Min-Max Payoffs in a Two-Player Location Game

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Abstract

We consider a two-player, sequential location game in d-dimensional Euclidean space with arbitrarily distributed consumer demand. The objective for each player is to select locations so as to maximize their market share—the mass of consumers in the vicinity of their chosen locations. At each stage, the two players (Leader and Follower) choose one location each from a feasible set in sequence. We first show that (i) if the feasible locations form a finite set in $\mathbf{R}^d$, Leader (the first mover) must obtain at least a $\frac{1}{d+1}$ fraction of the market share in equilibrium in the single-stage game, and there exist games in which Leader obtains no more than $\frac{1}{d+1}$; (ii) in the original Hotelling game (uniformly distributed consumers on the unit interval), Leader obtains $\frac{1}{2}$ even in the multiple stage game, using a strategy which is oblivious of Follower's locations. Furthermore, we exhibit a strategy for Leader, such that even if she has no information about the number of moves, her payoff must equal at least half the payoff of the single-stage game.

Keywords: *Location; Hotelling game; Condorcet paradox; Competitive location; Centerpoint theorem*

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1 Introduction

Starting with the classic Hotelling model [11], there is an extensive literature on location games. These games have been applied in several different contexts, including firms competing in a market (Gabscewicz and Thisse [9] provide a survey), political competition among parties or candidates (see Shepsle [16] for a survey), and facility location (surveyed by Eiselt, Laporte, and Thisse [8]).

In this paper, we consider min-max payoffs in a sequential location game with two players, named Leader and Follower. Given a demand distribution and a feasible set of locations, each player picks a feasible location in every stage with Leader moving first. After players have chosen their locations, each consumer buys one unit of the product from the closest player, breaking ties uniformly at random. We consider the game without prices, with each player maximizing its market share. We allow players to locate at previously occupied locations; therefore, it is immediate that Leader has a first-mover disadvantage in this game. By replicating the moves of Leader, Follower obtains a payoff no worse than $\frac{1}{2}$. Hence, we focus on the min-max (or worst-case) payoff of Leader.

Location games without pricing are commonly applied to, e.g., political contests and the facility location problem. As Osborne and Pitchik [15] show, the (simultaneous-move) game with prices may not possess a pure strategy equilibrium. With mixed strategy equilibria, the range of possible outcomes may be quite large. Further, characterizing the set of mixed strategy equilibria can be difficult. For a similar reason, we consider the sequential rather than simultaneous location game. On a side note, Prescott and Vischer [13] show that the outcomes of a sequential location game can differ significantly from those that obtain in a simultaneous move game.

We first examine a class of games in which the set of feasible locations is finite, and contained in $\mathbf{R}^d$. Without loss of generality, consumers are distributed over $\mathbf{R}^d$ (so there are d attributes of the product a consumer cares about). Since minimizing Leader's payoff is equivalent to maximizing Follower's payoff, the min-max payoff of Leader is equivalent to her payoff in a Nash equilibrium. In the single-stage game (with each player choosing just one location), we characterize completely the set of feasible min-

max payoffs for Leader over all choices of consumer distribution and location set. All Nash equilibria of the single-stage game are also subgame-perfect equilibria.

Dasci and Laporte [5] study a similar game where the Leader is already established as a monopolist with multiple locations, and Follower is contemplating entering the market. Both players are allowed to select multiple locations. They provide approximately optimal location and pricing strategies for both firms. In contrast, the focus of our work is on providing exact bounds for the disadvantage the Leader is faced with, in a game with no pricing.

1.1 Our results

We show that there exists a location game in $\mathbf{R}^d$ such that observed market shares are a result of a Nash equilibrium of this game if and only if the share of the first mover is between $\frac{1}{d+1}$ and $\frac{1}{2}$, and the shares of the players sum to 1. That is, over all location games in d-dimensional Euclidean space, the minimum payoff to Leader in a Nash equilibrium is $\frac{1}{d+1}$, and the maximum is $\frac{1}{2}$. Further, for any $y \in [\frac{1}{d+1}, \frac{1}{2}]$, there exist instances of the game such that Leader's equilibrium payoff is exactly y. For example, with a location set in $\mathbf{R}^2$, Leader must obtain at least $\frac{1}{3}$ of the payoff.

