes the impact of the proportion of outside directors on the board on the CEO’s decision. Section 6 provides conditions under which higher nominal board independence results in lower real board independence. Section 7 studies extensions, including allowing the board to make the replacement decision, and Section 8 concludes. All proofs are included in the appendix.

from 10 years ago.
2 Related Literature

The effect of board independence has been examined in both empirical and theoretical studies on corporate governance. Most empirical studies of corporate boards (e.g., Linck et al., 2008; Faleye et al., 2011; Ferreira et al., 2011) consider the fraction of outsiders on the board of directors as being equivalent to the independence of the board of directors. However, the empirical evidence on the impact of board independence on firm performance is ambiguous (see the review paper by Adams et al., 2010). There are two possible theoretical explanations.

First, increasing board independence may not always be beneficial for the firm or its shareholders. Arya et al. (1998), in discussing substitutes for the role of earnings management in their model, argue that a board that is fully independent from the CEO can be more active than is efficient ex ante, due to the limited commitment of the board in making CEO replacement decisions. Laux (2008) models and refines the argument of Arya et al. (1998). Drymiotes (2007) models the monitoring role of the board as improving the precision of performance measurement and shows that putting insiders whose interests are partially aligned with the CEO on the board can serve as a commitment device that facilitates effective monitoring by the board. Kumar and Sivaramakrishnan (2008) models the monitoring role as generating independent information on the firm’s economic prospects used in a CEO’s compensation contract and shows that a more dependent director increases her monitoring effort ex ante to offset the wealth cost from her poorer contract choice ex post. Adams and Ferreira (2007) build a model in which both the board’s monitoring and advising roles depend on the CEO’s information. An overly independent board inhibits the communication between the CEO and the board, since the CEO may withhold information to avoid stricter monitoring. Harris and Raviv (2008) shows that insider-controlled boards may be optimal in some cases, since insider-control better exploits insiders’ information.8 Baldenius et al. (2014) shows that shareholders may prefer an advisor-heavy (monitor-light) board to prevent

8Harris and Raviv (2008) builds on Dessein (2002)’s model of delegation as an alternative to cheap talk communication in which there is a tradeoff between “loss of control under delegation and loss of information under communication.”
CEOs from entrenching themselves by choosing “complex” projects. In this paper, we also consider the dual advising and monitoring roles of the board of directors. However, departing from these papers, our focus is on whether regulation requiring a higher proportion of outside directors on the board (nominal independence) leads to increased real independence.

Second, as this paper argues, the fraction of outsiders on the board may not be a good measure of real independence, which depends on board tenure and is endogenously determined. There are other papers on the endogenous determination of the composition of the board of directors, although none of them study the effect of director tenure on board independence. For example, Hermalin and Weisbach (1998) consider the determination of the composition of the board of directors as the outcome of a bargaining game between the CEO and the board of directors. Schmeiser (2012) models the composition of the board of directors as determined by director voting and examines the impact of majority rule on shareholder value.

3 The Model

Consider a three period model of a firm consisting of a CEO and a board of directors. The size of the board (the number of directors) is denoted by $N$ and is a constant. Of the $N$ directors, $O$ directors are outsiders. We measure the nominal independence level of the board as the proportion of outside directors, $o = \frac{O}{N}$, which is also constant. We distinguish the nominal independence level of the board from its real independence. The real independence of the board is the extent to which the board cares about shareholders’ interests as opposed to the CEO’s interests. We assume that the real independence level of the board of directors depends not only on the proportion of outside directors but also on how long these outside directors stay on the board. The real independence level of an outside director decreases over time as long as the outside director stays on the board. In particular, we assume the real independence level of any insider is 0, while the real independence level of an outsider
who has been on the board for $\tau$ periods (or has a tenure of $\tau$) is $\eta^\tau$, where $\eta \in (0, 1)$. The
real independence of the board of directors in period $t$, $i_t$, is defined as the average real
independence of all board members, which is given by:

$$
\begin{align*}
    i_t &= 0 \cdot (1 - o) + \sum_{k=1}^{O} \frac{\eta^\tau_k}{N} \\
    &= \frac{\sum_{k=1}^{O} \eta^\tau_k}{N},
\end{align*}
$$

(1)

where $\tau_k$ is the number of periods that an outsider $k$ has stayed on the board.

Each period $t = 0, 1, 2$, the firm’s profits depend on the outcome of a business decision, $s_t \in \mathbb{R}$, taken by the CEO. In our model, the board of directors can affect how the CEO
takes these periodic decisions in two different ways. First, the board can provide expertise
to improve the efficiency of the periodic decisions. Second, the board can monitor the CEO
to ensure that he maximizes the board’s preferences as opposed to his own.

The expertise of the board depends on the expertise of its members. In particular, each
individual director $j$ has an expertise $\epsilon_{j,t} \in (0, 1)$ in period $t$. The expertise of the board of
directors in period $t$, $e_t$, is then defined as the average expertise of all board members, which
is given by:

$$
e_t = \frac{\sum_{j=1}^{N} \epsilon_{j,t}}{N}.
$$

(2)

The board’s expertise level, $e_t$, measures the probability with which the board can learn
a parameter, $\tilde{p}_t$, which affects the efficiency of the business decision in period $t$. We assume
that as long as $\tilde{p}_t$ is learned by the board, it is also learned by the CEO. This is a simplifying
assumption used to avoid having to introduce communication between the board and the
CEO but can also be justified as describing practice, since CEOs almost always serve on the
board themselves. We assume that the prior distribution of $\tilde{p}_t$ is a normal distribution with
mean $\mu_p$ and variance $\sigma_p^2$, i.e.,

$$
\tilde{p}_t \sim N \left( \mu_p, \sigma_p^2 \right).
$$

(3)
The optimal decision \( s_t \) for both the shareholders and the CEO depends on the preference parameter, \( \tilde{p}_t \). In particular, the utility of the shareholders is given by:

\[
u_t^S = - (s_t - \tilde{p}_t)^2 ,
\]

while the CEO’s preferences are given by:

\[
u_t^M = - [s_t - (\tilde{p}_t + b)]^2 ,
\]

where \( b > 0 \) is common knowledge and measures the CEO’s bias relative to the preference of shareholders.

The board of directors not only provides advice to the CEO but also monitors the CEO’s strategic decision. The purpose of the board’s monitoring is to influence the decision \( s_t \). Specifically, the board monitors the CEO with intensity \( m_t \in [0, 1] \) in period \( t \), incurring a monitoring cost \( c(m_t) \) and obtains the right to determine the decision \( s_t \) if the monitoring succeeds, which happens with probability \( m_t \). With probability \( 1 - m_t \), the monitoring fails and the CEO retains the right to decide \( s_t \). The cost associated with the monitoring intensity satisfies \( c'(m_t) \geq 0, c''(m_t) > 0 \).

Following the existing literature (e.g., Adams and Ferreira, 2007; Drymiotes, 2007; Kumar and Sivaramakrishnan, 2008; Laux, 2008), we represent the preferences of the board by considering the board of directors as an average agent. To capture the influence of real independence, \( i_t \), on the preferences of the board, we assume the board’s utility function is a weighed average of the manager’s utility with weight \( 1 - i_t \) and the shareholder’s utility with the weight \( i_t \). Thus, the utility function of a board with real independence level \( i_t \) is:

\[
u_t^B = (1 - i_t) u_t^M + i_t u_t^S - c(m_t) .
\]
Plugging in the expressions for $u^M_t$ and $u^S_t$, the utility of the board can be written as:

$$u^B_t = - \left\{ s_t - \left[ \tilde{p}_t + (1 - i_t) b \right] \right\}^2 - c(m_t) - i_t (1 - i_t) (2\tilde{p}_t + b)^2.$$ 

Since the last term does not depend on the decision $s_t$, the objective function of the board can be normalized to:

$$u^B_t = - \left\{ s_t - \left[ \tilde{p}_t + (1 - i_t) b \right] \right\}^2 - c(m_t). \quad (7)$$

Both the CEO and the shareholders, and therefore the board, have inter-temporally additive utility and discount future utility with a discount factor $\beta$.

Previous research has recognized the influence of incumbent CEOs on director selection (e.g., Hermalin and Weisbach, 1998; Shivdasani and Yermack, 1999; Withers et al., 2012). In Periods 1 and 2, we assume the CEO can choose the fraction of directors to be replaced with new outside directors with relevant experience who can provide valuable advice about a strategic decision. The assumption that the CEO makes replacement decision is a simplifying assumption intended to capture the influence of the CEO on the board’s decision on director selection. In Section 7, we also present an extension of our model in which the board of directors instead of the CEO makes the replacement decision. The board composition is taken as given at the beginning of Period 0. The replacement decisions are denoted by the variables $r_1$ and $r_2$, which represent the proportion of board directors that are replaced in Periods 1 and 2 respectively. Here, we assume the CEO never replaces an insider because of the dependence of insiders on the CEO as well as the importance of some unmodeled firm specific expertise of inside directors, which implies $r_t \in [0, o], t = 1, 2$.

Replacement decisions affect both the real independence and the expertise level of the board of directors. Specifically, the influence of the replacement decision $r_t$ on the real independence $i_t$ is given by the expression:
\[
\dot{i}_t(i_{t-1}, r_t; o) = (o - r_t) \eta_o \frac{i_{t-1}}{o} + r_t \cdot 1 + (1 - o) \cdot 0 \\
= \dot{i}_{t-1} \eta + r_t \left(1 - \eta_o \frac{i_{t-1}}{o}\right),
\]

where \(\eta_o\) is the average real independence of existing outside directors, 1 is the real independence of new outside directors, and 0 is the real independence of inside directors. Also, the level of expertise of the board in period \(t\) after the replacement decision \(r_t\), is given by

\[
e_t(r_t) = e_o (1 - r_t) + r_t \epsilon = e_o + r_t \Delta,
\]

where \(\epsilon\) is the expertise of new outside directors and \(e_o\) is the expertise of existing board members before replacement, with \(e_o < \epsilon\) and \(\Delta = \epsilon - e_o\). We assume that all incumbent directors have a generalist level of expertise \(e_o\) that is applicable to all decisions \(s_t\). However, the CEO can obtain a higher level expertise that is decision specific in a given period by replacing current directors with new outside directors of expertise \(\epsilon > e_o\). In subsequent periods, all directors that remain on the board revert back to the generalist expertise level \(e_o\). This is a simplifying assumption intended to capture the idea that the board periodically needs the new skills of new directors to deal with new business strategies, which can be justified based on the criticism that the longer-tenured directors “can no longer keep current with respect to industrial or technological developments,”\(^{9}\) which suggests a persistent demand for outside directors with new skills and experience to guide the company in changing economic environments. In explaining the association between director turnover and firm value, Anderson and Chun (2014) argue “new directors

\(^{9}\)In discussing activists’ criticism on long director tenure, Katz and McIntosh (2014) states “critics posit that older directors—who are typically the longer tenured directors—can no longer keep current with respect to industrial or technological developments and are unable to offer new insights into corporate issues; they fear that these directors may hold fossilized positions that are no longer relevant in the changing economic and business environment.”
New strategic opportunity appears
CEO chooses replacement decision \( r_t \in (0, o) \)

Board real independence \( i_t \), and expertise \( e_t \) update
Board chooses monitoring effort \( m_t \in [0, 1] \)

\[ e_t \quad 1 - e_t \]

Board learns information
Board does not learn information

\[ m_t \quad 1 - m_t \]

Board chooses project \( s_t \)
CEO chooses project \( s_t \)

Payoff realized
Board real independence decreases to \( n_{i_t} \)

Figure 1: Time-line of Period \( t \)

bring fresh perspectives and new skills, and they may be more likely than established members to challenge orthodoxy and raise previously unasked questions.” We discuss the implications of expertise persistence in Section 7.

Overall, the sequence of events in Period \( t = 1, 2 \), is as shown in Figure 1. (Period 0 is a dummy period since there is no replacement decision at the start of Period 0.) At the beginning of Period \( t, t = 1, 2 \), the CEO chooses the fraction of directors to be replaced with new outside directors with relevant experience on the new strategic decision. Then, the board chooses the monitoring intensity, \( m_1 \). After that, the board learns the private information \( \tilde{p}_1 \) with probability \( e_1 \). Next, with probability \( m_1 \), the monitoring succeeds and the board chooses the project. With probability \( 1 - m_1 \), monitoring fails, and the CEO chooses the
project. After the project is chosen, the payoffs of the CEO, the board of directors and the shareholders are realized, and the real independence of outside directors on the board decreases.

4 Board’s Problem

In Period $t$, given the resultant real independence level of the board, $i_t$, and the resultant expertise level of the board, $e_t$, the monitoring intensity of the board affects only the payoffs of the CEO and shareholders in Period $t$. Moreover, the board’s utility is a weighted average of the CEO’s and the shareholders’ utility. Therefore, the board’s optimal monitoring intensity is a short-term decision that only maximizes the board’s utility in Period $t$, $u_t^B$.

