Abstract

The average distance of a connected graph $G$ is the average of the distances between all pairs of vertices of $G$. We present a linear time algorithm that determines, for a given interval graph $G$, a spanning tree of $G$ with minimum average distance (MAD tree). Such a tree is sometimes referred to as a minimum routing cost spanning tree.

Keywords: Graph Algorithms, Interval Graphs, Spanning Tree.

1 Introduction

The average distance $\mu(G)$ of a finite, connected graph $G = (V, E)$ is the average over all unordered pairs of vertices of the distances,

$$\mu(G) = \left( \frac{|V|}{2} \right)^{-1} \sum_{\{u,v\} \subset V(G)} d_G(u, v),$$

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where $d_G(u, v)$ denotes the distance between the vertices $u$ and $v$. A minimum average distance spanning tree of $G$ (MAD tree in short) is a spanning tree of $G$ of minimum average distance. MAD trees, also referred to as minimum routing cost spanning trees, are of interest in the design of communication networks [6]. One is interested in designing a tree subnetwork of a given network, such that on average, one can reach every node from every other node as fast as possible. In general, the problem of finding a MAD tree is NP-hard [6]. A polynomial-time approximation scheme is due to [1]. Hence it is natural to ask for which restricted graph classes a MAD tree can be found in polynomial time. In [3], an algorithm is exhibited that computes a MAD tree of a given distance-hereditary graph in linear time. In [5] it is shown that a MAD tree of a given outerplanar graph can be found in polynomial time.

In this paper, we show that for a given interval graph $G$ a MAD tree can be computed in time $O(|E|)$. If the interval representation of $G$ is known and the left and right boundaries of the intervals are ordered, then a MAD tree of $G$ can be found in time $O(|V|)$. In section 2, we present some structural results on MAD trees. In section 3, we sketch the algorithm, and in section 4, we give some concluding remarks.

2 Structure of MAD Trees in Interval Graphs

We always assume that the graph under consideration is connected. The distance of a vertex $v$, $d_G(v)$, is defined as $\sum_{x \in V} d_G(v, x)$. The total distance of a graph $H$ denoted $d(H)$ is the sum of all pairwise distances between nodes in $H$, i.e. $d(H) = \sum_{\{u, v\} \subseteq V(H)} d_H(u, v)$.

A median vertex of $G$ is a vertex $c$ for which $d_G(c)$ is minimum. The eccentricity of a vertex $v$ is defined as $e_G(v) = \max_{w \in V} d_G(v, w)$.

The neighbourhood of a vertex $v$ in $G$ is defined in two ways: the open version $N_G(v) = \{u : uv \in E\}$ and the closed version $N_G[v] = \{v\} \cup N_G(v)$.

The following lemma applies not only to interval graphs but to all connected graphs. Part (i) improves on a result in [2], which states that, if $T$ is a MAD tree of a given connected graph $G$, then there exists a vertex $c$ in $T$ such that every path in $T$ starting at $c$ is induced in $G$. We now prove that $c$ can be chosen to be a median vertex of $T$.

Lemma 1 (i) If $T$ is a MAD tree of $G$ and $c$ is a median vertex of $T$ then every $T$-path starting at $c$ is an induced path in $G$ (i.e. has no diagonals in $G$).

(ii) $T$ and $c$ can be chosen such that there is no vertex $c' \neq c$ such that $N_G[c]$ is strictly contained in $N_G[c']$.

Proof: (i) Let $P = v_1, \ldots, v_k$ be a path in $T$ with $c = v_1$. Suppose $P$ is not induced in $G$. Then there are $i$ and $j$, such that $i < j - 1$ and $v_i v_j \in E(G)$. Let $T_1$ and $T_2$ be the connected components of $T - v_{j-1} v_j$ containing $v_{j-1}$ and $v_j$, respectively. It is known (see [2]) that $d_T(c) \leq d_T(v_i) < d_T(v_{j-1})$. Since the vertices in $T_2$ are further away (in $T$) from $v_i$ than from $v_{j-1}$, we obtain $d_{T_1}(v_i) < d_{T_1}(v_{j-1})$. 

