Simpler analysis of LP extreme points for traveling salesman and survivable network design problems

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1. Introduction

We consider two well-studied combinatorial optimization problems, the SURVIVABLE NETWORK DESIGN PROBLEM (SNDP) and the SYMMETRIC TRAVELING SALESMAN PROBLEM (STSP). Given an undirected graph \( G = (V, E) \) and connectivity requirements \( r_{uv} \) for all undirected pairs \( u, v \in V \) of vertices, a Steiner network is a subgraph of \( G \) in which there are at least \( r_{uv} \) edge-disjoint paths between \( u \) and \( v \) for all pairs \( u, v \in V \). The SURVIVABLE NETWORK DESIGN PROBLEM is a general network design problem where we are given an edge-weighted graph \( G = (V, E) \) and connectivity requirements \( r_{uv} \mid (u, v) \in \binom{V}{2} \), and the task is to find a minimum-cost Steiner network.

A Hamiltonian cycle in graph \( G = (V, E) \) is a connected subgraph of \( G \) that has degree 2 at every vertex of \( V \). In the SYMMETRIC TRAVELING SALESMAN PROBLEM (STSP), we are given an edge-weighted undirected graph \( G = (V, E) \), and the goal is to compute a minimum-cost Hamiltonian cycle.

Linear programming methods have been successfully used in solving both these problems in practice [1,8]. Strong theoretical results have also been obtained by analyzing linear programming (LP) relaxations for these problems [7,8]. We present a common generalization of these problems and its natural LP relaxation.

Using this LP and a new counting argument, we prove the following results in Section 2.

**Theorem 1.1.** Given any extreme point \( x \) of the LP relaxation (LP_{sndp}) for SURVIVABLE NETWORK DESIGN, there exists an edge \( e \) such that \( x_e \geq \frac{1}{2} \).

**Theorem 1.2.** Given any extreme point \( x \) of the LP relaxation (LP_{stsp}) for the SYMMETRIC TRAVELING SALESMAN, there exists an edge \( e \) such that \( x_e = 1 \).

**Theorem 1.1** was originally proved by Jain [9], and **Theorem 1.2** by Boyd and Pulleyblank [3]. In fact [3] showed that any extreme point of the LP relaxation to the STSP has at least three 1-edges.

We also consider the element connectivity SURVIVABLE NETWORK DESIGN PROBLEM (SNDP_{elt}) in Section 3. This is a well-known generalization of the usual (edge-connectivity) SNDP, where the input is an edge-weighted undirected graph \( G = (V, E) \), a set \( U \subseteq V \) of terminals, and connectivity requirements \( r_{uv} \) for all undirected pairs \( u, v \in U \) of terminals. The vertices \( V \setminus U \) and edges \( E \) of the graph are called elements. The goal in SNDP_{elt} is to find the minimum-cost subgraph that contains at least \( r_{uv} \) element-disjoint paths between \( u \) and \( v \) for every \( u, v \in U \). Using the new counting argument, we provide a shorter proof of the following theorem for its natural LP relaxation considered in Fleischer et al. [5].

**Theorem 1.3.** Given any extreme point \( x \) of the LP relaxation (LP_{elt}) for element connectivity SURVIVABLE NETWORK DESIGN, there exists an edge \( e \) such that \( x_e \geq \frac{1}{2} \).
This result is originally due to Fleischer et al. [5], where they used it to obtain a 2-approximation algorithm for SNDP_{el}. Recently, Chuzhoy and Khanna [4] gave a very elegant reduction from the (even more general) vertex-connectivity SNDP to the element-connectivity SNDP; using this 2-approximation for SNDP_{el}, they obtained an $O(k^3 \log n)$-approximation algorithm for the vertex-connectivity SNDP (here $k$ is the maximum requirement and $n$ is the number of vertices).

Our proofs are based on a new counting argument that involves distributing fractional tokens. This idea was used earlier in Bansal et al. [2] for degree-bounded network design problems in directed graphs, and also appears implicit in the proofs of Gabow et al. [6] for the $k$-edge connected subgraph problem.