Within each period, we proceed by backward induction in solving the board’s optimal monitoring problem. If monitoring is successful, the board has full control over project choice. The project choice depends on whether the board successfully acquires the information about the true optimal project. So, if the board observes the realization of $\tilde{p}_t$, it faces the maximization problem:

$$\max_{s_t} - \{s_t - [\tilde{p}_t + (1 - i_t)b]\}^2 - c(m_t). \tag{10}$$

Otherwise, if the board does not know $\tilde{p}_t$, it solves:

$$\max_{s_t} - \mathbb{E}[s_t - [\tilde{p}_t + (1 - i_t)b]]^2 - c(m_t). \tag{11}$$

Therefore, the board chooses the optimal project according to:

$$s_t^B = \begin{cases} 
\tilde{p}_t + (1 - i_t)b & \text{if the board learns } \tilde{p}_t \\
\mathbb{E}[\tilde{p}_t] + (1 - i_t)b = \mu_p + (1 - i_t)b & \text{if the board does not learn } \tilde{p}_t.
\end{cases} \tag{12}$$

This implies that, if the board learns the true optimal project, the board chooses the project
with a bias \((1 - i_t)b\) away from the optimal project \(\tilde{p}_t\) from the shareholder’s perspective, and, therefore, obtains utility \(u_t^B = -c(m_t)\). Otherwise, if the board does not learn the true optimal project, the board chooses the project with a bias \((1 - i_t)b\) from the expected optimal project \(\mathbb{E}[\tilde{p}_t]\) and, therefore, earns a utility \(u_t^B = -\{\tilde{p}_t - \mathbb{E}[\tilde{p}_t]\}^2 - c(m_t)\).

If monitoring is not successful, the CEO retains control over the project choice decision, \(s_t\). The CEO’s project choice also depends on whether the board successfully learns the true optimal project. Recall that, if the board learns what the optimal project is, so does the CEO. Thus, the CEO chooses:

\[
s_t^M = \begin{cases} \tilde{p}_t + b & \text{if board learns } \tilde{p}_t \\ \mathbb{E}[\tilde{p}_t] + b = \mu_p + b & \text{if board does not learn } \tilde{p}_t. \end{cases} \tag{13}
\]

If the board learns \(\tilde{p}_t\), the CEO chooses the project with a bias \(b\) from \(\tilde{p}_t\). Otherwise, the CEO chooses the project with a bias \(b\) from the expected optimal project \(\mathbb{E}[\tilde{p}_t]\). The utility of the board resulting from the CEO’s choice is \(u_t^B = -i_t^2b^2 - c(m_t)\) if the CEO learns the project type, and \(u_t^B = -\{\mathbb{E}[\tilde{p}_t] - \tilde{p}_t + i_t b\}^2 - c(m_t)\) if the CEO does not learn the project type.

Anticipating the effects of its monitoring effort, the board chooses the optimal monitoring intensity to maximize its expected utility by solving:

\[
\max_{m_t \in [0,1]} m_t \left[ \epsilon_t \cdot 0 - (1 - \epsilon_t) \sigma_p^2 \right] + (1 - m_t) \left[ -\epsilon_t i_t^2 b^2 - (1 - \epsilon_t) \left( \sigma_p^2 + i_t^2 b^2 \right) \right] - c(m_t)
= \max_{m_t \in [0,1]} - (1 - \epsilon_t) \sigma_p^2 - (1 - m_t) i_t^2 b^2 - c(m_t). \tag{14}
\]

We further simplify the problem by assuming that the cost of the monitoring effort is quadratic, i.e., \(c(m_t) = k \frac{m_t^2}{2}\), for \(m_t \in [0,1]\). Solving the above program for the board, we obtain the optimal monitoring effort and state it in the following lemma:

**Lemma 1.** The optimal monitoring effort by the board of directors is \(m_t^* (i_t, \epsilon_t) = \frac{\epsilon_t i_t^2 b^2}{k}\). The monitoring effort is increasing and convex in the real independence level, i.e., \(\frac{\partial m_t^*(i_t, \epsilon_t)}{\partial i_t} = \frac{\epsilon_t i_t^2 b^2}{k}\).
Lemma 1 states that the optimal monitoring effort of the board is increasing in its real independence level, \( i_t \). Indeed, the higher the real independence level of the board is, the larger is the conflict over the project choice between the board and the CEO and, thus, the more the board values its control over the project choice.

Given the optimal project choices by the CEO and the board and the optimal monitoring intensity chosen by the board, we can derive the expected payoff for shareholders in each period. The expression for the shareholders’ expected payoff is:

\[
\mathbb{E} \left[ u_s^t(i_t, e_t) \right] = -\sigma_p^2 - b^2 + \underbrace{e_t \sigma_p^2}_{\text{advisory benefits}} + \underbrace{(2i_t - i_t^2) b^2 m^*_{i_t}(i_t, e_t)}_{\text{monitoring benefits}}. \quad (15)
\]

In this expression, the term \( e_t \sigma_p^2 \) measures the advisory benefits shareholders derive from board expertise, and the term \( (2i_t - i_t^2) b^2 m^*_{i_t}(i_t, e_t) \) measures the monitoring benefits that shareholders derive from board monitoring.

**Lemma 2.** In Period \( t \), the shareholders’ expected payoff is increasing in both the expertise level of the board, \( e_t \), and the real independence level of the board, \( i_t \).

Lemma 2 states the fairly obvious result that shareholders always prefer a board with a higher real independence and a higher expertise. This is true because the advisory benefit is increasing in the expertise of the board, and the monitoring benefit is increasing in the real independence of the board.

## 5 CEO’s Problem

In Section 4, we investigated the board’s optimal monitoring effort given its real independence level and expertise level. In this section, we study the CEO’s optimal
strategy regarding outside director replacement given that the CEO rationally anticipates the optimal monitoring effort chosen by the board. The outside director replacement decision affects the real independence level and the expertise of the board, so we first examine how the real independence and the expertise of the board determines the CEO’s expected payoff in each period.

Let $E[u_M^t(i_t, e_t)]$ denote the CEO’s expected payoff in Period $t$ when the real independence of the board is $i_t$ and the expertise level of the board is $e_t$. Given the optimal monitoring effort chosen by the board, we have:

$$E[u_M^t(i_t, e_t)] = -\sigma_p^2 + e_t \sigma_p^2 - i_t^2 b^2 m^*(i_t, e_t),$$

(16)

where the term $e_t \sigma_p^2$ represents the advisory benefits derived by the CEO from board expertise. That is, this term reflects the improvement in the decision derived from the information about $\tilde{p}_t$ available through board expertise. The term $i_t^2 b^2 m^*(i_t, e_t)$ represents the monitoring costs imposed on the CEO by board monitoring. It reflects the divergence of $s_t$ from the CEO’s preferred choice enforced by the board’s monitoring.

**Lemma 3.** In each period $t$, the CEO’s expected payoff is increasing in the expertise level of the board, $e_t$, and decreasing in the real independence level of the board, $i_t$. Moreover, the monitoring costs imposed on the CEO are increasing and convex in the real independence of the board, $i_t$.

Lemma 3 states that the CEO prefers a board with a higher expertise level, congruent with the preferences of shareholders, but prefers a board with a lower real independence level, incongruent with shareholders. The intuition is that the advisory benefits derived by the CEO are increasing in board expertise, and the monitoring costs imposed on the CEO are increasing in the real independence level of the board. Due to the conflict with shareholders, when the CEO decides on director replacement, the CEO chooses a composition of the board that maximizes his or her own benefit, which may harm shareholder value. Moreover, the
monitoring cost imposed on the CEO is increasing and convex in the real independence of the board, which implies the CEO has a strong preference for a board with a lower real independence level. Indeed, increasing the real independence of the board has two effects on the CEO’s payoff. On the one hand, increased real independence increases the gap between the CEO’s and the board’s preferred projects, making the CEO worse off when the board takes control over project selection. On the other hand, increased real independence also increases the monitoring intensity of the board (according to Lemma 1), making the CEO worse off as the board obtains control over the project choice more often.

After analyzing how the real independence level and expertise level of the board affects the CEO’s expected payoff in each period, we now investigate the CEO’s optimal replacement decision of independent directors on the board at the beginning of Period 1 and Period 2 for the three-date problem. Since it is a sequential decision problem for the CEO, we proceed by backward induction first solving the CEO’s optimal replacement decision at the beginning of Period 2. In the following analysis, we focus on the interior solutions of the CEO’s optimization programs, since they reflect tradeoff of the incentives faced by the CEO. In footnotes, we provide conditions under which an interior solution exists. At Section 7, we discuss the potential implications of non-interior solutions.

5.1 CEO’s Optimal Replacement Decision in Period 2

At the beginning of Period 2, considering the effect of outsider replacement on both the real independence and the expertise of the board, the CEO takes the replacement decision maximizing his expected utility in Period 2, which we denote by $\mathbb{E} \left[ u^M_2 (i_2, e_2) \right]$. The manager’s problem in Period 2 is:

$$
\max_{r_2} \mathbb{E} \left[ u^M_2 (i_2, e_2) \right] = -(1 - e_2) \sigma^2_p - \frac{b^4 i_2^4}{k} \tag{17}
$$

s.t. $i_2 (i_1, r_2; o) = i_1 \eta + r_2 \left( 1 - \eta \frac{i_1}{o} \right)$

$e_2 (r_2) = e_o + r_2 \Delta$. 

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On the one hand, a more intense replacement decision increases the expertise of the board, which is desirable to the CEO. On the other hand, a more intense replacement decision also increases the real independence of the board, which is undesirable to the CEO. The optimal replacement decision is then derived by balancing this tradeoff.

Define $r^*_2(i_1; o)$ and $i^*_2(i_1; o)$ as the CEO’s optimal replacement decision in Period 2 and the resultant real independence of the board respectively.

Lemma 4. Restricting attention to interior solutions, the CEO’s optimal replacement decision in Period 2 and the resultant real independence of the board, $r^*_2(i_1; o)$ and $i^*_2(i_1; o)$, are given by $i^*_2(i_1; o) = \left( \frac{\sigma^2_p \Delta}{\frac{1}{k} \left( 1 - \frac{\eta i_1 o}{\sigma} \right)^3} \right)^{\frac{1}{3}}$ and $r^*_2(i_1; o) = \frac{i^*_2(i_1, o) - i_1 o}{\left( 1 - \frac{\eta i_1 o}{\sigma} \right)}$.

The marginal benefit of replacement comes from increasing the expertise of the board and has the expression $\sigma^2_p \Delta$, where $\Delta$ measures the marginal effect of replacement on board expertise in Period 2 and $\sigma^2_p$ measures the marginal effect of board expertise on the manager’s utility in Period 2. The marginal cost of replacement is due to an increase in the real independence of the board and reflected in the expression $\frac{4b^3 i^2}{k} \left( 1 - \frac{\eta i_1 o}{\sigma} \right)$. In this last expression, the term $\left( 1 - \frac{\eta i_1 o}{\sigma} \right)$ represents the difference between the real independence of the new and the existing outsiders and measures the marginal effect of replacement on board real independence. The term $\frac{4b^3 i^2}{k}$ measures the marginal effect of real independence on the manager’s utility in Period 2. By balancing the marginal cost and the marginal benefit, we derive the optimal replacement decision and the resulting real independence in Period 2 as a function of $i_1$ and $o$. Perhaps surprisingly, we find that the resultant real independence of the board in Period 2 is increasing in the ratio $\frac{i_1 o}{\sigma}$.

Corollary 1. The optimal real independence of the board in Period 2, $i^*_2(i_1, o)$, is increasing in the ratio $\frac{i_1 o}{\sigma}$.

---

10 Here, for the solution to be interior it suffices to assume a non-extreme improvement in expertise from replacement $\Delta$, i.e., $\Delta \in \left( \frac{4b^3 i^2}{k} \left( 1 - \frac{\eta i_1 o}{\sigma} \right) \frac{(\eta i_1)^2}{\sigma^2}, \frac{4b^3}{k} \left( 1 - \frac{\eta i_1 o}{\sigma} \right) \frac{\sigma^3}{\sigma^2} \right)$. 

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Note that $\log_\eta \left[ \eta \frac{i}{o} \right]$ is the average tenure of the existing board members viewed collectively as an “average agent” at the beginning of Period 2 before the replacement decision. Corollary 1 states that a higher average board tenure before replacement results in a lower real independence after replacement in Period 2. A longer average tenure of existing board members (i.e., a lower ratio $\frac{i}{o}$) increases the real independence gap between the existing outside directors and the new directors, $(1 - \eta \frac{i}{o})$. The marginal cost of the replacement decision due to increased real independence is increasing in both the resultant real independence and the real independence gap, while the marginal benefit of the replacement decision from increasing board expertise is constant. The optimal replacement decision balances the marginal cost and the marginal benefit. Therefore, a longer average tenure of existing board results in a lower real independence of the board.