This result provides an upper bound for the size of the first-mover disadvantage in such a game. Entry timing games are often characterized by a trade-off between factors that imply a first-mover advantage (for example, in the political context, an early entrant has more time to raise money) and those that lead to a disadvantage. Our result implies that, keeping all other things the same, if the payoff increase as a result of a first-mover advantage exceeds $\frac{1}{2} - \frac{1}{d+1} = \frac{d-1}{2(d+1)}$ (so that the total payoff exceeds $\frac{1}{2}$), players should seek immediate entry in the single-stage game. Furthermore, our result provides an additional reason for the existence of a first-mover disadvantage in market entry order, complementing the results of Golder and Tellis [10] and Boulding and Christen [2].

We subsequently consider a multi-stage game in which the two players move sequentially at each stage, with Leader picking a location first, followed by Follower. Obtaining general results on the first-player payoff for multi-stage games may not be feasible. In particular, Leader's payoff need not be monotone in the number of stages. We provide

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two examples to demonstrate this. In one, we construct a game in which Leader obtains $\frac{1}{2}$ in a Nash equilibrium of the single-stage game, but only $\frac{1}{3}$ when the game is extended to two stages. Conversely, we exhibit a game in which Leader's payoff converges to $\frac{1}{2}$ as the number of stages grows. We then exhibit a slightly suboptimal result—we provide a simple strategy for Leader in the game with multiple stages, where Leader can obtain at least $\frac{1}{2(d+1)}$ payoff. The result holds even with information asymmetry where Leader does not know the number of stages but Follower does.

In the original Hotelling game (with the location set being the unit interval, and consumers uniformly distributed over this interval) with a known number of stages, we show that in the n-move game, for any n, the min-max payoff of Leader is $\frac{1}{2}$. In fact, we demonstrate a set of locations such that, if Leader occupies each location in this set, regardless of Follower's moves, she obtains a payoff of at least $\frac{1}{2}$.

1.2 Related work

Location games similar to the ones we consider have also been studied in computational geometry, under the label "Voronoi games." In these games, the location set is continuous, and the consumers are assumed to be uniformly distributed over some compact set. Co-location of players is not permitted. Cheong et al. [3], show that when the Voronoi game is played on a square with uniform demand, the number of moves is large enough, and Follower locates all his points after observing all of Leader's moves, Follower obtains a payoff of at least $\frac{1}{2} + \alpha$ for a fixed constant α. Some of the results we obtain here are cited as open questions by Cheong et al. In particular, we characterize the value of the sequential game, and the corresponding optimal strategies, in a high dimensional space. For the Voronoi game on the uniform line and uniform circle, Ahn et al. [1] show that Leader has a strategy which guarantees her a payoff of strictly more than $\frac{1}{2}$, while Follower can get a payoff arbitrarily close to $\frac{1}{2}$ without actually getting $\frac{1}{2}$. Variations of the original single-move Hotelling game with multiple players have also been considered under the name of "competitive facility location." Eiselt et al. [8] and Dasci and Laporte [5] provide excellent surveys of some of this work.

1.3 Paper outline

We define some preliminaries in the following section. Section 3 contains our main result characterizing the first-mover disadvantage in the one-round game. We also discuss a few examples illustrating the effect of changing some of the settings of the game, and briefly examine a special case of the game with multiple stages. In Section 4, we consider the location game where the number of rounds is known only to Follower and provide weaker bounds for the payoffs of the two players. We conclude with some final remarks in Section 5.

2 Preliminaries

Consider $\mathbf{R}^d$ with $d \geq 1$, endowed with the Euclidean distance function, δ. Consumers are distributed on $\mathbf{R}^d$, with distribution $F(\cdot)$ defined over the Borel $\sigma-$algebra on $\mathbf{R}^d$. Without loss of generality, the total mass of consumers is normalized to 1.

There are two players, Leader (she) and Follower (he). $L \subset \mathbf{R}^d$ denotes a compact set of points at which players may locate. The game has n stages. At each stage, the players move in sequence. First, Leader chooses a location in L, and then Follower responds. At any stage, either player is allowed to choose a location already occupied by either of the players. The game is therefore represented as a 4-tuple, (n, d, L, F).