**Notation.** For any subset $F \subseteq E$ of edges, the characteristic vector $\chi(F) \in \{0, 1\}^E$ (also denoted $\chi_F$) contains a 1 corresponding to each edge $e \in F$, and a 0 otherwise. For any assignment $x : E \to \mathbb{R}_+$ of non-negative real values to the edges and any subset $F \subseteq E$, $x(F)$ denotes the sum $\sum_{e \in F} x_e$.

### 2. The STSP and the edge-connectivity SNDP

Given a subset $S \subseteq V$, let $\delta(S) = \{(u, v) \in E \mid u \in S, v \notin S\}$ denote the set of edges with exactly one end-point in $S$. We also denote $\delta(v)$ by $\delta(v)$. Then, the classical LP relaxation (LP\text{\textsubscript{stsp}}) for the STSP has the following constraints:

- $x(\delta(S)) \geq 2 \ \forall \emptyset \subseteq S \subseteq V$ (cut constraints)
- $x(\delta(v)) = 2 \ \forall v \in V$ (degree constraints)
- $0 \leq x_e \leq 1 \ \forall e \in E$.

We now consider the LP relaxation (LP\text{\textsubscript{sndp}}) for the SNDP. A function $f$ from subsets of $V$ to the integers is called \textit{weakly supermodular} if $f(V) = f(\emptyset) = 0$ and, for all $S, T \subseteq V$, one of the following holds.

$$f(S) + f(T) \leq f(S \cup T) + f(S \cap T), \text{ or } f(S) + f(T) \leq f(S \setminus T) + f(T \setminus S).$$

It is easy to see that the function $f$ defined by $f(S) = \max_{r \subseteq \delta(S)} r_u$ for each subset $S \subseteq V$ is weakly supermodular. It can be verified that the above function encodes the connectivity requirements $f_{ru}$. We state the LP relaxation [9] for any network design problem with weakly supermodular connectivity requirement (which contains the SNDP as a special case).

- $x(\delta(S)) \geq f(S) \ \forall S \subseteq V$ (cut constraints)
- $0 \leq x_e \leq 1 \ \forall e \in E$.

Now we present the LP relaxation of a generalization of both the SNDP and the STSP. The input consists of an undirected graph $G = (V, E)$ with edge-costs $c : E \to \mathbb{R}_+$, a weakly supermodular function $f : 2^V \to \mathbb{Z}$, and a designated subset $W \subseteq V$ of vertices. The LP corresponding to this is as follows.

\[
\text{(LP)} \quad \begin{align*}
\text{minimize} & \quad \sum_{e \in E} c_e x_e \\
\text{subject to} & \quad x(\delta(S)) \geq f(S) \quad \forall S \subseteq V \\
& \quad x(\delta(v)) = f(v) \quad \forall v \in W \\
& \quad 0 \leq x_e \leq 1 \quad \forall e \in E.
\end{align*}
\]

Note that the first set of constraints above enforces the connectivity requirements $f$, the second set of constraints enforces the degree constraints on $W$, and the last set of constraints ensures that only a subgraph is chosen.

Given graph $G$, edge-costs $c$ and connectivity requirements $\{r_{ru} \mid (u, v) \in \binom{V}{2}\}$, the LP relaxation (LP\text{\textsubscript{sndp}}) of this SNDP instance is obtained by setting, in (LP), $f(S) = \max_{r \subseteq \delta(S)} r_u$ for each subset $S \subseteq V$ and $W = \emptyset$. For an instance of the STSP given by graph $G$ and edge-costs $c$, the corresponding LP relaxation (LP\text{\textsubscript{stsp}}) is obtained by setting $f(S) = 2$ for each $\emptyset \not\subseteq S \subseteq V$, $f(\emptyset) = f(V) = 0$, and $W = V$.

We prove the following theorem, which implies Theorems 1.1 and 1.2.