2
Consider the tree 

$$T' = T - v_jv_{j-1} + v_iv_j.$$ 

Since the distances between any two vertices are the same in $T$ and $T'$, unless one vertex is in $T_1$ and the other vertex is in $T_2$, we have 

$$d(T') - d(T) = |V(T_2)|(d_{T_1}(v_i) - d_{T_2}(v_{j-1})) < 0,$$

contradicting the minimality of $d(T)$. Hence $P$ is an induced path in $G$.

(ii) If there is a vertex $c'$ with $N_G[c] \subset N_G[c']$ and $N_G[c] \neq N_G[c']$, then joining all $T$-neighbours of $c$ to $c'$ instead of $c$ yields a spanning tree $T'$ with 

$$d(T') \leq d(T),$$

in which $c$ is an end vertex and $c'$ is a median vertex. QED

From now on we assume that $G$ is an interval graph. Each vertex $v$ corresponds to an interval $I(v) = [l(v), r(v)]$, such that two vertices $v$ and $w$ are adjacent if and only if $I(v) \cap I(w) \neq \emptyset$.

**Proposition 1** Let $v_0, \ldots, v_k$ be an induced path of the interval graph $G$. Then

1. $l(v_1) < l(v_2) < \ldots < l(v_k)$ and $r(v_0) < \ldots < r(v_k-1)$ or
2. $r(v_1) > \ldots > r(v_k)$ and $l(v_0) > \ldots > l(v_k-1)$.

In the first case, we call such a path an R-path, in the second case, we call it an L-path.

Consequently, each path of the MAD tree $T$ of $G$ starting at a median vertex $c$ of $T$ is an L-path or an R-path.

For a vertex $v$ of $G$ let $h(v)$ be a neighbour $x$ of $v$ such that $r(x)$ is maximum; if $r(v) = \max_{w \in V} r(w)$, we say $h(v)$ is undefined. Similarly, let $k(v)$ be a neighbour $y$ of $v$ such that $l(y)$ is minimum. Again, if $l(v) = \min_{w \in V} l(w)$, we say that $k(v)$ is undefined. We also define $h^0(v) = k^0(v) = v$ for all $v$. For $i \geq 2$, $h^i(v)$ is defined as $h(h^{i-1}(v))$. Analogously, we define $k^i(v)$.

A component of a graph is trivial if it contains only one vertex, otherwise it is called nontrivial.

For a given vertex $v$ of $G$ and an integer $i \geq 2$ let $V^R_i(v)$ ($V^L_i(v)$) be the set of all vertices at distance $i$ from $v$, whose intervals lie completely to the right (left) of the interval of $v$. We also let $V^L_1(v) = V^L(v) = N_G(v)$.

**Theorem 1** If $G$ is an interval graph then there is a MAD tree $T$ of $G$ with a median vertex $c$, such that for each $i$, $1 \leq i \leq e_{x_G}(c)$,

\[
\tag{*} \left\{ \begin{array}{l}
each v \in V^R_i(c) \text{ is adjacent in } T \text{ to } h^{i-1}(c) \\
each v \in V^L_i(c) \text{ is adjacent in } T \text{ to } k^{i-1}(c). \end{array} \right.
\]

**Proof.** Let $T$ be a MAD tree of $G$ and let $c$ be a median vertex of $T$, where $c$ is chosen according to Lemma 1(ii).

By Lemma 1(i), each $G$-neighbour $v$ of $c$ is also in $T$ adjacent to $c$, since otherwise the $c - v$ path in $T$ would have a diagonal. Hence $(*)$ holds for $i = 1$. 

\[3\]
Suppose that (⋆) does not hold for some $i$. Let $i$ be the smallest such number. For the rest of this proof, we omit the reference to the median vertex $c$ in the notation for $V^R_i$.