Let s_i denote the location chosen by Leader at stage i, and t_i the location chosen by Follower. Let S_i and T_i denote the first i moves of the two players respectively, with $S_0 = T_0 = \emptyset$. A pure strategy for Leader at stage i is a map $a_i : S_{i-1} \times T_{i-1} \to L$. Similarly, a pure strategy for Follower at stage i is a map $b_i : S_i \times T_{i-1} \to L$. A pure strategy for Leader in the game as a whole is denoted $A = (a_1, \ldots, a_n)$ and similarly for Follower.

After each player has chosen its n locations, each consumer buys 1 unit of the good from the closest location. If the closest location is not unique, the consumer randomizes with equal probability over the set of closest locations.

Given a multiset Y of locations chosen by the players and some point v in $\mathbf{R}^d$, we define $\delta(v, Y) = \min_{y \in Y} \delta(v, y)$ as the distance between v and the point in Y closest

to v. Let $\kappa_Y(v) = |\{y \in Y : \delta(v,y) = \delta(v,Y)\}|$ be the number of points in Y which are at minimum distance from v. The demand gathered by a point $y \in Y$ is defined as $r(y, Y \setminus \{y\}) = \int_{v \in \mathbf{R}^d : \delta(v,y) = \delta(v,Y)} \frac{1}{\kappa_Y(v)} dF(v)$. Now let S and T be the locations chosen by Leader and Follower respectively, at the end of the game. Then, we write Leader's payoff as $r(S,T) = \sum_{s \in S} r(s, S \cup T \setminus \{s\})$. Follower's payoff is $r(T,S) = 1 - r(S,T)$. Note that by definition, for any location x and set of locations Y, we have $r(x,Y) \leq r(x,y)$ $\forall y \in Y$. Occasionally, we use $r_1(x,y)$ to denote Leader's payoff if she locates at x and Follower locates at y. We also define $r_2(x,y) = 1 - r_1(x,y)$, and suppress the dependence on x and y when the dependence is unambiguous.

The strategy choices of the two players, a and b, imply chosen locations, $S(a,b)$ and $T(a,b)$ respectively. Notationally, for convenience, we often suppress the dependence of S, T on a, b. Leader's min-max payoff is defined as $\underline{r}_1 = \max_a \min_b \; r(S(a,b), T(a,b))$. Since this is a constant-sum game, a strategy of Follower that minimizes the payoff of Leader must maximize the payoff of Follower. Hence, when n is known to both players, the strategies that lead to Leader earning its min-max payoff constitute a Nash equilibrium of the game.

Without loss of generality, we assume that L spans $\mathbf{R}^d$. Otherwise, we can project the d-dimensional space orthogonally to the subspace spanned by L. The orthogonal projection π has the property that for any two location points $l_1, l_2 \in L$ and a demand point $x \in \mathbf{R}^d$, $\delta(l_1, x) \leq \delta(l_2, x) \Leftrightarrow \delta(l_1, \pi(x)) \leq \delta(l_2, \pi(x))$. Thus payoffs and equilibrium strategies in the game remain unaffected.

3 One-round location game

We begin by examining the single-stage game. In focusing on Leader's min-max payoff, we essentially bound the size of the first mover disadvantage in this model. Recall that when the number of stages is known to both players, the min-max payoff of Leader is identical to its payoff in a Nash equilibrium. We therefore state our result in terms of Nash equilibrium payoffs.

We first consider the case of a finite location set. (Finiteness of the location set is nec-

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essary to prove Theorem 1 below, as we show in Section 3.2 following the theorem.) The demand distribution $F(\cdot)$ may be continuous. Let $\mathcal{G}_d$ denote the set of all location games in d-dimensional Euclidean space with a finite location set. Let $G_d(1) = (1, d, L, F)$ denote a game in $\mathcal{G}_d$.