**Theorem 2.1.** Let $x$ be a basic feasible solution to (LP) where $f$ is weakly supermodular.

A. There exists an edge $e \in E$ such that $x_e \geq \frac{1}{2}$.

B. Moreover, if $f(S)$ is even for each subset $S \subseteq V$, then there exists an edge $e \in E$ such that $x_e = 1$.

The first part of Theorem 2.1 was at the heart of the iterative 2-approximation algorithm for the SNDP [9].

Before the proof of Theorem 2.1, we state some properties of tight constraints of extreme points. Two sets $X, Y$ are intersecting if $X \cap Y \neq \emptyset$ and $X \cap Y = \emptyset$. A family of sets is \textit{laminar} if no two sets are intersecting. The proof of the following lemma is immediate from the uncrossing lemma in Jain [9].

**Lemma 2.2** [9]. Let $x$ be a basic feasible solution to (LP) with $f$ being weakly supermodular, such that $0 < x_e < 1$ for all edges $e \in E$. Then, there exists a laminar family $\mathcal{L}$ of subsets such that

1. $x$ is the unique solution to $(x(\delta(S)) = f(S), \forall S \in \mathcal{L})$;
2. the vectors $x|_S \in \mathcal{L}$ for $S \in \mathcal{L}$ are linearly independent; and
3. $|E| = |\mathcal{L}|$.

**Proof.** Lemma 4.3 in [9] proves this lemma when $W = \emptyset$; that proof is based on standard \textit{uncrossing} arguments. In the general case, there are additional \textit{equalities} for singleton vertex-sets corresponding to $W$. Let $(LP)$ denote the polytope given by just the first and third sets of constraints in (LP), i.e., without equality constraints on $W$. Note that the polytope $(LP)$ is a face of polytope (LP'). Hence any extreme point in (LP) is also an extreme point in (LP'), for which the lemma from [9] applies. \hfill $\square$

We now prove Theorem 2.1. Let $x$ be any basic feasible solution to (LP).

**Proof of Theorem 2.1 (A).** We first prove that $x_e \geq \frac{1}{2}$ for some edge $e \in E$. Suppose for the sake of contradiction that $x_e < \frac{1}{2}$ for each $e \in E$. If $x_e = 0$ for some $e \in E$, we can remove edge $e$ from the graph $G$ and variable $x_e$ from (LP). The residual solution $x$ remains a basic feasible solution to the modified (LP). Thus we assume without loss of generality that $x_e > 0$ for all $e \in E$, and so Lemma 2.2 applies.

We will show a contradiction to Lemma 2.2 by means of a new counting argument. The counting argument proceeds as follows. We assign one token to each edge in $E$, and then reassign the tokens such that we can collect strictly more than one token per set in the laminar family $\mathcal{L}$: this would imply $|E| > |\mathcal{L}|$, which is the desired contradiction.

For any sets $S, R \in \mathcal{L}$, we say that $S$ is the parent of $R$ (or equivalently, that $R$ is a child of $S$) if $S$ is the smallest set in $\mathcal{L}$ containing $R$. Each edge $e = (u, v) \in E$ is given a unit token, which it reassigns as follows.

1. (**Rule 1**) Let $S$ be the smallest set in $\mathcal{L}$ containing $u$, and $R$ be the smallest set in $\mathcal{L}$ containing $v$. Then $e$ assigns $x_e$ tokens to each of $S$ and $R$.
2. (**Rule 2**) Let $T$ be the smallest set in $\mathcal{L}$ containing both $u$ and $v$. Then $e$ assigns $1 - 2x_e$ tokens to $T$. 