5.2 CEO’s Optimal Replacement Decision in Period 1

At the beginning of Period 1, the average real independence level of the board is $\eta o$, and the average expertise level of the board for the strategic decision in Period 1 is $e o$. Based on the real independence and expertise of the board, the CEO takes the outside director replacement decision at the beginning of Period 1 anticipating the optimal replacement decision in the following period. The replacement decision in Period 1 affects the real independence and expertise of the board in both Period 1 and Period 2, so it affects the expected utility of the CEO in both periods. We use $U_M (r_1; o)$ to denote the CEO’s expected discounted sum of utilities in Period 1 and Period 2 if the CEO’s replacement decision at the beginning of Period 1 is $r_1$ and the nominal board independence is $o$. Let $\beta$ be the CEO’s discount factor. The CEO’s problem regarding the optimal outside director replacement decision at
the beginning of Period 1 can be stated as follows:

\[
\begin{align*}
\max_{r_1} U_M (r_1; o) &= \mathbb{E} \left[ u_1^M (i_1, e_1) \right] + \beta \mathbb{E} \left[ u_2^M (i_2, e_2) \mid r_2 = r_2^* (i_1, o) \right] \\
\text{s.t.} & \quad i_1 (r_1; o) = \eta o + r_1 (1 - \eta) \\
& \quad e_1 (r_1) = e_o + r_1 \Delta,
\end{align*}
\]

(18)

where \( \mathbb{E} \left[ u_1^M (i_1, e_1) \right] \) is the expected utility in Period 1, and \( \mathbb{E} \left[ u_2^M (i_2, e_2) \mid r_2 = r_2^* (i_1, o) \right] \) is the expected utility in Period 2 anticipating the CEO’s optimal replacement decision at the beginning of Period 2 given by expression (17).

**Lemma 5.** Restricting attentions to interior solutions, the CEO’s optimal replacement decision in Period 1, \( i_1^* (o) \), is determined by:

\[
\sigma^2 \Delta - \left[ \frac{4b^3 i_1^3}{k} + \beta \frac{\eta \left( 1 - \frac{i_2^* (i_1, o)}{\sigma} \right) \Delta}{(1 - \eta \frac{1}{\sigma})^2} \right] (1 - \eta) = 0.
\]

(19)

The marginal benefit of replacement from increasing the expertise of the board is \( \sigma_p^2 \Delta_1 \), where \( \Delta_1 \) measures the marginal effect of the replacement decision on board expertise, and \( \sigma_p^2 \) measures the marginal effect of board expertise on the manager’s utility. The marginal cost of replacement due to an increase in real independence of the board is

\[
\left[ \frac{4b^3 i_1^3}{k} + \beta \frac{\eta \left( 1 - \frac{i_2^* (i_1, o)}{\sigma} \right) \Delta_2}{(1 - \eta \frac{1}{\sigma})^2} \right] (1 - \eta).
\]

Specifically, \( (1 - \eta) \), which represents the difference in the real independence between the new outsiders and the existing outside directors at the beginning of Period 1, measures the marginal effect of the replacement decision on the board real independence. The marginal effect of the real independence on the manager’s

\[\text{Here, for the solutions to be interior it suffices to assume that the real independence decays quickly enough with director tenure, i.e., } \eta \text{ is small enough, and that the improvement in expertise from replacement } \Delta \text{ is not extreme. A small } \eta \text{ guarantees that the real independence decays quickly enough with board tenure so that the CEO is not willing to replace all incumbent outsiders. The assumption of a non-extreme } \Delta \text{ serves to guarantee that the optimal replacement decisions are interior solutions. We show in the appendix that to obtain an interior solution it is sufficient to assume: } \eta < \min \left\{ \frac{3}{4}, 1 - \sqrt{\frac{\beta}{3}} \right\}, \Delta \in \left( \frac{4b^3 o^2 (1 - \eta)}{k \sigma^2}, \frac{\eta^3}{(1 + \eta)} \right). \]
utility in Period 1 is captured by \( \frac{4b^2i_1}{k} \), and the marginal effect of the real independence in Period 1 on the manager’s utility in Period 2 is captured by \( \eta \left( 1 - \frac{i_1^*(o)}{o} \right) \Delta / \left( 1 - \eta \frac{i_1}{o} \right)^2 \). By balancing the marginal cost and the marginal benefit, the resultant real independence of the board in Period 1 is determined by an implicit function of the nominal independence denoted as \( i_1^*(o) \). Interestingly, we find that the ratio of the optimal real independence of the board in Period 1, \( i_1^*(o) \), and the nominal independence, \( o \), is decreasing in nominal independence \( o \).

**Lemma 6.** The ratio \( \frac{i_1^*(o)}{o} \) is decreasing in the nominal independence \( o \).

Lemma 6 states that, as the nominal independence of the board increases, the resultant ratio \( \frac{i_1^*(o)}{o} \) after the replacement decision of the CEO at the beginning of period 1 decreases, where \( i_1^*(o) \) is the interior solution to the CEO’s optimal replacement decision in Period 1 defined in Lemma 5. Lemma 6 implies that a higher nominal board independence results in a longer average tenure of the outside directors after replacement, since the term \( \log_\eta \left( \frac{i_1}{o} \right) \) measures the average tenure of existing outside directors after replacement at the beginning of period 1. If, alternatively, the resultant average tenure of the board with higher nominal independence were lower after replacement at the beginning of period 1, it would immediately imply that the real independence in period 1 is higher for the board with higher nominal independence. Moreover, by Corollary 1, the board with lower average tenure would end up with a higher real independence in period 2. Therefore, the board with higher nominal independence would have a higher real independence in both periods. Then, the marginal cost of replacement would be higher in both period 1 and period 2 for the board with higher nominal independence. However, this cannot be true in equilibrium because the marginal benefit of the replacement decision is the same regardless of the nominal independence.
6 Nominal versus Real Independence

In Section 5, we derived the optimal director replacement strategy of the CEO. Given the optimal replacement strategy, we now study how the nominal board independence or the fraction of outside directors affects the CEO’s decision about replacing directors and, thus, the real board independence. Recall that the replacement decision made by the CEO does not change the fraction of outside directors on the board, so the nominal independence of the board stays the same.

Proposition 1. With interior solutions for optimal replacement decisions in both periods, and regulation $R'$ that mandates $O'$ outside directors and regulation $R$ that mandates $O$ outside directors, $O' > O$, the average real independence level of the board at the end of Period 2 is lower for the board with $O'$ outsiders than for the board with $O$ outsiders.

Proposition 1 states that the more outside directors a regulation requires to be on the board initially (the higher the initial nominal board independence is), the lower the real independence level of the board is in Period 2. In other words, a higher nominal board independence leads to a lower real independence of the board with time. At the beginning of Period 1 before replacement, the average tenure of the outside directors on the board is $\eta$ which is the same across different boards with different numbers of outsiders. So, the real independence of the board with $O'$ outsiders is also higher, which makes the CEO more reluctant to replace existing directors on the board with new outsiders whose real independence level is higher than the existing directors. Therefore, a higher initial nominal independence level of the board results in a higher average tenure of the board, hence, at the beginning of Period 2 before replacement. The higher average tenure of the board at the beginning of Period 2 again makes the CEO more reluctant to replace existing directors, since

\[\frac{4h^3(\sigma')^3(1-\eta)}{s^2} \cdot \frac{\eta^3}{1-\beta(1+\eta)} + \frac{4h^3(1-\eta)}{s^2k} \cdot \Delta \in \left(\frac{4h^3(\sigma')^3(1-\eta)}{k\sigma^2} \cdot \frac{\eta^3}{1-\beta(1+\eta)} + \frac{4h^3(1-\eta)}{s^2k}\right).\] The set of parameters that satisfy these conditions is not empty if

\[\frac{\sigma}{\sigma'} > \frac{\eta}{\sqrt{1-\beta(1+\eta)}}.\]
the gap of the real independence between the existing board members and new outsiders is larger. Therefore, a higher initial nominal independence level of the board results in a lower real independence level of the board by the end of Period 2.

Although we restricted attention to interior solutions, we must stress that our result can be overturned when the solution is not interior. Essentially, a higher nominal board independence leads to a higher real independence if the CEOs in both firms make the same extreme replacement decisions. We discuss this point further in Section 7.4.

7 Extensions

In this section, we consider three extensions. In the first extension, we extend the time horizon to infinite periods, because it gives a natural setup to study the relaxed assumption on improvements in board expertise. We also relax the assumption that the improvement in board expertise from replacement is constant in the number of new outside directors hired. In the last extension, we let the board of directors rather than the CEO make the replacement decision. In the last subsection, we discuss some of other assumptions of our analysis and their role in driving our results.

7.1 Infinite Horizon Problem of the CEO

In the main model, we consider a two period model to illustrate the key tradeoffs. To check the robustness of our result and also develop conditions under which our results are overturned, we extend the time horizon to infinite periods and look at whether our results continue to hold. Here, all the myopic decisions taken by the board and the CEO (which include the monitoring effort of the board and the project choice of the board and the CEO) in each period from the main model remain the same and, therefore, the results for those decisions from the main model continue to hold. We only change the analysis of the replacement decision by moving to an infinite horizon.
Define $V_M(i; o)$ as the present value of the expected utility of the CEO if the current real board independence is $i$ and the nominal board independence is $o$. By the definition of the value function, we have:

$$
V_M(i; o) = \max_{r \in [0, o]} - (1 - e_o - \Delta r) \sigma^2 - \frac{(bi')^4}{k} + \beta V_M(i'; o),
$$

$$s.t. i' = \eta i + r \left(1 - \eta \frac{i}{o}\right),
$$

which can be rewritten as:

$$
V_M(i; o) = \max_{i' \in [\eta, o]} - \left(1 - e_o - \Delta \frac{i' - \eta i}{1 - \eta} \right) \sigma^2 - \frac{(bi')^4}{k} + \beta V_M(i'; o).
$$

We define $I(i; o)$ as the solution to the problem above, which is also called the policy function. Since it is a deterministic problem and the utility function is continuous and bounded, there is a unique policy function $I(i; o)$. We search for a fixed point of the policy function, $i^*(o)$, which satisfies $I(i^*(o); o) = i^*(o)$, as the stable real independence. If the current real independence is equal to $i^*(o)$, it will remain the same after the CEO’s optimal replacement decision in next period. By showing how the fixed point varies with the nominal board independence, $o$, we are able to show the impact of nominal board independence on the real board independence.

**Proposition 2.** $\forall o, o', s.t., 0 < o < o' < 1$, a higher nominal independence results in a lower real independence, i.e., $i^*(o') < i^*(o)$, if the improvement in board expertise from replacement is not too large, i.e., $\Delta \in \left(0, \frac{b^4}{k\sigma^2}4o^3(1 - \eta)\right)$. If the improvement in board expertise from replacement is large enough, i.e., $\Delta \geq \frac{b^4}{k\sigma^2}4(o')^3(1 - \eta)$, a higher nominal independence results in a higher real independence, i.e., $i^*(o') > i^*(o)$.

Proposition 2 confirms Proposition 1 can be generalized to an infinite horizon as long as the real independence of outsiders decreases with tenure, i.e., $\eta < 1$, and the improvement in board expertise from replacement, $\Delta$, is not too large.
7.2 Assumptions on the Impact of Replacement on Expertise

So far, we have assumed that the marginal improvement in board expertise from replacement is constant. That is, we assumed that the contribution of each new director to the board expertise is the same. Now, we relax this assumption by allowing the marginal improvement in board expertise to be decreasing in the size of the replacement. A possible reason for such a decrease in the expertise contribution could be that new directors have different levels of expertise and are hired prioritizing directors with higher expertise. An alternative reason could be that new directors have overlapping skills and, therefore, it becomes harder to find new directors with different skills the more new directors are hired. Specifically, we assume:

\[ e_t = e_0 + G(r), \]

where \( G'(r) \geq 0, G''(r) \leq 0 \) and \( G'(0) = \Delta \). Our original assumption in the main model is a special case of this assumption, i.e., \( G''(r) \equiv 0 \). With the new more general assumption, we find that our main results may or may not hold depending on the scale of \( \frac{G''(r)}{G'(r)} \). As in our analysis in section 7.1, we study an infinite horizon model and look at the fixed point of real independence in order to study the impact of regulating nominal independence on the real board independence.

Proposition 3. \( \forall o, o', s.t., 0 < o < o' < 1 \), if the real independence of outsiders decreases quickly with tenure, i.e., \( \eta < \frac{3}{4 + \beta} \), and the improvement in board expertise from replacement is not too large, i.e., \( G'(r) \in \left(0, \frac{b^4}{k_\sigma^2} k_\sigma^4 (1 - \eta)\right) \), we have:

1. \( i^* (o') < i^* (o) \) if \( \frac{G''(r)}{G'(r)} > -\left(\frac{1}{1-\eta} + \frac{\beta}{1-\beta\eta}\right) \);  
2. \( i^* (o') > i^* (o) \) if \( \frac{G''(r)}{G'(r)} < -\left(\frac{1-\beta}{1-\eta}\right) \).