It is clear that $\underline{r}_1 \leq \frac{1}{2}$, since Follower can ensure $r_2 = \frac{1}{2}$ via the strategy $b = a$, which replicates each move of Leader. How low can the min-max payoff of Leader be? The following example shows that, when the location set is in $\mathbf{R}^2$, Leader's payoff can be as low as $\frac{1}{3}$.
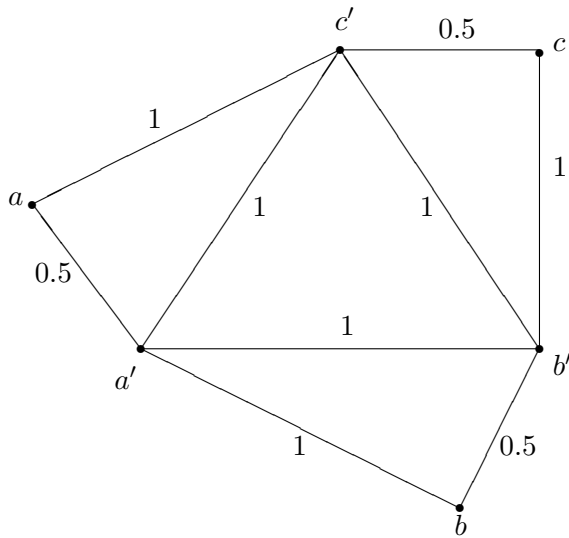

Figure 1: A location game in the Euclidean plane. Points a, b, and c have demands x, $\frac{1}{2}(1-x)$ and $\frac{1}{2}(1-x)$ respectively, and, $L = \{a', b', c'\}$. Lines are labeled by the Euclidean distance between their endpoints.

Example 1 Consider the game given by Figure 1, with $L = \{a', b', c'\}$, and $f(a) = f(b) = f(c) = \frac{1}{3}$ (that is, $x = \frac{1}{3}$), where $f(v)$ denotes the density of demand at v. Follower's best response is as follows: If Leader chooses a', Follower chooses b'; if Leader chooses b', Follower chooses c'; otherwise, Follower chooses a'. Given this, Leader is indifferent over $\{a', b', c'\}$. Regardless of the location she chooses, Leader obtains a

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payoff of $\frac{1}{3}$, with Follower obtaining $\frac{2}{3}$.

In fact, we show that this game represents the worst case for Leader over all such location games in $\mathbf{R}^2$. That is, there does not exist a demand distribution and a finite location set in $\mathbf{R}^2$, such that Leader obtains a Nash equilibrium payoff strictly less than $\frac{1}{3}$ in this single-move location game. The result extends more generally: in $\mathbf{R}^d$, Leader must obtain at least $\frac{1}{d+1}$, and there exists a game in which she obtains exactly $\frac{1}{d+1}$ (so the bound is tight).

Theorem 1 *There exists a location game $G_d(1) \in \mathcal{G}_d$ such that r_1, r_2 are payoffs in a Nash equilibrium of $G_d(1)$ if and only if $r_1 \in [\frac{1}{d+1}, \frac{1}{2}]$ and $r_2 = 1 - r_1$.*

Proof: It is immediate from the definition of the game that, in any equilibrium, $r_1 + r_2 = 1$. We now prove that $r_1 \in [\frac{1}{d+1}, \frac{1}{2}]$.

"If" part:

Given a value $x \in [\frac{1}{d+1}, \frac{1}{2}]$, we construct a game $G_d(1)$ for which $r_1 = x$. For $x = \frac{1}{d+1}$, this essentially reconstructs Example 1 in d dimensions. We first construct the game in the $(d+1)$-dimensional Euclidean space (for ease of exposition), then project it down to the d-dimensional Euclidean space.

The set of location points is a simplex given by $L = \{l_1, l_2, \ldots, l_{d+1}\}$, where point l_i is at position $+1$ on the ith co-ordinate axis. There are $d + 1$ demand points v_i. Let f represent the density of demand. Set $f(v_1) = x \in [\frac{1}{d+1}, \frac{1}{2}]$ and $f(v_i) = \frac{1}{d}(1 - x)$ for all $i > 1$. Fix $\epsilon > 0$ such that $\epsilon \ll 1$. Demand point v_i has i^{th} co-ordinate $1 - \epsilon$, and for $j \neq i$, the jth co-ordinate is $\epsilon[(j - i) \mod (d + 1)]$. This induces the following distance function between demand points and location points:

$$\delta^2(l_i, v_j) = \begin{cases} 2 - 2\epsilon[1 + (i - j) \mod (d + 1)] + \hat{d}\epsilon^2 & : i \neq j \\ \hat{d}\epsilon^2 & : i = j. \end{cases}$$

where $\hat{d} = 1 + \sum_{i=1}^{d} i^2$.