Theorem 2.1

Lemma 2.2

Proof of Theorem 2.1

We now show that each set in \( L \) receives at least one token. Let \( S \in L \) have \( k \) children \( R_1, \ldots, R_k \) in \( L \). We have the following tight inequalities for the extremal point \( x \).

\[
x(S) = f(S) \quad \text{and} \quad x(R_i) = f(R_i) \quad \forall \ 1 \leq i \leq k.
\]

Subtracting, we obtain

\[
x(S) - \sum_{i=1}^{k} x(R_i) = f(S) - \sum_{i=1}^{k} f(R_i) \Rightarrow
\]

\[
x(A) - x(B) = 2x(C) = f(S) - \sum_{i=1}^{k} f(R_i)
\]

where

\[
A = \{ e : e \cap (\cup_i R_i) = \emptyset, |e \cap S| = 1 \}
\]

\[
B = \{ e : e \cap (\cup_i R_i) = 1, |e \cap S| = 2 \}
\]

\[
C = \{ e : e \cap (\cup_i R_i) = 2, |e \cap S| = 2 \}.
\]

Observe that \( A \cup B \cup C = \emptyset \); otherwise, we have the dependence \( x_{\delta(S)} = \sum_{i=1}^{k} A(R_i) \). Also, \( S \) receives \( x_e \) tokens for each edge \( e \) in \( A \) (by Rule 1), \( 1 - x_e \) tokens for each edge \( e \) in \( B \) (by Rules 1 & 2), and \( 2 - 2x_e \) tokens for each edge \( e \) in \( C \) (by Rule 2). Hence, the total number of tokens received by \( S \) is exactly

\[
\sum_{e \in A} x_e + \sum_{e \in B} (1 - x_e) + \sum_{e \in C} (1 - 2x_e)
\]

\[
= x(A) + |B| - x(B) + |C| - 2x(C)
\]

\[
= |B| + |C| + f(S) - \sum_{i=1}^{k} f(R_i).
\]

Observe that, for every edge \( e \in E \), \( x_e, 1 - x_e, 1 - 2x_e > 0 \) since \( 0 < x_e < \frac{1}{2} \); combined with the fact that \( A \cup B \cup C = \emptyset \), the number of tokens assigned to \( S \) is strictly positive (using the first expression in Eq. (1)). On the other hand, the last expression in (1) implies that the number of tokens assigned to \( S \) is integral. Thus every \( S \in L \) gets at least one token in this assignment (Fig. 1).

Now we show that there are some unassigned tokens, thereby showing the strict inequality \( |L| < |E| \). Let \( R \) be a maximum-cardinality set in \( L \); note that none of the sets in \( L \setminus R \) contains \( R \) and \( R \neq V \) since \( f(V) = 0 \). Consider any edge \( e \in \delta(R) \); the token by Rule 2 for edge \( e \) is unassigned as there is no set such that \( \{T \cap e\} = 2 \). This gives us the desired contradiction, and proves the first part of Theorem 2.1. \( \square \)

Proof of Theorem 2.1 (B). We now consider the case when \( f(S) \) is even for each \( S \subseteq V \), and show that, for any basic feasible solution \( x \) to (LP), there is always an edge \( e \in E \) with \( x_e = 1 \). The proof follows the same approach as above but with a scaled token assignment. For the sake of contradiction, we assume that \( x_e < 1 \) for each \( e \in E \). As before, we can assume without loss of generality that \( x_e > 0 \).

Again, we will show a contradiction to Lemma 2.2 by showing that \( |L| < |E| \). The counting argument proceeds as follows. We assign one token to each edge \( e = (u, v) \in E \), which it redistributes as follows.

1. (Rule 1) Let \( S \) be the smallest set in \( L \) containing \( u \), and \( R \) be the smallest set in \( L \) containing \( v \). Then \( e \) assigns \( \frac{x_e}{2} \) tokens to each of \( S \) and \( R \).

2. (Rule 2) Let \( T \) be the smallest set in \( L \) containing both \( u \) and \( v \). Then \( e \) assigns \( 1 - x_e \) tokens to \( T \).