We first show that only one vertex in $V^R_i$ has $T$-neighbours in $V^R_i$. Suppose that there exist vertices $v_i, v'_i \in V^R_i$, $v_{i-1} \neq v'_{i-1} \in V^R_{i-1}$ such that $v_i v_{i-1}, v'_i v'_{i-1} \in E(T)$. Since, by the minimality of $i$, $v_{i-1}$ and $v'_{i-1}$ are adjacent in $T$ to $h^{i-2}(c)$, we have $v_i \neq v'_i$ (Else there would be a cycle in $T$). Without loss of generality, we may assume that $r(v_{i-1}) \geq r(v'_{i-1})$. Let $B$ be the component of $T - v_{i-1}v_i$ not containing $c$ and let $B'$ be the component of $T - h^{i-2}(c)v'_{i-1}$ not containing $c$. Moreover let

$$\text{child}(v'_{i-1}) = \{w \in B' \mid v'_{i-1}w \in E_T\}.$$ 

Note that $\text{child}(v'_{i-1})$ is exactly the set of children of $v'_{i-1}$ when $T$ is rooted at $c$. Now $\text{child}(v'_{i-1}) \subseteq N_{G}(v_{i-1})$ since every child of $v'_{i-1}$ is on an $R$-path from $c$ and $r(v_{i-1}) \geq r(v'_{i-1})$. Therefore,

$$T' = T - \{v'_{i-1}w \mid w \in \text{child}(v'_{i-1})\} + \{v_{i-1}w \mid w \in \text{child}(v'_{i-1})\}$$

is a spanning tree of $G$. Since the distance between any two vertices are the same in $T$ and $T'$, unless one of the vertices is in $B$ and the other one is in $B'$, the difference between the total distances is

$$d(T') - d(T) = -2|V(B)||V(B')| - 1 < 0$$

contradicting the minimality of $d(T)$. Hence at most one vertex $v$ in $V^R_{i-1}$ has $T$-neighbours in $V^R_i$. Hence only one vertex in $V^R_{i-1}$ has $T$-neighbours in $V^R_i$. Since each vertex in $V^R_i$ that is adjacent in $G$ to a vertex $v$ in $V_{i-1}$ is also adjacent in $G$ to $h^{i-1}(c)$, we can join each vertex in $V^R_i$ not to $v$, but to $h^{i-1}(c)$ without increasing the total distance of $T$. Analogously, we can achieve that each vertex in $V^L_i$ is adjacent in $T$ to $k^{i-1}(c)$. Hence $T$ satisfies condition (⋆).

QED

Theorem 1 suggests the following polynomial time algorithm. Fix a vertex $c$ of $G$, determine $h^i(c)$ and $k^i(c)$ for $i = 1, 2, \ldots$ and construct a spanning tree of $G$ in which each vertex in $V^R_i(c)$ is adjacent to $h^{i-1}(c)$ and each vertex in $V^L_i(c)$ is adjacent to $k^{i-1}(c)$ for $i \geq 1$. Construct such a tree for each $c \in V(G)$. Among those $n$ trees select a tree with minimum total distance. By Theorem 1, this tree is a MAD tree of $G$.

**Theorem 2** A MAD tree of an interval graph can be determined in polynomial time.

A set of intervals and the corresponding interval graph $G$ is shown below. The vertices $c, h^i(c)$ and $k^i(c), i = 1, 2$, are labelled. Thick lines indicate the edges of a MAD tree $T$ satisfying the condition (⋆) of Theorem 1.

\[\text{Image of interval graph}\]

4
3 Linear-time Computation of a MAD Tree

We assume that an interval representation of $G = (V, E)$ is known and that the left borders $l(v)$ and right borders $r(v)$, $v \in V$ are sorted. As in the previous section, we assume that $h(v)$ is a neighbour $x$ of $v$, such that $r(x)$ is maximum and that $k(v)$ is a neighbour $y$ of $v$, such that $l(y)$ is minimum. In [7] it is shown that $h$ and $k$ can be determined in time linear in the number of vertices (even in logarithmic time in parallel with a linear workload).