For any demand point v_j, we can define a precedence relation $\prec_j$ as $l_i \prec_j l_{i'}$ if demand point v_j prefers l_i over $l_{i'}$, that is, $\delta(l_i, v_j) < \delta(l_{i'}, v_j)$. It follows that for every j, we have

$$l_j \prec_j l_{(j+1) \mod (d+1)} \prec_j l_{(j+2) \mod (d+1)} \prec_j \cdots \prec_j l_{(j-1) \mod (d+1)}$$

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This precedence relation is identical to that induced by a Condorcet voting paradox [4] instance with $d+1$ voters and $d+1$ choices.

It now follows that $r_1(l_i, l_{(i-1)\mod(d+1)}) = x$ for $i = 1$, and $r_1(l_i, l_{(i-1)\mod(d+1)}) = \frac{1}{d}(1-x)$ for $i > 1$. For $x \in [\frac{1}{d+1}, \frac{1}{2}]$, we have $x \geq \frac{1}{d}(1-x)$. Leader's equilibrium strategy, therefore, is to choose l_1, and the resulting payoff is $r_1 = x$.

Finally, we obtain our d-dimensional instance by orthogonally projecting the demand points to the d-dimensional hyperplane formed by the points in L. Such a projection reduces each $\delta^2(l_i, v_j)$ by the same amount, and hence preserves the precedence relation $\prec_j$.

"Only if" part:

Note first that, for any $G_d(1) \in \mathcal{G}_d$, as observed earlier, we have $r_1 \leq \frac{1}{2}$ in any Nash equilibrium.

For any subset S of $\mathbf{R}^d$, let $F(S) = \int_{v \in S} dF(v)$ represent the total demand of points in S. In order to prove our result, we need to define the concept of *centerpoints*. A point $p_0 \in \mathbf{R}^d$ is a centerpoint if every closed half-space H that contains p_0 has demand $F(H) \geq \frac{1}{d+1}$. The following theorem may be found in Matoušek [12] (also Edelsbrunner [6]).

Theorem 2 [Centerpoint Theorem] *For any mass distribution F in $\mathbf{R}^d$, there exists a point p_0 such that any closed half-space containing p_0 has at least $\frac{1}{d+1}$ of the mass.*

The centerpoint of a distribution need not be unique; in Example 1, any point in the convex hull of a, b and c is a centerpoint. However, at least one centerpoint is guaranteed to exist. In the remainder of this proof, we prove the following (stronger) claim using centerpoints:

Claim 1 *Let p_0 be a centerpoint of the distribution F, and let L_0 be the set of location points at minimum distance from p_0. Then there exists a point $l \in L_0$ such that $r_1(l, l') \geq \frac{1}{d+1}$ for all $l' \in L$.*

If $p_0 \in L_0$, the claim follows immediately, so suppose $p_0 \notin L_0$. We consider two cases: *Case (i) $L_0 = \{l_0\}$*, that is, there is a unique location point closest to the centerpoint p_0. Consider any other location point l', and let $H^o(l_0, l') = \{v \in \mathbf{R}^d : \delta(l_0, v) < \delta(l', v)\}$ be

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the open half-space consisting of points closer to l_0 than to l'. Since $\delta(l_0, p_0) < \delta(l', p_0)$, there is a closed half-space containing p_0 which is fully contained in $H^o(l_0, l')$. Therefore, $r_1(l_0, l') \geq F(H^o(l_0, l')) \geq \frac{1}{d+1}$, and locating at l_0 ensures that Leader earns at least $\frac{1}{d+1}$ payoff.

Case (ii) $|L_0| > 1$. Define a precedence relation on L_0 as follows: $l \prec l'$ if and only if $r_1(l, l') < \frac{1}{d+1}$. We need to show that there exists a point $l \in L$ such that there is no $l' \in L$ with $l \prec l'$. We begin by proving that $\prec$ is *acyclic* on L_0; that is, there is no sequence of elements $(l_1, l_2, \ldots, l_k)$ in L_0 with $l_1 \prec l_2 \prec l_3 \prec \ldots \prec l_k \prec l_1$.