We now show that each set in \( L \) receives at least one token. As before, let \( S \in L \) have children \( R_1, \ldots, R_k \) \((k \geq 0)\). We have the following tight inequalities.

\[
x(S) = f(S) \quad \text{and} \quad x(R_i) = f(R_i) \quad \forall \ 1 \leq i \leq k.
\]

Dividing by two and subtracting, we obtain

\[
\frac{1}{2} \left[ x(S) - \sum_{i} x(R_i) \right] = \frac{1}{2} \left[ f(S) - \sum_{i} f(R_i) \right]
\]

where the edge sets \( A, B, C \) are exactly as in the earlier case. Observe that \( A \cup B \cup C = \emptyset \); there is a dependence in the constraints for \( S \) and its children. Also, \( S \) receives \( \frac{x_e}{2} \) tokens for each edge \( e \) in \( A \) (Rule 1), \( 1 - \frac{x_e}{2} \) tokens for each edge \( e \) in \( B \) (Rules 1 & 2), and \( 1 - x_e \) tokens for each edge \( e \) in \( C \) (Rule 2). Hence, the total number of tokens received by \( S \) is

\[
\sum_{e \in A} \frac{x_e}{2} + \sum_{e \in B} \left(1 - \frac{x_e}{2} \right) + \sum_{e \in C} \left(1 - x_e \right)
\]

\[
= \frac{x(A)}{2} + |B| - \frac{x(B)}{2} + |C| - x(C)
\]

\[
= |B| + |C| + f(S) - \sum_{i} f(R_i).
\]

Following the same reasoning as before, this quantity is a positive integer (here \( f \) is an even-valued function, so the number of tokens is still integral). Thus every set \( S \in L \) receives at least one token in this assignment. Finally, note that some tokens corresponding to the maximal sets in \( L \) are unassigned. This shows the strict inequality \( |L| < |E| \), and gives us the desired contradiction. This proves the second part of Theorem 2.1. \( \square \)

3. The element-connectivity SNDP

In this section, we consider the element-connectivity survivable network design problem (SNDP\(_{es} \)). In this problem, we are given an undirected graph \( G = (V, E) \) with edge-costs \( c : E \rightarrow R_+ \), a set \( U \subseteq V \) of terminals, and connectivity requirements \( r_u \) for all undirected pairs \( u, v \in U \) of terminals. Vertices in \( V \setminus U \) are called non-terminals. The edges and non-terminals of the graph are called elements. The goal in SNDP\(_{es} \) is to find a minimum-cost subgraph that contains at least \( r_u \) element-disjoint paths between \( u \) and \( v \) for every \( u, v \in U \). Fleischer et al. [5] used iterative rounding to obtain a 2-approximation algorithm for this problem. They [5] showed that SNDP\(_{es} \) can be formulated as a suitable integer program (defined formally below), such that an extreme point solution to its LP-relaxation contains an edge with solution value at least half. We give a short proof of this result using a new counting argument generalizing the results in the previous section.
A set-pair is an ordered tuple \((S, S')\) where \(S, S' \subseteq V\). Let \(\mathcal{F}\) denote some family of set-pairs. A two-set function \(f : \mathcal{F} \to \mathbb{Z}_+\) is called weakly two-supermodular if, for any \((S, S')\) and \((T, T') \in \mathcal{F}\), at least one of the following holds.

1. \((S \cap T, S' \cap T') \subseteq \mathcal{F}\) and \((S \cap T, S' \cap T') \subseteq \mathcal{F}\), and we have
   \[ f(S \cap T, S' \cap T') + f(S \cup T, S' \cup T') \geq f(S, S') + f(T, T'). \]
2. \((S \cap T, S' \cap T') \subseteq \mathcal{F}\) and \((S \cup T, S' \cap T') \subseteq \mathcal{F}\), and we have
   \[ f(S \cap T, S' \cap T') + f(S \cup T', S' \cap T') \geq f(S, S') + f(T, T'). \]

For any set-pair \((S, S')\), let \(E(S, S') = \{ (u, v) \in E \mid u \in S, v \in S'\}\) denote the edges with one end-point in \(S\) and the other in \(S'\). For any assignment \(x : E \to \mathbb{R}_+\) and set-pair \((S, S')\), we abbreviate \(x(E(S, S'))\) by just \(x(S, S')\). The LP-relaxation for \(\text{SNDP}_{\text{el}}\) considered in [5] is the following.