We consider any vertex $v$ and assume that the median vertex $c$ is left (right) of $v$, i.e., that $r(c) \leq r(v)$ ($l(c) \geq l(v)$).

Let $T$ be a MAD tree according to Theorem 1 rooted at $c$ and let $v \neq c$ be a vertex, $v \in V(c)$, say. Consider $T_v$, the subtree of $T$ rooted at $v$. Then either $v$ is a leaf of $T$ and $T_v$ is trivial, or $v = h^i(c)$ for some $i$. In that case, $T_v$ contains some $G$-neighbours of $v$, in particular $h(v) = h^{i+1}(c)$, and all vertices in $V_R^i(v) = V^R_{i+2}(c) = V^R_i(v)$ (which are in $T$ adjacent to $h^{i+1}(c) = h(v)$), $V_L^i(c) = V_L^i(c)$ (which are in $T$ adjacent to $h^{i+2}(c) = h^2(v)$), and so on, as long as they are defined. The fact that the main part of $T_v$, namely $T_{h(v)}$, only depends on whether $v$ is to the left or right of $c$, but not on the actual choice of $c$, is the key to our algorithm.

**Definition 1** Let $G$ be a connected interval graph with $h$ and $k$ as defined above, and $v \in V(G)$. If $h(v)$ ($k(v)$) is not defined then let $T_v^R$ ($T_v^L$) be the empty tree. If $h(v)$ is defined then let $T_v^R$ be the tree with vertex set $\{h(v)\} \cup \bigcup_{j \geq 2} V_R^j(v)$, where a vertex $x \in V^R_j(v)$ is adjacent to $h^{j-1}(v)$. Analogously, if $k(v)$ is defined then let $T_v^L$ be the tree with vertex set $\{k(v)\} \cup \bigcup_{j \geq 2} V_L^j(v)$, where a vertex $x \in V^L_j(v)$ is adjacent to $k^{j-1}(v)$. Note that both $T_v^R$ and $T_v^L$ do not contain $v$!
Hence the tree $T = T_c$ consists of $c$ as root, the neighbours of $c$ in the original graph $G$ as neighbours in $T_c$, and $T^{L}_c$ and $T^{R}_c$ appended on $k(c)$ and $h(c)$ respectively. We do not determine these trees $T^{R}_v$ and $T^{L}_v$ explicitly. Instead, we compute the following quantities.

1. the number of neighbours $num^R(v)$ ($num^L(v)$) of $h(v)$ ($k(v)$) that are not neighbours of $v$, i.e. the number of children of $h(v)$ ($k(v)$) in $T^R_v$ ($T^L_v$),
2. $|T^R_v|$ ($|T^L_v|$), the number of vertices of the tree $T^R_v$ ($T^L_v$),
3. the total distance $t^R(v)$ ($t^L(v)$) of $T^R_v$ ($T^L_v$), and
4. the distance $d^R(v)$ ($d^L(v)$) of $h(v)$ ($k(v)$) in $T^R_v$ ($T^L_v$).

The numbers $num^L(v)$ and $num^R(v)$ can be determined overall in $O(n)$ time (see, e.g. [7]). For any particular $v$, $d^R(v)$ can be determined in $O(1)$ time if $d^R(h(v))$ and $num^R(v)$ are known. Also $t^R(v)$ can be determined in $O(1)$ time if $d^R(v)$, $num^R(v)$ and $t^R(h(v))$ are known. Analogous statements hold for the left counterparts.

In more detail, we proceed as follows.