For a contradiction suppose $L' = \{l_1, l_2, \ldots, l_k\} \subseteq L_0$ is a set of location points forming such a cycle. Let H_i denote the closed half-space of points at least as close to l_{i+1} as to l_i; that is, $H_i = \{v \in \mathbf{R}^d : \delta(l_{i+1}, v) \leq \delta(l_i, v)\}$. Let $H_i^=$ denote the corresponding hyperplane of points equidistant between l_i and l_{i+1}. Then, $l_i \prec l_{i+1}$ implies that $F(H_i) - \frac{1}{2}F(H_i^=) > \frac{d}{d+1}$.

Now let ν_i be the vector $l_{i+1} - l_i$. Then, each half-space is given by $H_i = \{u \in \mathbf{R}^d : u \cdot \nu_i \geq 0\}$. We refer to l_{i+1} and l_i as the location points defining ν_i. Since L' is a cycle, we also have $\sum_{i=1}^m \nu_i = 0$. That is, a positive linear combination of these vectors ν_i sums to zero.

Now, Carathéodory's theorem (see Eckhoff [7]) implies that there exists a positive combination of at most $d + 1$ of the vectors ν_i that sums to zero. Without loss of generality, let these be $\nu_1, \ldots, \nu_{d'}$ $(2 \leq d' \leq d + 1)$, and let α_i be positive reals such that $\sum_{i=1}^{d'} \alpha_i \nu_i = 0$.

Next consider any point x lying in $X = \cap_{i=1}^{d'} H_i$. Since $x \cdot \nu_i \geq 0$ for all i, $1 \leq i \leq d'$, we must have $0 \leq x \cdot \alpha_i \nu_i = -\sum_{j \leq d', j \neq i} x \cdot \alpha_j \nu_j \leq 0$, implying that $x \cdot \nu_i = 0$ for all $i \leq d'$. In other words, $X = \cap_{i=1}^{d'} H_i^=$. Note that $p_0 \in X$; therefore X is not empty.

For each half-space H_i, we have $F(H_i) > \frac{d}{d+1} + \frac{F(X)}{2}$. Taking complements, $F(\overline{H_i}) < \frac{1}{d+1} - \frac{F(X)}{2}$. Therefore, we have $F(\cup_{i<d'} \overline{H_i} \cup X) < \frac{d'-1}{d+1} - \frac{d'-1}{2}F(X) + F(X) \leq \frac{d}{d+1} + \frac{1}{2}F(X)$. Taking complements once again, $F(\cap_{i<d'} H_i \backslash X) > \frac{1}{d+1} - \frac{F(X)}{2}$. But $\cap_{i<d'} H_i \backslash X$ is disjoint from $H_{d'}$, which has demand greater than $\frac{d}{d+1} + \frac{F(X)}{2}$. This along with $F(X) > 0$ contradicts the fact that the total demand is 1. Therefore, we have a contradiction, and the cycle L' cannot exist.

We have shown that the relation $\prec$ is acyclic. An acyclic relation on a finite set must contain a point l_0 which is not preceded by any other point $l' \in L_0$. Such a point can be found by starting at any point $l \in L_0$, and moving to any point $l' \in L_0$ such that $l \prec l'$. Since $\prec$ is acyclic and L_0 is finite, this process must terminate at an l_0 such that there is no point $l' \in L_0$ with $l_0 \prec l'$.

If Leader locates at l_0 and Follower locates at any point $l' \in L_0$, then $r_1(l_0, l') \geq \frac{1}{d+1}$ because l' does not precede l_0. If Follower locates at some point $l' \notin L_0$, then the argument for Case (i) ($|L_0| = 1$) shows that $r_1(l_0, l') \geq \frac{1}{d+1}$. This completes the proof of the "only if" part, as well as Theorem 1. ∎

3.1 Choosing the best location point

In Theorem 1 we show that locating at one of the points closest to a centerpoint guarantees a payoff of at least $\frac{1}{d+1}$ in the one move game. The following example shows that this does not hold in general for an arbitrary location point closest to a centerpoint, thus necessitating a proof as given above.