Proposition 3 states that a higher nominal independence results in a lower real independence if the expertise contribution of new directors does not decrease much with the number of new directors \( \frac{G''(r)}{G'(r)} \) is large enough). However, a higher nominal
independence results in a higher real independence if the expertise contribution of new directors decreases significantly with the number of new directors ($\frac{G''(r)}{r}$ is small enough). The intuition is that if and only if new directors contribute enough to board expertise, the manager of a firm with a board with a lower nominal independence is willing to replace significantly more incumbent board members with new directors, which results in a higher nominal independence. Otherwise, the replacement decision is similar for firms with different nominal board independence levels, and, thus, a higher nominal independence results in a higher real independence. For example, if only one new director is available to provide specialist expertise, the CEO wants to hire that new director anyway as long as the expertise is valuable enough. In such cases, a higher nominal independence would result in a higher real independence.

7.3 Replacement Decision by the Board of Directors

In our previous analysis, we assume that the CEO decides whether to replace an existing outside director to reflect the influence of CEO on director succession. We recognize that the hiring and firing decision is taken directly by (the nomination committee of) the board instead of by the CEO. Now, we relax this assumption by allowing the board of directors, viewed as an average agent, to make the replacement decision. The objective of the board of directors is to maximize the present value of its expected future payoffs. As in our analysis in Section 7.1, we study an infinite horizon model with the marginal improvement in board expertise from replacement being a constant $\Delta$ and solve for the fixed point of real independence in order to study the impact of regulating nominal independence on real board independence.

Define $V_B(i; o)$ as the present value of the board’s expected utility if the current real board independence is $i$ and the nominal board independence is $o$. By the definition of the
value function, we have:

\[
V_B(i; o) = \max_{r \in [0,o]} - (1 - e_o - \Delta r) \sigma^2 - \left(1 - \frac{(bi')^2}{k}\right) (bi')^2 - \frac{k}{2} \left(\frac{(bi')^2}{k}\right)^2 + \beta V_B(i'; o),
\]

s.t. \(i' = \eta i + r \left(1 - \frac{i}{\eta o}\right)\), \quad (23)

which can be rewritten as:

\[
V_B(i; o) = \max_{i' \in [\eta i, o]} - \left(1 - e_o - \Delta \frac{i' - \eta i}{1 - \eta^2 o}\right) \sigma^2 - \left((bi')^2 - \frac{(bi')^4}{2k}\right) + \beta V_B(i'; o). \quad (24)
\]

As we did in the analysis of the CEO’s problem, we define \(I(i; o)\) as the solution to the problem above, which is again called the policy function. We also look at how the fixed point varies with the nominal board independence, \(o\), in order to show the impact of nominal board independence on real board independence.

**Proposition 4.** \(\forall o, o', s.t., 0 < o < o' < 1\), if the CEO’s bias relative to the preference of shareholders is not too large, i.e., \(\frac{b^2}{k} < \frac{1}{3}\):

1. If the real independence level of directors decreases quickly with tenure, i.e., \(\eta < \min\left\{\frac{2}{5}, \frac{1 - 3 \eta^2}{2 + \beta - (4 + \beta) \frac{k^2}{\sigma^2}}\right\}\), and the improvement in board expertise from replacement is not too large, i.e., \(\Delta \in \left(0, \frac{2 \eta^2}{\sigma^2} o \left(1 - \frac{(bo)^2}{k}\right) (1 - \eta)\right)\), then higher nominal independence results in lower real independence, i.e., \(i^* (o') < i^* (o)\).

2. If the real independence level of directors decreases slowly with tenure, i.e., \(\eta > \frac{1}{2 + \beta}\), and the improvement in board expertise from replacement is large enough, i.e., \(\Delta \geq \frac{2 \eta^2}{\sigma^2} o' \left(1 - \frac{(bo')^2}{k}\right) (1 - \eta)\), then higher nominal independence results in higher real independence, i.e., \(i^* (o') > i^* (o)\).

Proposition 4 states that when the board of directors makes the replacement decision, a higher nominal independence level of the board results in a lower real independence of the board in the long-term if the real independence level of outsiders decreases quickly with
tenure and the improvement in board expertise from replacement is not too large. However, a higher nominal independence level of the board leads to a higher real independence of the board if the real independence level of outsiders does not decrease quickly with tenure and the improvement in board expertise from replacement is large enough. The intuition for this result is that if the real independence level of the directors decreases quickly with tenure (i.e., $\eta$ is small enough), the incumbent board has a low real independence level and, thus, it is more aligned with the CEO. Therefore, similar to the impact of replacement on the CEO’s utility, the marginal cost of replacement for the existing board is higher if either the real independence of the board is higher or the average tenure is longer. Given the same real independence, the board with a higher nominal independence has a longer average tenure. Since the marginal benefit is fixed, in order to satisfy the equilibrium condition, the marginal cost of replacement should be the same across the boards with different nominal independence. Therefore, in the fixed point, a higher nominal independence results in a lower real independence of the board. Alternatively, if the real independence level of outsiders does not decreases quickly with tenure, as long as the improvement in board expertise from replacement is large enough, periodical director replacements will maintain the real independence of the board close to the nominal independence. In this case, a higher nominal independence level of the board leads to a higher real independence of the board.

7.4 Discussion of Other Assumptions

In this subsection, we return to our main (three period) model and discuss some of our other assumptions and their role in driving our results.

In the above analysis, we restricted attention to interior solutions. Now, we discuss the conditions under which an interior solution exists and what happens if there is no interior solution. The existence of interior solutions requires only that the real independence decays quickly enough with director tenure, i.e., $\eta$ is small enough, and that the improvement in
expertise from replacement $\Delta$ is not extreme.

To illustrate when the restriction to interior solutions is restrictive and when it is not, we now provide a numerical example. The values of the parameters used in the numerical example are summarized in Table 1. Given the parameters in Table 1, the sufficient condition for interior solutions for a pair of $(O, O')$ can be represented by an interval of the expertise improvement from replacement $\Delta \in (\Delta_{\min}, \Delta_{\max})$. The values of $\Delta_{\min}$ and $\Delta_{\max}$ corresponding to different pairs of $(O, O')$ are summarized in Table 2. From Table 2, the size of the interval $(\Delta_{\min}, \Delta_{\max})$ is large if $O$ and $O'$ are close to each other, which implies our result is permissive for close $O$ and $O'$. We can also find that the interval $(\Delta_{\min}, \Delta_{\max})$ is not empty for $O' = N = 10$, i.e., a regulated board with all outsiders can end up with a lower real independence than a regulated board with some insiders. Indeed, if the real independence decays quickly over time (e.g., $\eta = 0.05$), there exists an interval of the expertise improvement from replacement such that the real independence of a regulated board with all outsiders is lower than the real independence of a regulated board with only one outsider at the end of Period 2.\textsuperscript{13} In this case, the sufficient conditions seem restrictive—to the point of resembling an existence result.

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</thead>
<tbody>
<tr>
<td>$b$</td>
<td>CEO’s bias on project choice</td>
<td>0.2</td>
</tr>
<tr>
<td>$k$</td>
<td>board’s monitoring effort tolerance</td>
<td>0.04</td>
</tr>
<tr>
<td>$N$</td>
<td>number of directors on board</td>
<td>10</td>
</tr>
<tr>
<td>$\beta$</td>
<td>discount factor</td>
<td>0.8</td>
</tr>
<tr>
<td>$\sigma_p$</td>
<td>ex-ante uncertainty of optimal project</td>
<td>0.2</td>
</tr>
<tr>
<td>$\eta$</td>
<td>decay rate of real independence</td>
<td>0.3</td>
</tr>
</tbody>
</table>

Table 1: The values of the parameters

If the sufficient conditions are violated, the interior solutions may not exist. On the one hand, if either the real independence does not decay quickly with the director tenure, i.e., $\eta$ is large enough, or the improvement in expertise from replacement $\Delta$ is too large, the CEO may find it optimal to replace all outside directors regardless of the nominal independence.

\textsuperscript{13}For example, given the values of the parameters in Table 1 except that $\eta = 0.05$ instead of 0.3, the interval is $\Delta \in (0.0005, 0.0038)$. 

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Table 2: The interval of the expertise improvement from replacement $\Delta \in (\Delta_{\text{min}}, \Delta_{\text{max}})$ to the satisfy sufficient conditions for interior solutions

<table>
<thead>
<tr>
<th>O</th>
<th>O'</th>
<th>$\Delta_{\text{min}}$</th>
<th>$\Delta_{\text{max}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>5</td>
<td>0.0119</td>
<td>0.1792</td>
</tr>
<tr>
<td>4</td>
<td>6</td>
<td>0.0206</td>
<td>0.1792</td>
</tr>
<tr>
<td>4</td>
<td>7</td>
<td>0.0327</td>
<td>0.1792</td>
</tr>
<tr>
<td>4</td>
<td>8</td>
<td>0.0489</td>
<td>0.1792</td>
</tr>
<tr>
<td>4</td>
<td>9</td>
<td>0.0696</td>
<td>0.1792</td>
</tr>
<tr>
<td>4</td>
<td>10</td>
<td>0.0954</td>
<td>0.1792</td>
</tr>
</tbody>
</table>

level. On the other hand, if the improvement in expertise from replacement $\Delta$ is too small, the CEO may optimally choose to keep all incumbent outsiders no matter what the nominal or real independence is. Although cases without interior solutions are not the focus of our paper, it is important to recognize that our result will sometimes be overturned when the solution is not interior. In particular, a higher nominal board independence leads to a higher real independence if the CEO in the firm with the board with $O'$ outsiders and the CEO in the firm with the board with $O$ outsiders make the same extreme replacement decisions, i.e., either replacing all or no outsiders. Even when only one of the solutions (either $O$ or $O'$) is non-interior, our result is sometimes overturned.

Another important assumption we make in our main model is that the improvement in board expertise is non-persistent, i.e., in each period, all directors that remain on the board revert back to the generalist expertise level $e_o$. This assumption is reasonable to the extent that new directors are hired to deal with a new business strategy or environment. However, the assumption was made primarily to simplify the analysis, and our main result may not hold if expertise is persistent enough. If, for example, director expertise is perfectly persistent, outside directors with a low expertise level would be gradually replaced by new outsiders. The average expertise level, therefore, should increase over time and converge to a final composition where all existing outside directors have a high expertise level and further replacements are no longer beneficial.\(^{14}\) In such a scenario, director tenure would keep growing, and outsiders would eventually lose their independence. Consequently, real

\(^{14}\)We thank an anonymous referee for pointing this out.
independence would converge to 0 regardless of the nominal independence level.

8 Conclusion

This paper offers a new perspective on board independence by focusing on the impact of director tenure on the “real independence” of the board. Instead of taking the fraction of outside directors directly as the measure of board independence, we define it as the “nominal independence” level of the board. The “real independence” level of the board of directors is determined by both the nominal independence level and the length of the relationship between the board members and the CEO. We assume the real independence of outside directors decreases over time if the outside directors stay on the board (no turnover), while the replacement of existing outside directors with new candidates increases the average real independence level of the board. We study a three period model in which the board both monitors and advises the CEO, and the CEO decides whether to replace some of the outside directors in each period. The CEO balances the trade-off between a higher expertise of the board introduced by new directors and a lower real independence of the board attained by keeping the same board. We find that the higher is the nominal independence level of the board (i.e., the higher fraction of outside directors), the more reluctant the CEO is in replacing existing directors. Therefore, a higher nominal independence may result in a lower real independence and a lower expertise of the board of directors in the long-run. In the extension section, we show our result can be generalized to an infinite horizon and relax some assumptions to investigate the conditions under which our result holds and discuss the conditions under which our result is overturned.

Although we do not directly study the optimal regulation of board composition, our results contribute some insights. The regulation on board composition is intended to protect shareholders by improving board independence. By requiring a higher proportion of outside directors on the board, it is likely to weaken the advisory role of the board due to the
superior (or at least different) firm-specific expertise of inside directors. Presumably, optimal regulations on board composition would tradeoff the benefits of higher board independence and the costs of weakening the advising role of the board. However, as shown in this paper, a regulation that requires a higher fraction of outside directors on the board can result in lower board turnover and, hence, decreases the independence of the board. In this case, there are only costs (no benefits) to increasing nominal independence. Building a richer model to study the optimal regulation of tenure is a natural next step.