**Theorem 1.3.** Let \(x\) be a basic feasible solution to \((\text{LP}_{\text{el}})\), where \(f : \mathcal{F} \to \mathbb{Z}_+\) is weakly two-supermodular; then there exists an \(e \in E\) such that \(x_e \geq \frac{1}{2}\).

Proof of Theorem 3.1. Suppose for a contradiction that the claim does not hold, and let \(x\) be an extreme point solution with \(x_e < \frac{1}{2}\) for all \(e \in E\). If \(x_e = 0\) for some \(e \in E\), we can remove edge \(e\) from the graph \(G\) and variable \(x_e\) from \((\text{LP}_{\text{el}})\). The residual solution \(x\) remains a basic feasible solution to the modified \((\text{LP}_{\text{el}})\). Thus we assume without loss of generality that \(x_e > 0\) for all \(e \in E\), and so Lemma 3.2 applies. We will derive a contradiction using a counting argument similar to the one in the previous section. Each edge \(e = (i, j) \in E\) is assigned one unit of token, which it distributes to nodes in \(L\) as follows.

1. Rule I: Assign \(x_e\) tokens to the smallest node \((S, S') \in \mathcal{L}\) such that either \(i \in S\) or \([i, j] \cap S' = \emptyset\).
2. Rule II: Assign \(x_{i,j}\) tokens to the smallest node \((T, T') \in \mathcal{L}\) such that either \(j \in T\) or \([i, j] \cap T' = \emptyset\).
3. Rule III: Assign \(1 - 2x_e\) tokens to the smallest node \((R, R') \in \mathcal{L}\) such that \([i, j] \cap \emptyset = \emptyset\).

Note that both \(x_e\) and \(1 - 2x_e\) are strictly positive for any edge \(e\). Additionally, by Lemma 4.8 in Fleischer et al. [5], it follows that each of Rules I, II and III assigns tokens to at most one node. Hence each edge in \(E\) distributes a total of at most one token.

We now show that each edge of \(L\) receives a total of at least one token. Consider any node \((S, S') \in \mathcal{L}\) with children \((R_i, R_i')\) for \(i = 1, 2\). If \((S, S')\) is a leaf then \(k = 0\). For each \(i \in \{k\}\), we have \((A) R_i' \supseteq S',\) since \((S, S') \geq (R_i, R_i');\) and \((B) R_i' \supseteq R_j\) for all \(j \in \{k\} \setminus \{i\}\), since \((R_i, R_i')\) and \((R_j, R_j')\) are incomparable, and they satisfy condition (C3). Additionally, the \(\{R_i\}_{i=1}^k\) are disjoint subsets of \(S\). Define the following edge-sets:

- \(H = \bigcup_{i=1}^k E(R_i, R_i' \setminus S')\)
- \(C = \{ e \in H \mid e \cap (\cup_i R_i) = 2\}\)
- \(B = \{ e \in H \mid e \cap (\cup_i R_i) = 1\}\)
- \(D = \bigcup_{i=1}^k E(R_i, S')\)
- \(A = E(S \setminus (\cup_i R_i), S').\)

Thus we can write \(\chi_{\text{el}} \chi(E(S, S')) = 2 \cdot (x(C) + x(B) + x(D)),\) and \(x(E(S, S')) = x(D) + x(A)\). Recall that the tight LP constraints imply that

\[ x(E(S, S')) = f(S, S') \] \[ x(E(R, R')) = f(R, R') \] \[ \forall 1 \leq i \leq k. \]

Subtracting, we obtain (since the \(f\)-values are all integral)

\[ x(E(S, S')) - \sum_{i=1}^k x(E(R_i, R_i')) = f(S, S') - \sum_{i=1}^k f(R_i, R_i') \in \mathbb{Z}. \]