Example 2 Consider the following instance of the location game in 3-dimensional Euclidean space, with the co-ordinates labeled x, y and z respectively. The demand is concentrated at 4 points: $p_1 = (1, 0, 0), p_2 = (-0.5, -\sqrt{3}/2, 0), p_3 = (-0.5, \sqrt{3}/2, 0)$ and $p_4 = (0, 0, 5)$. The demands at p_1, p_2 and p_3 are $0.25 - \epsilon$, where $0 < \epsilon \ll 1$. The demand at p_4 is $0.25 + 3\epsilon$. The set of location points consists of a set L' of several points at distance 1 from p_4 with the z-co-ordinate at least 5.5, and a single location point $l_0 = (0, 0, 4)$.

The only centerpoint of this demand distribution is at p_4. All location points are equidistant from it, since they are all at distance 1. However, if Leader locates at any point in L', then Follower can locate at l_0 resulting in a payoff of only $\frac{1}{8} + 1.5\epsilon$ for Leader.

Therefore, if there is more than one location point closest to the set of centerpoints, one cannot arbitrarily locate at any one of them. By Theorem 1, there must exist a point closest to a centerpoint, such that locating at that point guarantees at least $\frac{1}{d+1}$ payoff for Leader; the point l_0 in Example 2 is such a point.

3.2 Finiteness of the location set

Finiteness of the location set, L, is used in the "only if" part of the theorem to show that the acyclicity of $\prec$ implies that we can find a sink node. The following example, a variant of the largest number game, indicates that there is no extension to a countably infinite set. Consider the unit interval, $[0, 1]$. Let $f(0) = 1$ (so that all demand is at the point 0). Let $L = \{\frac{1}{n}\}_{n \in \mathbf{Z}_+}$, where $\mathbf{Z}_+$ is the set of positive integers. For any point l_1 chosen by Leader, Follower can find a point closer to 0, and obtain a payoff of 1.

3.3 Non-monotonicity of payoffs

Consider the game in $\mathcal{G}_d$ constructed in the "If" part of Theorem 1, with $x = \frac{1}{d+1}$. Let us study how the payoff of Leader changes as the number of moves n increases (with both players knowing n). While the number of moves is less than $d + 1$, Leader can weakly increase her payoff by picking at each stage a location where she has not located yet. When the number of moves is $d + 1$ or more, the strategy of first locating at all points in L and then replicating Follower's previous move guarantees a payoff which converges from below to $\frac{1}{2}$ as n increases.

Given the last remark above, one might conjecture that, in any instance the multi-stage game, the min-max payoff of Leader is weakly increasing in the number of moves, n. However, the following example demonstrates that this is not always true.

Example 3 Consider two replicas of the game in Example 1, with location sets $L_i = \{a'_i, b'_i, c'_i\}$ for $i = 1, 2$. The demand density is $\frac{1}{6}$ at each of the points in $D_i = \{a_i, b_i, c_i\}$, for $i = 1, 2$. Further, let a'_j be the closest location point in L_j to the demand points D_i, for $i = 1, 2$ and $j \neq i$. Let $\delta(a_i, a'_j) > 2$ for $i = 1, 2$ and $j \neq i$, so that the points in L_j are sufficiently far from the points in D_i.

Suppose $n = 1$, so that each player moves just once. Leader's optimal action is to choose either a'_1 or a'_2. If Leader chooses a'_1, Follower's best response is to choose any of $\{a'_1, a'_2, b'_2, c'_2\}$, with a corresponding best response set if Leader chooses a_1. In either case, Leader obtains a payoff of $\frac{1}{2}$.

Now, suppose $n = 2$. Without loss of generality, suppose Leader chooses a location

in L_1 with her first move. Conditional on choosing a point in L_1, locating at a'_1 is an optimal action for Leader. Now, Follower responds by locating at b'_1. Consider Leader's best response. If she chooses any point in L_2, Follower will choose the corresponding point in L_2 such that it obtains $\frac{2}{3}$ of the demand closest to each of L_1 and L_2, and hence captures a payoff of $\frac{2}{3}$ in the game. If instead, Leader chooses any point in L_1, Follower will then choose a'_2, obtaining all of the demand closest to L_2, and at worst $\frac{1}{3}$ of the demand closest to L_1, for an overall payoff no worse than $\frac{2}{3}$. Hence, Leader can obtain no more than $\frac{1}{3}$ in the 2-move game.