A related issue that may be interesting to explore in future research is the impact of board regulation on the CEO’s incentive to come up with new business strategies. In our setting, the new business strategy is exogenous. If the new business strategy is endogenous and depends on the CEO’s effort, will regulations that mandate higher nominal independence discourage the CEO from coming up with new business strategies?
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APPENDIX

A Proof

Proof of Lemma 1:

Proof. By the first order condition, we have

\[ km^*_t - i^2_t b^2 = 0 \]

\[ \Rightarrow m^*_t (i_t, e_t) = \frac{i^3_t b^2}{k}. \]

Therefore, we have

\[ \frac{\partial m^*_t (i_t, e_t)}{\partial i_t} = \frac{2i_t b^2}{k} > 0. \]

Proof of Lemma 2:

Proof. The expected payoff of the shareholder given the optimal monitoring intensity of the board is,

\[ E \left[ u^S_t (i_t, e_t) \right] = -m^*_t (i_t, e_t) \left[ e_t (1 - i_t)^2 b^2 + (1 - e_t) \left( \sigma_p^2 + (1 - i_t)^2 b^2 \right) \right] \]

\[ - (1 - m^*_t (i_t, e_t)) \left[ e_t b^2 + (1 - e_t) \left( \sigma_p^2 + b^2 \right) \right], \]

which can be simplified to,

\[ E \left[ u^S_t (i_t, e_t) \right] = -(1 - e_t) \sigma_p^2 - (1 - m^*_t (i_t, e_t)) b^2 - m^*_t (i_t, e_t) (1 - i_t)^2 b^2 \]

\[ = -\sigma_p^2 - b^2 + e_t \sigma_p^2 + m^*_t (i_t, e_t) (2i_t - i^2_t) b^2 \]

\[ = -\sigma_p^2 - b^2 + e_t \sigma_p^2 + \frac{b^4}{k} (2i^3_t - i^4_t). \]

Therefore, we have

\[ \frac{\partial E \left[ u^S_t (i_t, e_t) \right]}{\partial i_t} = 2 (3 - 2i_t) \frac{i^2_t b^4}{k} \]

\[ > 0. \]
Moreover,
\[
\frac{\partial E[u^S(i_t, e_t)]}{\partial e_t} = \sigma_p^2 > 0.
\]

\begin{proof}
Proof of Lemma 3:
Proof. The expected payoff of the CEO given the optimal monitoring intensity of the board
is
\[
E[u^M_m (i_t, e_t)]
= -m^*_t (i_t, e_t) \left[ e_t i^2_t b^2 + (1 - e_t) \left( \sigma_p^2 + i^2_t b^2 \right) \right]
- (1 - m^*_t (i_t, e_t)) (1 - e_t) \sigma_p^2
= - (1 - e_t) \sigma_p^2 - m^*_t (i_t, e_t) i^2_t b^2
= - (1 - e_t) \sigma_p^2 - \frac{i^4_t b^4}{k}.
\]
Then, we have
\[
\frac{\partial E[u^M_m (i_t, e_t)]}{\partial i_t} = - \frac{4i^3_t b^4}{k} < 0
\]
\[
\frac{\partial E[u^M_m (i_t, e_t)]}{\partial e_t} = \sigma_p^2 > 0
\]
\[
\frac{\partial^2 E[u^M_m (i_t, e_t)]}{\partial i_t \partial e_t} = 0
\]
\[
\frac{\partial^2 E[u^M_m (i_t, e_t)]}{\partial i^2_t} = - \frac{12i^2_t b^4}{k} < 0
\]
\[
\frac{\partial^2 E[u^M_m (i_t, e_t)]}{\partial e^2_t} = 0,
\]
which implies \( E[u^M_m (i_t, e_t)] \) decreases in \( i_t \) and increases in \( e_t \), and \( \frac{\partial^2 E[u^M_m (i_t, e_t)]}{\partial i_t \partial e_t} = 0 \), \( \frac{\partial^2 E[u^M_m (i_t, e_t)]}{\partial i^2_t} < 0 \).
\end{proof}

Proof of Lemma 4:

Proof. The manager’s Problem in Period 2 is:
\[
\max_{r_2} E \left[ u^M_2 (i_2, e_2) \right] = -(1 - e_2) \sigma^2 - \frac{b^4 i_2^4}{k}
\]

s.t. \( i_2 (i_1, r_2; o) = i_1 \eta + r_2 \left( 1 - \frac{i_1}{o} \right) \)

\( e_2 (r_2) = e_o + r_2 \Delta. \)

By the first order condition, we have:

\[
\frac{\partial E \left[ u^M_2 (i_2, e_2) \right]}{\partial r_2} = \sigma^2 \Delta - \frac{4b^4 i_2^3}{k} \left( 1 - \frac{i_1}{o} \right) = 0.
\]

The interior solution determined by the first order condition is as follows:

\[
i^*_2 (i_1, o) = \left( \frac{\sigma^2 \Delta}{\frac{4b^4}{k} \left( 1 - \eta i_1^2 o \right)} \right)^{\frac{1}{3}}
\]

\[
r^*_2 (i_1, o) = \frac{i^*_2 (i_1, o) - i_1 \eta}{(1 - \eta i_1^2 o)}.
\]

In order to guarantee that the interior solution is valid, i.e., \( r^*_2 (i_1, o) \in [0, o] \) or equivalently, \( i^*_2 (i_1, o) \in [\eta i_1, o] \), we need to assume that the condition for interior solution is satisfied:

\[
\eta i_1 < \left( \frac{\sigma^2 \Delta}{\frac{4b^4}{k} \left( 1 - \eta i_1^2 o \right)} \right)^{\frac{1}{3}} < o
\]

\[
\eta i_1^3 \left( 1 - \frac{i_1}{o} \right) < \frac{\sigma^2 \Delta}{\frac{4b^4}{k}} < \left( 1 - \frac{i_1}{o} \right)
\]

\[
\frac{4b^4}{k} \left( \frac{i_1}{o} \right)^3 \left( 1 - \frac{i_1}{o} \right) < \frac{\sigma^2 \Delta}{\sigma^2} < \frac{4b^4}{k} \left( 1 - \frac{i_1}{o} \right),
\]

which implies \( \Delta \) should be intermediate, i.e., \( \Delta \in \left( \frac{4b^4}{k} \left( 1 - \eta i_1^2 o \right)^{-\frac{3}{2}}, \frac{4b^4}{k} \left( 1 - \eta i_1^2 o \right)^{-\frac{1}{2}} \right). \)

Proof of Corollary 1

Proof. Since \( i^*_2 (i_1, o) = \left( \frac{\sigma^2 \Delta}{\frac{4b^4}{k} \left( 1 - \eta i_1^2 o \right)} \right)^{\frac{1}{3}} = \left( \frac{\sigma^2 \Delta}{\frac{4b^4}{k}} \right)^{\frac{1}{3}} \left( 1 - \frac{i_1}{o} \right)^{-\frac{1}{2}}, \) we have
\[
\frac{\partial i^*_2(i_1, o)}{\partial i^*_1} = -\frac{1}{3} \left(1 - \eta \frac{i_1}{o}\right)^{-\frac{4}{3}} \left(-\eta\right) \left(\frac{\sigma^2 \Delta}{4b^4}\right)^{\frac{1}{3}}
\]
\[
= \frac{1}{3} \left(\frac{\sigma^2 \Delta}{4b^4}\right)^{\frac{1}{3}} \left(1 - \eta \frac{i_1}{o}\right)^{-\frac{4}{3}} \eta
\]
\[
= \frac{1}{3} \eta \left(1 - \eta \frac{i_1}{o}\right)
\]
which implies \(i^*_2(i_1, o)\) is increasing in \(\frac{i_1}{o}\).

**Proof of Lemma 5:**

*Proof.* By the Envelope Theorem, we have:

\[
\frac{\partial E\left[u^*_2(i_2, e_2) | r_2 = r^*_2(i_1, o)\right]}{\partial i_1} = \frac{\partial E\left[u^*_2(i_2(i_1, r_2; o), e_2(r_2))\right]}{\partial i_1} \bigg|_{r_2 = r^*_2(i_1, o)}
\]
\[
= - \frac{4b^4 \left[i^*_2(i_1, o)\right]^3}{k} \eta \left(1 - \frac{r_2}{o}\right) \bigg|_{r_2 = r^*_2(i_1, o)}.
\]

Substituting \(r^*_2(i_1, o) = \frac{i^*_2(i_1, o) - i_1 o}{1 - \eta \frac{i_1}{o}}\) and using the result in Lemma 4 that \(\sigma^2 \Delta - \frac{4b^4 \left[i^*_2(i_1, o)\right]^3}{k} \left(1 - \eta \frac{i_1}{o}\right) = 0\), we can obtain:

\[
\frac{\partial E\left[u^*_2(i_2, e_2) | r_2 = r^*_2(i_1, o)\right]}{\partial i_1} = - \frac{\sigma^2 \Delta}{\left(1 - \eta \frac{i_1}{o}\right)^2} \left(1 - \frac{i^*_2(i_1, o)}{\eta \left(1 - \eta \frac{i_1}{o}\right)}\right)
\]
\[
< 0.
\]
Therefore, we have the first order condition to the optimal replacement decision problem at the beginning of Period 1 as follows:

\[
\frac{\partial U_M}{\partial r_1} = \sigma^2 \frac{de_1 (r_1)}{dr_1} - \frac{4b^4i^3}{k} \frac{di_1 (r_1; \omega)}{dr_1} + \beta \frac{\partial E [u'_M (i_2, e_2) | r_2 = r^*_2 (i_1, \omega)]}{\partial i_1} \frac{di_1 (r_1; \omega)}{dr_1} \\
= \sigma^2 \Delta - \left[ \frac{4b^4i^3}{k} + \beta \frac{\sigma^2 \Delta \eta \left( 1 - \frac{i^*_2 (i_1, \omega)}{\omega} \right)}{(1 - \eta_{i_1}^o)^2} \right] (1 - \eta) \\
= 0.
\]

Meanwhile, the second order condition below should also be valid.

\[
\frac{\partial^2 U_M}{\partial r_1^2} = \frac{\partial \left( \frac{\partial U_M}{\partial r_1} \right)}{\partial r_1} \\
= \frac{\partial}{\partial i_1} \left( \sigma^2 \Delta - \left[ \frac{4b^4i^3}{k} + \beta \frac{\sigma^2 \Delta \eta \left( 1 - \frac{i^*_2 (i_1, \omega)}{\omega} \right)}{(1 - \eta_{i_1}^o)^2} \right] (1 - \eta) \right) \frac{di_1 (r_1; \omega)}{dr_1} \\
= (1 - \eta) \frac{\partial}{\partial i_1} \left( \sigma^2 \Delta - \left[ \frac{4b^4i^3}{k} + \beta \frac{\sigma^2 \Delta \eta \left( 1 - \frac{i^*_2 (i_1, \omega)}{\omega} \right)}{(1 - \eta_{i_1}^o)^2} \right] (1 - \eta) \right) \\
= - (1 - \eta)^2 \sigma^2 \Delta \left( \frac{12b^4i^2}{k\sigma^2 \Delta} + \beta \eta \left( \frac{2 - \frac{7}{3} \frac{i^*_2 (i_1, \omega)}{\omega}}{(1 - \eta_{i_1}^o)^3} \right) \right) \\
< 0.
\]

One sufficient condition for second order condition is: \( \eta < 1 - \sqrt{\frac{3}{7}} \), which can be proved as follows.
Since $\eta_1 < i_2^* (i_1, o) = \left(\frac{\sigma^2 \Delta}{4b^4 k (1 - \eta_1 o)}\right)^\frac{1}{3} < o$ and $i_1^* (o) \in [\eta o, o]$, we have

$$
\frac{12b^4 i_1^2}{k \sigma^2 \Delta} + \beta \eta \frac{2 - \frac{7}{3} \frac{i_2^* (i_1, o)}{o}}{(1 - \eta_1 o)^3} = \frac{3i_1^2}{[i_2^* (i_1, o)]^3 (1 - \eta_1 o)} + \beta \eta \frac{2 - \frac{7}{3} \frac{i_2^* (i_1, o)}{o}}{(1 - \eta_1 o)^3} > \frac{3i_1^2}{\sigma^3 (1 - \eta_1 o)} + \beta \eta \frac{2 - \frac{7}{3}}{(1 - \eta_1 o)^3} = \frac{3}{o \left(1 - \eta_1 o\right)} \left(\frac{i_1^2}{\sigma^2} - \frac{1}{9} \beta \eta^2 \frac{1}{(1 - \eta_1 o)^3}\right)
$$

$$
= \frac{3}{o \left(1 - \eta_1 o\right)} \left(\eta^2 - \frac{1}{9} \beta \eta^2 \frac{1}{(1 - \eta)^2}\right)
$$

$$
= \frac{3\eta^2 o^2}{\sigma^3 (1 - \eta_1 o)} \left(1 - \frac{\beta}{9 (1 - \eta)^2}\right),
$$

which is positive if $\eta < 1 - \sqrt[3]{\frac{\beta}{3}}$.