Adding \(|B| + |C|\) (an integer) to the above expression, we obtain

\[ \sum_{e \in A} x_e + \sum_{e \in B} (1 - x_e) + \sum_{e \in C} (1 - 2x_e) \in \mathbb{Z}. \]

Note that \(A \cup B \cup C \neq \emptyset;\) otherwise, \(\chi(E(S, S')) = \sum_{i=1}^k x(E(R_i, R_i'))\), contradicting the linear independence in Lemma 3.2. Since \(0 < x_e < \frac{1}{2}\) for all \(e \in E\), the left-hand side above is strictly positive, and

\[ \sum_{e \in A} x_e + \sum_{e \in B} (1 - x_e) + \sum_{e \in C} (1 - 2x_e) \geq 1. \]

We now show that the tokens assigned to \((S, S')\) total to at least the left-hand side in Inequality (2).
• Edge $e = (u, v) \in A$. Let $u \in S \setminus (\cup_i R_i)$ and $v \in S'$. We claim that the token assigned by Rule I goes to $(S, S')$. Clearly, $(S, S')$ is the smallest set-pair with $u \in S$. For any descendant $(T, T')$ of $(S, S')$, we must have $T' \supseteq S' \ni v$; thus we cannot have $u, v \not\in T'$. Hence $(S, S')$ receives $x_e$ tokens from $e$.

• Edge $e = (u, v) \in C$. Let $u \in R_i$ and $v \in R_j$ for $i, j \in [k], i \neq j$. We claim that the token assigned by Rule III goes to $(S, S')$. Clearly $u, v \not\in S'$. Furthermore, for any child $(R_i, R_j')$ of $(S, S')$ we have $R_i' \supseteq R_i \ni u$ or $R_j' \supseteq R_j \ni v$. Hence $(S, S')$ receives $1 - 2x_e$ tokens from $e$.

• Edge $e = (u, v) \in B$. Let $u \in R_i$ and $v \in R_i' \setminus S'$ for some $i \in [k]$. We first claim that the token assigned by Rule III goes to $(S, S')$. Clearly $u, v \not\in S'$. We show that $(u, v) \cap R_i' = \emptyset$ for every child $(R_i, R_i')$ of $(S, S')$.

1. Suppose $\ell = i$; then $v \in R_i'$.
2. Suppose $\ell \in [k] \setminus \{i\}$; then $u \in R_i \subseteq R_i'$.

That is, $(S, S')$ receives the token by Rule III. We next claim that the token assigned by Rule II also goes to $(S, S')$. Note that $v \not\in \cup_i R_i$, so no descendant $(T, T')$ of $(S, S')$ can have $v \in T$. As seen above, $(S, S')$ is the smallest node with $u, v \not\in S'$; i.e., $(S, S')$ receives the token by Rule II. Hence $(S, S')$ receives in total $1 - x_e$ tokens from $e$.

Thus each node of $L$ receives at least a unit token.

We now show that there is some positive amount of unused tokens. Let $(P, P') \in L$ be any maximal node in $L$. Note that there is at least one maximal node $(P, P') \in L$ and $E(P, P') \neq \emptyset$. We claim that the token of any edge $(u, v) \in E(P, P')$ given by Rule III is unused. Let $u \in P$ and $v \in P'$. For any descendant $(T, T')$ of $(P, P')$, we have $T' \supseteq P' \ni v$; so $T \cap \{u, v\} \neq \emptyset$. Any node $(Q, Q') \in L$ that is not a descendant of $(P, P')$ is incomparable to $(P, P')$, and we have $Q' \supseteq P \ni u$. Thus $\{u, v\} \cap S' \neq \emptyset$ for all $(S, S') \in L$, i.e., the Rule III token of edge $(u, v)$ is unassigned. Thus there is a positive amount of unused tokens. However, this implies that $|E| > |L|$, which contradicts Lemma 3.2.

This completes the proof of Theorem 3.1. $\square$

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References