**Determine** $num^R(v)$: If $h(v)$ is not defined then $num^R(v) = 0$. Otherwise, $num^R(v)$ is the number of vertices $w$, such that $r_v < l_w \leq r_{h(v)}$. This can be determined in overall linear time.

**Determine** $|T^R_v|$: If $h(v)$ is not defined, $|T^R_v| = 0$. Otherwise,

$$|T^R_v| = |T^R_{h(v)}| + num^R(v).$$

**Determine** $d^R(v)$: If $h(v)$ is not defined then $d^R(v) = 0$. Otherwise

$$d^R(v) = d^R(h(v)) + |T^R_{h(v)}| + num^R(v) - 1.$$

**Determine the distance** $t^R(v)$: If $h(v)$ is not defined then $t^R(v) = 0$. Otherwise

$$t^R(v) = t^R(h(v)) + d^R(v) + (num^R(v) - 1) \left((num^R(v) - 2) + 2|T^R(h(v))| + d^R(h(v))\right).$$

**Determine the total distance of** $T_c$: First we determine the degree $\delta(c)$ of $c$. This can be done in overall time $O(n)$, for all $c$ (One has to count the number of right borders between the left border $l_c$ and the right border $r_c$ of $c$ and the number of intervals passing the right border of $c$). Then we define $n(c)$ to be the total number of neighbours of $c$ excluding itself as well as $h(c)$ and $k(c)$ if they are defined.

The total distance of $T_c$ is

$$(n(c))^2 + t^L(c) + t^R(c) + (1 + 2n(c) + d^L(c))|T^R(c)| + (1 + 2n(c) + d^R(c))|T^L(c)| + 2|T^L_c||T^R_c| + (d^L(c) + d^R(c))(1 + n(c)).$$
Thus we compute the numbers \( t^R(v) (t^L(v)) \), \( d^R(v) (d^L(v)) \), \( |T^L_v| (|T^R_v|) \), and \( \text{num}^R(v) (\text{num}^L(v)) \) in overall linear time.

We obtain the total distances and thus the average distances of the trees \( T_c \) with assumed median vertices \( c \) in overall linear time, and we only have to select one \( T_c \) with minimum average distance. We determine this particular \( T_c \) explicitly using Theorem 1. Also this can be done in linear time.

**Theorem 3** If an interval representation of an interval graph \( G \) with \( n \) vertices is given and the left and the right boundaries \( l(v) \) and \( r(v) \) are sorted then a MAD tree of \( G \) can be determined in \( O(n) \) time.

### 4 Conclusions

It remains an open problem to decide whether there exists a polynomial time algorithm to find a MAD tree of a vertex weighted interval graph. If \( w \) is a real valued weight function on the vertex set of \( G \), then the average distance of \( G \) with respect to \( w \) is defined (see [4]) as

\[
\left( \frac{w(V)}{2} \right)^{-1} \sum_{\{u,v\} \subseteq V(G)} w(u)w(v)d_G(x, y),
\]

where \( w(V) \) is the total weight of the vertices in \( G \).

Theorem 1 does not hold (and hence the algorithm presented above does not work) if there is a weight function on the vertices of \( G \). To see this, consider the path with four vertices \( v_1, \ldots, v_4 \) of weight 1, 2, 1, 5 together with a vertex \( v_5 \) of weight 0, that is adjacent to \( v_1, \ldots, v_4 \). An interval representation of this graph is as follows: \( v_1 : [0, 1], v_2 : [0, 3], v_3 : [2, 5], v_4 : [4, 5], v_5 : [0, 5] \). This graph has a unique MAD tree, the path \( v_1, v_2, v_3, v_4, v_5 \). It is easy to check that \( T \) does not contain a vertex satisfying the conclusion of Theorem 1.

New techniques seem necessary to solve the edge-weighted counterpart of the MAD tree problem for Interval graphs. It would also be interesting to know whether our algorithm can be extended to the class of strongly chordal graphs.

### References