The second order condition implies $\sigma^2 \Delta = \left[\frac{4b^4 o^3}{k} + \frac{\sigma^2 \Delta \eta (1 - \frac{i_2^* (i_1, o)}{o})}{(1 - \eta_1 o)^2}\right] (1 - \eta)$ is decreasing in $i_1$. Meanwhile, to guarantee that the interior solution exists i.e., $i_1^* (o) \in [\eta o, o]$, we should have the following two conditions:

1. $\frac{\partial U_M}{\partial r_1}$ is negative at $i_1 = o$, which can be stated as:

$$
\sigma^2 \Delta - \left[\frac{4b^4 o^3}{k} + \frac{\sigma^2 \Delta \eta (1 - \frac{i_2^* (o, o)}{o})}{(1 - \eta_1 o)^2}\right] (1 - \eta) < 0
$$

$$
\sigma^2 \Delta \left(1 - \beta \frac{\eta \left(1 - \frac{1}{o} \left(\frac{\sigma^2 \Delta}{4b^4 k (1 - \eta_1 o)}\right)^\frac{1}{3}\right)}{(1 - \eta)}\right) - \frac{4b^4 o^3 (1 - \eta)}{k} < 0.
$$

A sufficient condition to guarantee the above inequality hold is as follows:

$$
\sigma^2 \Delta - \frac{4b^4 o^3 (1 - \eta)}{k} < 0
$$

$$
\Delta < \frac{4b^4 o^3 (1 - \eta)}{\sigma^2 k}.
$$

2. $\frac{\partial U_M}{\partial r_1}$ is positive at $i_1 = \eta o$, which can be stated as:
\[
\sigma^2 \Delta = \left[ \frac{4b^4 (\eta o)^3}{k} + \beta \frac{\sigma^2 \Delta \eta \left( 1 - \frac{i_2^*(\eta o, o)}{o} \right)}{(1 - \eta^2)^2} \right] (1 - \eta) \geq 0
\]

\[
\sigma^2 \Delta \left( 1 - \beta \left( 1 - \frac{\eta}{(1 + \eta)} \right) \right) - \frac{4b^4 (\eta o)^3 (1 - \eta)}{k} \geq 0
\]

A sufficient condition to guarantee the above inequality hold is as follows:

\[
\sigma^2 \Delta \left( 1 - \beta \frac{\eta}{(1 + \eta)} \right) - \frac{4b^4 (\eta o)^3 (1 - \eta)}{k} \geq 0
\]

\[
\Delta \geq \frac{4b^4 \sigma^3 (1 - \eta)}{k \sigma^2} \frac{\eta^3}{1 - \beta \frac{\eta}{(1 + \eta)}}.
\]

Therefore, a sufficient condition for the second order condition is

\[\Delta \in \left( \frac{4b^4 \sigma^3 (1 - \eta)}{k \sigma^2} \frac{\eta^3}{1 - \beta \frac{\eta}{(1 + \eta)}}, \frac{4b^4 \sigma^3 (1 - \eta)}{k \sigma^2} \frac{\eta^3}{1 - \beta \frac{\eta}{(1 + \eta)}} \right), \text{ which is nonempty if } \eta < 1 - \frac{\sqrt{3}}{3}.\]

In Lemma 4, we have shown that to guarantee that an interior solution in Period 2, the expertise improvement should satisfy: \[\Delta \in \left( \frac{4b^4}{k} \left( 1 - \eta \frac{i_1}{o} \right) \frac{(\eta o)^3}{\sigma^2}, \frac{4b^4}{k} \left( 1 - \eta \frac{i_1}{o} \right) \frac{\sigma o^3}{\sigma^2} \right). \]

A sufficient condition to guarantee \[\Delta \in \left( \frac{4b^4}{k} \left( 1 - \eta \frac{i_1}{o} \right) \frac{(\eta o)^3}{\sigma^2}, \frac{4b^4}{k} \left( 1 - \eta \frac{i_1}{o} \right) \frac{\sigma o^3}{\sigma^2} \right), \text{ which does not depend on the value of } i_1 \in (\eta o, o), \text{ is } \eta < \frac{3}{4}, \Delta \in \left( \frac{4b^4}{k} \left( 1 - \eta \right) \frac{(\eta o)^3}{\sigma^2}, \frac{4b^4}{k} \left( 1 - \eta^2 \right) \frac{\sigma o^3}{\sigma^2} \right).\]

Therefore, an overall sufficient condition to guarantee an interior solution in both periods is \[\eta < \min \left\{ \frac{3}{4}, 1 - \frac{\sqrt{3}}{3} \right\}, \Delta \in \left( \frac{4b^4 \sigma^3 (1 - \eta)}{k \sigma^2} \frac{\eta^3}{1 - \beta \frac{\eta}{(1 + \eta)}}, \frac{4b^4 \sigma^3 (1 - \eta)}{k \sigma^2} \frac{\eta^3}{1 - \beta \frac{\eta}{(1 + \eta)}} \right).\]

\[\square\]

**Proof of Lemma 6:**

We show that \[\frac{i_1^*(o)}{o} \text{ is decreasing in } o \text{ by showing contradiction. In equilibrium, } i_1^*(o) \text{ is determined by the first order condition:}\]

\[\sigma^2 \Delta = \left[ \frac{4b^4 (i_1)^3}{k} + \beta \frac{\sigma^2 \Delta \eta \left( 1 - \frac{i_2^*(i_1, o)}{o} \right)}{(1 - \eta \frac{i_1}{o})^2} \right] (1 - \eta) = 0 \tag{25}\]

in which the LHS is decreasing in \(i_1\) due to the second order condition. If \(o' > o\) and \[\frac{i_1^*(o')}{o'} \geq \frac{i_1^*(o)}{o}, \text{ denote } \hat{i}_1(o') = \frac{i_1^*(o)}{o} o', \text{ we have} \]

\[i_1^*(o') \geq \frac{i_1^*(o)}{o} o' = \hat{i}_1(o') > i_1^*(o)\]

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Therefore, we obtain

\[
0 = \sigma^2 \Delta - \left[ \frac{4b^4 (i_1^* (o))^3}{k} + \beta \frac{\sigma^2 \Delta \eta \left( 1 - \frac{i_2^* (i_1^* (o), o)}{o} \right)}{1 - \eta i_1^* (o) / o} \right] (1 - \eta)
\]

\[
= \sigma^2 \Delta - \left[ \frac{4b^4 (i_1^* (o))^3}{k} + \beta \frac{\sigma^2 \Delta \eta \left( 1 - \frac{i_2^* (i_1' (o'), o')}{1 - \eta i_1' (o') / o'} \right)}{1 - \eta i_1' (o') / o'} \right] (1 - \eta)
\]

\[
> \sigma^2 \Delta - \left[ \frac{4b^4 (i_1' (o'))^3}{k} + \beta \frac{\sigma^2 \Delta \eta \left( 1 - \frac{i_2^* (i_1' (o'), o')}{1 - \eta i_1' (o') / o'} \right)}{1 - \eta i_1' (o') / o'} \right] (1 - \eta)
\]

\[
\geq \sigma^2 \Delta - \left[ \frac{4b^4 (i_1' (o'))^3}{k} + \beta \frac{\sigma^2 \Delta \eta \left( 1 - \frac{i_2^* (i_1' (o'), o')}{1 - \eta i_1' (o') / o'} \right)}{1 - \eta i_1' (o') / o'} \right] (1 - \eta)
\]

\[
= 0,
\]

where the first inequality comes from \( \hat{i}_1 (o') > i_1^* (o) \), and the second inequality comes from the second condition that LHS (25) is decreasing in \( i_1 \) and \( i_1^* (o') \geq \hat{i}_1 (o') \), which is a contradiction. Therefore, we have \( \frac{i_1^* (o)}{o} \) is decreasing in \( o \).

\[\square\]

**Proof of Proposition 1:**

*Proof.* By Lemma 6, we have the ratio \( \frac{i_1^* (o)}{o} \) is decreasing in the nominal independence \( o \). Moreover, by Corollary 1, we have the resultant real independence in Period 2 \( i_2^* (i_1, o) \) is increasing in the ratio \( \frac{i_1}{o} \). Therefore, we have the real independence in Period 2 \( i_2^* (i_1^* (o), o) \) is decreasing in the nominal independence \( o \).

\[\square\]

**Proof of Proposition 2:**

*Proof.* Since the policy function \( I (i; o) \)is the solution to maximize the value function \( V_M (i; o) \), by the first order condition, we have:

\[
\frac{1}{1 - \eta i_2^*} \Delta \sigma^2 - \frac{4b^4 (I (i; o))^3}{k} + \beta \frac{\partial V_M (i; o)}{\partial i} \bigg|_{i = I (i; o)} = 0,
\]

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and by Envelope Theorem, we have:

$$\frac{\partial V_M(i; o)}{\partial i} = \frac{\eta \left( \frac{i(i; o)}{o} - 1 \right)}{(1 - i_o)} \Delta \sigma^2.$$ 

If there is a fixed point, i.e., \(i^*(o)\) s.t. \(I(i^*(o); o) = i^*(o)\), it should satisfy both the first order condition for the optimal replacement decision to determined the value function and the the first derivative formula of the value function derived using the Envelope Theorem. Therefore, the fixed point \(i^*(o)\) satisfies:

$$\frac{1}{1 - i_o^2} \Delta \sigma^2 - \frac{4 b^4 i^3}{k} + \frac{\eta \left( \frac{i}{o} - 1 \right)}{(1 - i_o^2)} \Delta \sigma^2 = 0$$

$$\left[ \beta \left( \frac{1 - \eta}{(1 - i_o^2)} \right) + \frac{1 - \beta}{(1 - i_o^2)} \right] \Delta \sigma^2 - \frac{4 b^4 i^3}{k} = 0$$

$$i^3 \left[ \frac{\beta(1 - \eta)}{(1 - i_o^2)^2} + \frac{1 - \beta}{(1 - i_o^2)} \right] = \frac{\Delta \sigma^2}{4b^4 i^3 k}$$

$$i^3 \left( \frac{1 - i_o^2}{(1 - \beta \eta) - (1 - \beta) i_o^2} \right)^2 = \frac{\Delta \sigma^2}{4b^4 i^3 k}. \quad (26)$$

Now, we show that equation (26) has a unique solution if \(\Delta \in \left(0, \frac{k^4}{4 \sigma_o^2} (1 - \eta) \right)\).

In order to prove the unique solution, we discuss two cases, i.e., \(\frac{\partial LHS(26)}{\partial i} \geq 0\) and \(\frac{\partial LHS(26)}{\partial i} < 0\).

First, we show that if \(\eta \leq \frac{3}{4 + \beta}, \frac{\partial LHS(26)}{\partial i} \geq 0\). Specifically, we have

$$\frac{\partial LHS(26)}{\partial i} = i^2 \left[ 3 \left( 1 - \eta_o^2 \right)^2 - 2 \eta_o^2 (1 - \eta_o^2) \right] \left[ (1 - \beta \eta) - (1 - \beta) \eta_o^2 \right] + (1 - \beta) \frac{\eta_o}{2} i^3 (1 - \eta_o^2)^2$$

$$= i^2 \left[ 3 \left( 1 - \eta_o^2 \right)^2 - 2 \eta_o^2 (1 - \eta_o^2) \right] \left[ (1 - \beta \eta) - (1 - \beta) \eta_o^2 \right]$$

$$= i^2 \left( 3 - 5 \eta_o^2 \right) \left( (1 - \beta \eta) - 2 (1 - \beta) \eta_o^2 (1 - 2 \eta_o^2) \right)$$

$$= \frac{4 b^4 i^2}{k} \left( 1 - \eta_o^2 \right) \left[ 4 (1 - \beta) \left( \eta_o^2 \right)^2 - (7 - 5 \beta \eta - 2 \beta) \eta_o^2 + 3 \right] \left( (1 - \beta \eta) - (1 - \beta) \eta_o^2 \right)$$

$$= \frac{4 b^4 i^2}{k} \left( 1 - \eta_o^2 \right) \left[ 4 (1 - \beta) \left( \eta_o^2 \right)^2 - (7 - 5 \beta \eta - 2 \beta) \eta_o^2 + 3 \right] \left( (1 - \beta \eta) - (1 - \beta) \eta_o^2 \right)$$

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in which
\[
\frac{\partial}{\partial \eta^i_o} \left[ 4 (1 - \beta) \left( \eta^i_o \right)^2 - (7 - 5\beta \eta - 2\beta) \eta^i_o + 3 (1 - \beta \eta) \right] \\
= 8 (1 - \beta) \left( \eta^{-i}_o \right) - (7 - 5\beta \eta - 2\beta) \\
< 8 (1 - \beta) \eta - (7 - 5\beta \eta - 2\beta) \\
= -7 + 8\eta - 3\beta \eta + 2\beta \\
< \max \{-5 + 5\eta, -7 + 8\eta\}.
\]

The last term is negative if \( \eta < \frac{7}{8} \). Therefore, if \( \eta \leq \frac{3}{4+\beta} < \frac{7}{8} \), we have
\[
\left[ 4 (1 - \beta) \left( \eta^i_o \right)^2 - (7 - 5\beta \eta - 2\beta) \eta^i_o + 3 (1 - \beta \eta) \right] \\
\geq 4 (1 - \beta) \eta^2 - (7 - 5\beta \eta - 2\beta) \eta + 3 (1 - \beta \eta) \\
= (4 + \beta) \eta^2 - (7 + \beta) \eta + 3 \\
= (\eta - 1) ((4 + \beta) \eta - 3) \\
\geq 0,
\]
which implies \( \frac{\partial LHS(26)}{\partial i} \geq 0 \).

If \( \eta \leq \frac{3}{4+\beta} \), and, thus, \( \frac{\partial LHS(26)}{\partial i} \geq 0 \), in order to guarantee that the fixed point exists, i.e., \( i^*(o) \in (0, o) \), we have:
\[
LHS(26)|_{i=0} < RHS(26) < LHS(26)|_{i=o} \\
0 < \Delta < \frac{b^4}{k\sigma^2}4o^3(1 - \eta).
\]

Otherwise if \( \Delta \geq \frac{b^4}{k\sigma^2}4o^3(1 - \eta) \), it will be corner solution that \( i^*(o) = o \).

Alternatively, if \( \eta > \frac{3}{4+\beta} \), we have 4 \( (1 - \beta) \left( \eta^i_o \right)^2 - (7 - 5\beta \eta - 2\beta) \eta^i_o + 3 (1 - \beta \eta) \) is positive when \( i = 0 \) and is negative when \( i = o \), which implies that \( \exists i^#(o) \in (0, o) \), s.t., \( \frac{\partial LHS(26)}{\partial i} \) is positive for \( i \in (0, i^#(o)) \) and is negative for \( i \in (i^#(o), o) \). Thus, \( LHS(26) \) is a concave function of \( i \in (0, o) \). Moreover, it is easy to show that \( LHS(26)|_{i=0} = 0 < LHS(26)|_{i=o} \). Hence, if \( RHS(26) < LHS(26)|_{i=o} \) or equivalently, \( \Delta < \frac{b^4}{k\sigma^2}4o^3(1 - \eta) \), equation (26) also has a unique solution for \( i^*(o) \) with \( \frac{\partial LHS(26)}{\partial i}|_{i=i^*(o)} > 0 \). Otherwise if \( \Delta \geq \frac{b^4}{k\sigma^2}4o^3(1 - \eta) \), we obtain a corner solution with \( i^*(o) = o \), or there are two interior solutions which satisfy equation (26), and the stable fixed point of real independence which can be achieved in the dynamics is closer to the 

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nominal independence $o$ and satisfies $\frac{\partial LHS(26)}{\partial i}|_{i=i^*(o)} < 0$.

Moreover,

\[
\frac{\partial}{\partial o} \left[ \frac{4b^4 i^3}{k} \left[ \frac{\beta (1-\eta)}{(1-\eta^2)^2} + \frac{(1-\beta)}{(1-\eta^2)} \right] \right] > 0.
\]

Therefore, $\forall o, o’, s.t., 0 < o < o’ < 1$, if $\Delta \in \left(0, \frac{b^4}{k\sigma^2} 4o^3 (1 - \eta)\right)$, we have $i^*(o’) < i^*(o)$, which implies higher nominal independence results in lower real independence. Otherwise if $\Delta \geq \frac{b^4}{k\sigma^2} 4(o’)^3 (1 - \eta)$, we have $i^*(o’) > i^*(o)$, which implies higher nominal independence results in higher real independence.

\[
\Box
\]

**Proof of Proposition 3:**

*Proof.* Similar to the proof of Proposition 2, if there is a fixed point, i.e., $i^*(o)$ s.t. $I(i^*(o) ; o) = i^*(o)$, it should satisfy:

\[
\frac{1}{1-\eta_o^2} G’ \left( \frac{i-\eta o}{1-\eta_o^2} \right) \sigma^2 - \frac{4b^4 i^3}{k} + \beta \eta \frac{(i-1)}{(1-\eta_o^2)^2} G’ \left( \frac{i-\eta o}{1-\eta_o^2} \right) \sigma^2 = 0,
\]

which can be rewritten as:

\[
\frac{(1-\beta\eta) - (1-\beta) \eta_o^2}{(1-\eta_o^2)^2} G’ \left( \frac{i-\eta o}{1-\eta_o^2} \right) \sigma^2 - \frac{4b^4 i^3}{k} = 0
\]

\[
\left[ \frac{\beta (1-\eta)}{(1-\eta_o^2)^2} + \frac{(1-\beta)}{(1-\eta_o^2)} \right] G’ \left( \frac{i-\eta o}{1-\eta_o^2} \right) \sigma^2 - \frac{4b^4 i^3}{k} = 0
\]

\[
\frac{1}{i^3} \left[ \frac{\beta (1-\eta)}{(1-\eta_o^2)^2} + \frac{(1-\beta)}{(1-\eta_o^2)} \right] G’ \left( \frac{1-\eta}{1-\eta_o^2} \right) = \frac{4b^4}{k\sigma^2}.
\]

(27)

We look at the sign of $\frac{\partial}{\partial i} LHS (27)$ and $\frac{\partial}{\partial o} LHS (27)$ to determine whether $i^*(o)$ is increasing or decreasing in $o$. 

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On one hand, we have

\[ \frac{\partial}{\partial i} LHS (27) \]
\[ = \frac{1}{i^3} \left[ \frac{\beta (1 - \eta)}{(1 - \eta_i^2) + (1 - \eta_i)} \right] G'' \left( \frac{1 - \eta}{1 - \eta_i^2} \right) \frac{1}{i^2} \frac{1 - \eta}{1 - \eta_i^2} \]
\[ + \frac{\partial}{\partial i} \left\{ \frac{1}{i^3} \left[ \frac{\beta (1 - \eta)}{(1 - \eta_i^2) + (1 - \eta_i)} \right] G' \left( \frac{1 - \eta}{1 - \eta_i^2} \right) \right\} \]
\[ = \frac{1}{i^3} \left[ \frac{\beta (1 - \eta)}{(1 - \eta_i^2) + (1 - \eta_i)} \right] G'' \left( \frac{1 - \eta}{1 - \eta_i^2} \right) \frac{1}{i^2} \frac{1 - \eta}{1 - \eta_i^2} \]
\[ + \left\{ -\frac{3}{i^4} \left[ \frac{\beta (1 - \eta)}{(1 - \eta_i^2) + (1 - \eta_i)} \right] \frac{1}{i^3} \frac{1}{i^2} \left[ \frac{2 \beta (1 - \eta)}{(1 - \eta_i^2) + (1 - \eta_i)} \right] \right\} G' \left( \frac{1 - \eta}{1 - \eta_i^2} \right) \]
\[ = \frac{1 - \eta}{i^4 (1 - \eta_i^2) + (1 - \beta)} \left[ \frac{\beta (1 - \eta)}{(1 - \eta_i^2) + (1 - \eta_i)} \right] G'' \left( \frac{1 - \eta}{1 - \eta_i^2} \right) \]
\[ + \frac{1}{i^4 (1 - \eta_i^2)} \left\{ \frac{2 \beta (1 - \eta)}{(1 - \eta_i^2) + (1 - \eta_i)} - 4 (1 - \beta) \right\} G' \left( \frac{1 - \eta}{1 - \eta_i^2} \right). \]

If \( \eta < \frac{3}{4 + \beta} \), we have

\[ \frac{2 \beta (1 - \eta)}{(1 - \eta_i^2) + (1 - \eta_i)} - 4 (1 - \beta) \]
\[ \leq \max \left\{ \frac{2 \beta (1 - \eta)}{(1 - \eta_i^2) + (1 - \eta_i)} - 4 (1 - \beta), 2 \beta (1 - \eta) - 5 \beta (1 - \eta) - 3 (1 - \beta) \right\} \]
\[ = \max \left\{ -\frac{3 + (4 + \beta) \eta}{(1 - \eta)}, -3 \beta (1 - \eta) + (1 - \beta) \right\} \]
\[ < 0. \]

Therefore, if \( \eta < \frac{3}{4 + \beta} \), we have \( \frac{\partial}{\partial i} LHS (27) < 0. \) Meanwhile, in order to guarantee that \( \exists i \in (0, o) \) s.t. \( LHS (27) = \frac{4b^4}{k_\sigma^2} \), we should have \( LHS (27) |_{i=0} > \frac{4b^4}{k_\sigma^2} \) and \( LHS (27) |_{i=o} < \frac{4b^4}{k_\sigma^2} \), which implies \( G' (r) \in \left( 0, \frac{b^4}{k_\sigma^2} \right) \).

On the other hand, we have
$$\frac{\partial}{\partial o} LHS (27)$$

$$= - \frac{1}{i^3} \left[ \beta (1 - \eta) \left( 1 - \eta^2_o \right)^2 + \frac{(1 - \beta)}{(1 - \eta^2_o)} \right] G'' \left( \frac{1 - \eta}{1 - \eta^2_o} \right) \frac{1}{i^3} \frac{1 - \eta}{1 - \eta^2_o}$$

$$+ \frac{\partial}{\partial o} \left\{ \frac{1}{i^3} \left[ \beta (1 - \eta) \left( 1 - \eta^2_o \right)^2 + \frac{(1 - \beta)}{(1 - \eta^2_o)} \right] \right\} G' \left( \frac{1 - \eta}{1 - \eta^2_o} \right)$$

$$= - \frac{\eta_i}{i^3} \left[ (1 - \eta) \left\{ \beta (1 - \eta) \left( 1 - \eta^2_o \right)^2 + (1 - \beta) \right\} G'' \left( \frac{1 - \eta}{1 - \eta^2_o} \right) \right.$$

$$+ \left\{ 2 \beta (1 - \eta) + (1 - \beta) \right\} G' \left( \frac{1 - \eta}{1 - \eta^2_o} \right) \}.$$

If \( \frac{G''(r)}{G'(r)} < -\frac{1 + \beta}{1 - \eta} \), since \( \frac{1}{1 + \frac{(1 - \beta)(1 - \eta^2_o)}{\beta(1 - \eta)}} \leq \frac{1}{1 + \frac{(1 - \beta)(1 - \eta^2_o)}{\beta(1 - \eta)}} = \beta \), we have

$$\frac{G'' \left( \frac{1 - \eta}{1 - \eta^2_o} \right)}{G' \left( \frac{1 - \eta}{1 - \eta^2_o} \right)} < - \frac{1 + \beta}{1 - \eta}$$

\( \leq - \left( \frac{1}{1 - \eta} + \frac{1}{(1 - \eta) \left[ 1 + \frac{(1 - \beta)(1 - \eta^2_o)}{\beta(1 - \eta)} \right]} \right) \)

$$= - \frac{2 \beta (1 - \eta) + (1 - \beta)}{(1 - \eta) \left[ \beta(1 - \eta) \left( 1 - \eta^2_o \right)^2 + (1 - \beta) \right]},$$

which implies \( (1 - \eta) \left[ \frac{\beta (1 - \eta)}{(1 - \eta^2_o)} + (1 - \beta) \right] G'' \left( \frac{1 - \eta}{1 - \eta^2_o} \right) + \left[ 2 \frac{\beta (1 - \eta)}{(1 - \eta^2_o)} + (1 - \beta) \right] G' \left( \frac{1 - \eta}{1 - \eta^2_o} \right) < 0 \)

and, thus, \( \frac{\partial}{\partial o} LHS (27) > 0 \).
If \( G''(r) > -\left(\frac{1}{1-\eta} + \frac{\beta}{1-\beta\eta}\right) \), since \( \frac{1}{1-\beta(1-\eta)\frac{1}{1-\eta}} \geq \frac{1}{1+\beta(1-\eta)} = \frac{\beta(1-\eta)}{1-\beta\eta} \), we have

\[
\frac{G''\left(\frac{1}{1-\eta}\right)}{G'(\frac{1}{1-\eta})} > -\left(\frac{1}{1-\eta} + \frac{\beta}{1-\beta\eta}\right)
\]

\[
\geq -\left(\frac{1}{1-\eta} + \frac{1}{(1-\eta)\left(1 + \frac{(1-\beta)(1-\eta\frac{1}{1-\eta})}{\beta(1-\eta)}\right)}\right)
\]

\[
= -\frac{2\beta(1-\eta)}{(1-\eta)\left(1 + \frac{(1-\eta\frac{1}{1-\eta})}{(1-\eta\frac{1}{1-\eta})}\right)}(1-\beta),
\]

which implies \((1-\eta)\left[\frac{\beta(1-\eta)}{(1-\eta\frac{1}{1-\eta})} + (1-\beta)\right]G''\left(\frac{1}{1-\eta}\right) + \left[2\frac{\beta(1-\eta)}{(1-\eta\frac{1}{1-\eta})} + (1-\beta)\right]G'\left(\frac{1}{1-\eta}\right) > 0\)

and, thus, \(\frac{\partial}{\partial \eta LHS (27)} < 0\).

Therefore, \(\forall o, o', s.t., 0 < o < o' < 1\), if \(\eta < 3 \delta_{4+\beta}\), \(G^*(r) \in \left(0, \frac{4\delta^4}{k^4}\right)\), we have

\(i^* (o') < i^* (o)\) if \(\frac{G''(r)}{G'(r)} > -\left(\frac{1}{1-\eta} + \frac{\beta}{1-\beta\eta}\right)\), and \(i^* (o') > i^* (o)\) if \(\frac{G''(r)}{G'(r)} < -\left(\frac{1}{1-\eta} + \frac{\beta}{1-\beta\eta}\right)\). Note that if \(G''\left(\frac{1}{1-\eta}\right) = 0\), we have the original result in which the formula is decreasing in both \(i\) and \(o\), so a higher \(o\) results in a lower \(i^* (o)\). Otherwise if \(G''\left(\frac{1}{1-\eta}\right)\) is small enough, the formula is decreasing in \(i\) but increasing in \(o\), so a higher \(o\) results in a higher \(i^* (o)\).

\[
\square
\]

**Proof of Proposition 4:**

*Proof.* By the first order condition of the value function, we have:

\[
\frac{1}{1-\eta^2} \Delta^2 - 2b^2 i^* \left(1 - \frac{(bi')^2}{k}\right) + \beta \frac{\partial V (i; o)}{\partial i} \bigg|_{i=I(i; o)} = 0,
\]

and by Envelope Theorem, we have:

\[
\frac{\partial V (i; o)}{\partial i} = \frac{\eta \left(I(i; o) - 1\right)}{(1-\eta\frac{1}{1-\eta})^2} \Delta^2.
\]

If there is a fixed point, i.e., \(i^* (o)\) s.t. \(I (i^*(o) ; o) = i^* (o)\), it satisfies both the first order condition for optimal replacement decision to determined the value function and the the first
derivative formula of the value function derived using the Envelope Theorem. Therefore, the fixed point \(i^*(o)\) should make the following equation hold:

\[
\frac{1}{1 - \eta_o^i} \Delta \sigma^2 - 2b^2i \left(1 - \frac{(bi)^2}{k}\right) + \beta \frac{\eta(o)}{(1 - \eta_o^i)^2} \Delta \sigma^2 = 0
\]

\[
\frac{(1 - \beta \eta) - (1 - \beta) \eta_o^i}{(1 - \eta_o^i)^2} \Delta \sigma^2 - 2b^2i \left(1 - \frac{(bi)^2}{k}\right) = 0
\]

\[
\frac{\beta (1 - \eta)}{(1 - \eta_o^i)^2 + (1 - \beta)} \Delta \sigma^2 - 2b^2i \left(1 - \frac{(bi)^2}{k}\right) = 0
\]

\[
i \left(1 - \frac{(bi)^2}{k}\right) \left(1 - \eta_o^i\right)^2 \frac{\Delta \sigma^2}{(1 - \beta \eta) - (1 - \beta) \eta_o^i} = \frac{\Delta \sigma^2}{2b^2i}.
\]  

(28)

We will prove that equation (28) has a unique solution if \(b^2i < \frac{1}{3}\), \(\eta < \min \left\{ \frac{2}{5}, \frac{1 - 3\beta}{2 + \beta - (4 + \beta)\frac{3^2}{k}} \right\}\), and \(\Delta \in \left(0, \frac{2b^2i}{\sigma^2o} \left(1 - \frac{(bi)^2}{k}\right) (1 - \eta)\right)\).

First, we prove that if \(b^2i < \frac{1}{3}\), \(\eta < \min \left\{ \frac{2}{5}, \frac{1 - 3\beta}{2 + \beta - (4 + \beta)\frac{3^2}{k}} \right\}\), we have \(\frac{\partial}{\partial i} LHS\) (28) > 0. Specifically, we obtain:

\[
\frac{\partial}{\partial i} LHS (28) = \frac{(1 - \beta) \eta_o^i \left(1 - \frac{(bi)^2}{k}\right) \left(1 - \eta_o^i\right)^2}{[(1 - \beta \eta) - (1 - \beta) \eta_o^i]^2}
\]

\[
+ \left(1 - \frac{i}{\eta_o^i}\right) \frac{(1 - 3\eta_o^i) (1 - \eta) + 2 (1 - \beta) (\eta_o^i)^2}{[(1 - \beta \eta) - (1 - \beta) \eta_o^i]^2}
\]

\[
+ \left(1 - \eta_o^i\right) \frac{(bi)^2}{k} \frac{-3 (1 - \beta \eta) + (7 - 5 \beta \eta - 2 \beta) \eta_o^i - 4 (1 - \beta) (\eta_o^i)^2}{[(1 - \beta \eta) - (1 - \beta) \eta_o^i]^2},
\]

where
\[
\frac{\partial}{\partial i} \left[ \left( 1 - 3\eta \frac{i}{o} \right) (1 - \beta \eta) + 2 (1 - \beta) \left( \eta \frac{i}{o} \right)^2 \right] = \eta \frac{1}{o} \left[ -3 (1 - \beta \eta) + 4 (1 - \beta) \left( \eta \frac{i}{o} \right) \right] < \eta \frac{1}{o} \left[ -3 (1 - \beta \eta) + 4 (1 - \beta) \eta \right] = \eta \frac{1}{o} \left[ -3 - \beta \eta + 4 \eta \right],
\]

which is negative if \( \eta < \frac{3}{4 - \beta} \). Therefore, if \( \eta < \frac{3}{4 - \beta} \), we have:

\[
\left( 1 - 3\eta \frac{i}{o} \right) (1 - \beta \eta) + 2 (1 - \beta) \left( \eta \frac{i}{o} \right)^2 \geq (1 - 3\eta) (1 - \beta \eta) + 2 (1 - \beta) \eta^2 = 1 - (3 + \beta) \eta + (2 + \beta) \eta^2 = (1 - (2 + \beta) \eta) (1 - \eta),
\]

and

\[
\frac{\partial}{\partial i} \left\{ i^2 \left[ -3 (1 - \beta \eta) + (7 - 5\beta \eta - 2\beta) \eta \frac{i}{o} - 4 (1 - \beta) \left( \eta \frac{i}{o} \right)^2 \right] \right\} = i \left[ -6 (1 - \beta \eta) + 3 (7 - 5\beta \eta - 2\beta) \eta \frac{i}{o} - 16 (1 - \beta) \left( \eta \frac{i}{o} \right)^2 \right]
= (1 - \beta) i \left[ -6 \frac{1 - \beta \eta}{(1 - \beta)} + 3 \left( 5 \frac{1 - \beta \eta}{(1 - \beta)} + 2 \right) \eta \frac{i}{o} - 16 \left( \eta \frac{i}{o} \right)^2 \right].
\]

\[
\frac{3(5 - \beta \eta + 2)}{32} > \frac{3(5 + 2)}{32} > \frac{2}{3} > \eta \text{ implies } -6 \frac{(1 - \beta \eta)}{(1 - \beta)} + 3 \left( 5 \frac{(1 - \beta \eta)}{(1 - \beta)} + 2 \right) \eta \frac{i}{o} - 16 \left( \eta \frac{i}{o} \right)^2 \text{ is increasing in } (0, o). \text{ So we have}
\]

\[
\frac{\partial}{\partial i} \left\{ i^2 \left[ -3 (1 - \beta \eta) + (7 - 5\beta \eta - 2\beta) \eta \frac{i}{o} - 4 (1 - \beta) \left( \eta \frac{i}{o} \right)^2 \right] \right\} \leq (1 - \beta) i \left[ -6 \frac{(1 - \beta \eta)}{(1 - \beta)} + 3 \left( 5 \frac{(1 - \beta \eta)}{(1 - \beta)} + 2 \right) \eta - 16\eta^2 \right]
= i \left[ -6 (1 - \beta \eta) + 3 (7 - 5\beta \eta - 2\beta) \eta - 16 (1 - \beta) \eta \right]
= i \left[ -6 + 21\eta - 16\eta^2 + \beta \eta^2 \right]
= i \left[ - (1 - \eta) (6 - 15\eta) - (1 - \beta) \eta^2 \right],
\]

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which is negative if $\eta < \frac{2}{5}$. Therefore, if $\eta < \frac{2}{5}$, we have

$$
\frac{(bi)^2}{k} \left[ -3 \left( 1 - \beta \eta \right) + \left( 7 - 5 \beta \eta - 2 \beta \right) \eta o - 4 \left( 1 - \beta \right) \left( \eta o \right)^2 \right] 
\geq \frac{(bo)^2}{k} \left[ -3 \left( 1 - \beta \eta \right) + \left( 7 - 5 \beta \eta - 2 \beta \right) \eta - 4 \left( 1 - \beta \right) \eta^2 \right] 
= \frac{(bo)^2}{k} \left[ -3 + (7 + \beta) \eta - (4 + \beta) \eta^2 \right] 
= -\frac{(bo)^2}{k} (1 - \eta) (3 - (4 + \beta) \eta),
$$

Overall, we have

$$
\frac{\partial}{\partial i} LHS \ (28) 
\geq (1 - \eta o) \left\{ (1 - (2 + \beta) \eta) (1 - \eta) - \frac{(bo)^2}{k} (1 - \eta) (3 - (4 + \beta) \eta) \right\} 
\frac{[(1 - \beta \eta) - (1 - \beta) \eta o \right]^2}{[(1 - \beta \eta) - (1 - \beta) \eta o \right]^2} 
= (1 - \eta o) \left\{ (1 - \eta) \left[ (1 - (2 + \beta) \eta) - \frac{(bo)^2}{k} (3 - (4 + \beta) \eta) \right] \right\} 
\frac{[(1 - \beta \eta) - (1 - \beta) \eta o \right]^2}{[(1 - \beta \eta) - (1 - \beta) \eta o \right]^2} 
= (1 - \eta o) \left\{ (1 - \eta) \left[ 1 - 3 \frac{b^2}{k} - \left( 2 + \beta - (4 + \beta) \frac{b^2}{k} \right) \eta \right] \right\} 
\frac{[(1 - \beta \eta) - (1 - \beta) \eta o \right]^2}{[(1 - \beta \eta) - (1 - \beta) \eta o \right]^2},
$$

which is positive if $\frac{b^2}{k} < \frac{1}{3}$, and $\eta < \frac{1 - \frac{3 b^2}{2 + \beta - (4 + \beta) \frac{b^2}{k}}}{\frac{1}{\eta o}}$. In order to guarantee that the fixed point exists, i.e., $i^* (o) \in (0, o)$, we have:

$$
LHS(26) |_{i=0} < RHS(26) < LHS(26) |_{i=0} 
\quad 0 < \Delta < \frac{2 b^2 o \left( 1 - \left( \frac{b o}{k} \right)^2 \right)}{\sigma^2} (1 - \eta).
$$

Second, it is easy to show that $\frac{\partial}{\partial o} LHS \ (28) = \frac{\partial}{\partial o} \left[ \frac{i \left( 1 - \left( \frac{b o}{k} \right)^2 \right)}{\beta \eta o \left( 1 - \frac{b o}{k} \right)^2} \right] > 0$.

Therefore, as $i^* (o)$ is determined by equation (28), we can conclude that $\forall o, o', s.t., 0 < o < o' < 1$, if $\frac{b^2}{k} < \frac{1}{3}$, $\eta < \min \left\{ \frac{2}{5}, \frac{1 - \frac{3 b^2}{2 + \beta - (4 + \beta) \frac{b^2}{k}}}{\frac{1}{\eta o}} \right\}$, and $\Delta \in \left( 0, \frac{2 b^2 o \left( 1 - \left( \frac{b o}{k} \right)^2 \right)}{\sigma^2} (1 - \eta) \right)$,
we have \(i^*(o') < i^*(o)\), which implies higher nominal independence results in lower real independence.

Now we derive the conditions under which our main result is overturned. First, we observe that:

\[
\frac{\partial}{\partial i} LHS(28) \bigg|_{i = o} = (1 - \eta) \left\{ (1 - (2 + \beta) \eta)(1 - \eta) - \frac{(bo)^2}{k} (1 - \eta)(3 - (4 + \beta) \eta) \right\} \frac{[(1 - \beta \eta) - (1 - \beta) \eta]^2}{[(1 - \beta \eta) - (1 - \beta) \eta]^2} \\
= (1 - (2 + \beta) \eta) - \frac{(bo)^2}{k} (3 - (4 + \beta) \eta),
\]

which is negative if \(\frac{b^2}{k} < \frac{1}{3} \), \(\eta > \frac{1}{2 + \beta}\). Since \(\frac{\partial}{\partial i} LHS(28)\) is continuous function of \(i\), it means that \(\exists i^#_B \in (0, o)\) s.t. \(\frac{\partial}{\partial i} LHS(28)\) is negative in \(i \in (i^#_B, o)\). Therefore, if \(\frac{b^2}{k} < \frac{1}{3}\), \(\eta > \frac{1}{2 + \beta}\) and \(\Delta \geq \frac{2b^2}{\sigma^2} o \left(1 - \frac{(bo)^2}{k}\right)(1 - \eta)\), the solution to equation (28) will be either corner solution that \(i^*(o) = o\), or there are multiple interior solutions which satisfy equation (26), and the stable fixed point of real independence which can be achieved in dynamics is closest to the nominal independence \(o\) and satisfies \(\frac{\partial LHS(26)}{\partial i} \bigg|_{i = i^*(o)} < 0\). Again, combining the above result with the fact that \(\frac{\partial}{\partial o} LHS(28) > 0\), we can conclude that \(\forall o, o', \text{s.t.}, 0 < o < o' < 1\), if \(\frac{b^2}{k} < \frac{1}{3}\), \(\eta > \frac{1}{2 + \beta}\), and \(\Delta \geq \frac{2b^2}{\sigma^2} o' \left(1 - \frac{(bo)^2}{k}\right)(1 - \eta)\), we have \(i^*(o') > i^*(o)\), which implies higher nominal independence results in higher real independence.

\[\square\]