3.4 Oblivious strategy for the Hotelling context

The above example suggests that there is no general result on the equilibrium payoffs as n increases. Since results on the general n-move game are difficult to obtain, we next study the game in Hotelling's original setting, where the demand is distributed uniformly over $[0, 1]$. Let $H(n) = (n, 1, [0, 1], U[0, 1])$ denote the Hotelling game with n rounds, $L = [0, 1]$, and $f(x) = 1$ for $x \in [0, 1]$. We first show that there is no second-mover advantage in $H(n)$. In particular, for any fixed n, there exists a set of location points S that Leader can choose which implies that her payoff is at least $\frac{1}{2}$, regardless of the strategy of Follower.

Theorem 3 *For the game $H(n)$, we have $\underline{r}_1 = \frac{1}{2}$.*

Proof: Consider $S = (s_1, s_2, \ldots, s_n)$, where $s_i = \frac{1}{2n} + \frac{(i-1)}{n}$. This divides the unit line into $n+1$ intervals—the two border intervals are of length $\frac{1}{2n}$, while the internal intervals are of length $\frac{1}{n}$.

Let Follower's chosen location points be given by $T = (t_1, \ldots, t_n)$. We will show that each point t_i gets payoff at most $\frac{1}{2n}$. This implies that $r_1 \geq \frac{1}{2}$. As observed earlier, Follower can obtain a payoff of $\frac{1}{2}$ by simply replicating each of Leader's moves (i.e. set $t_i = s_i$ for each i). First note that, even in the absence of any points t_i, the total demand captured by each point s_i individually is at most $\frac{1}{n}$ for any i.

Consider the point t_i. Suppose $t_i = s_j$ for some j. Clearly, the market share of point t_i is at most $\frac{1}{2n}$ from our observation above. Next suppose that t_i lies in one of the

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border intervals. Again, since the length of these intervals is $\frac{1}{2n}$, the market share of t_i is at most $\frac{1}{2n}$.

Finally, consider the case when t_i lies in some interval (s_j, s_{j+1}). If there is at least one other point t_k in this interval, t_i and t_k may share the total demand in that interval, each getting at most $\frac{1}{2n}$. If t_i is the only point in this interval, then, it gets $\frac{1}{2}(s_{j+1} - t_i)$ demand from the left and $\frac{1}{2}(t_i - s_j)$ demand from the right. Combining the two, we have that t_i gets at most $\frac{1}{2n}$ of the demand. Thus Follower obtains a payoff no greater than $\frac{1}{2}$. ∎

A similar result was obtained independently by Ahn et al. [1], in the context of Voronoi games, which differ from our location games in that co-location is not allowed in Voronoi games.

Note that Leader's strategy in Theorem 3 is oblivious of Follower's strategy T. Thus, Leader's strategy guarantees her a payoff of at least $\frac{1}{2}$ even when both players move simultaneously at each round, or indeed, even if the order of moves is completely arbitrary.

4   Asymmetric information

Next, we consider an asymmetric-information version of the location game. In this game, the number of stages, n, is known to Follower but not to Leader. Instead, Leader merely knows that $n \in N$, where N is some feasible set for the number of stages.

In terms of min-max payoffs, this changes the flavor of the game completely. The min-max payoff of Leader now contains an additional uncertain element, the number of stages in the game. As a result, the min-max payoffs in the game can no longer be thought of as equilibrium payoffs. Given location sets S, T for the two players, and a known number of stages n, let $r_1(S, T, n) = r(S_n, T_n)$ denote Leader's payoff in the game. Then, when Leader does not know the number of stages, but only that it lies in some set N, her min-max payoff is given by $\underline{r}_1(N) = \max_a \min_{n \in N} \min_{b_n} r_1(S(a, b_n), T(a, b_n), n)$.

4.1 Hotelling context: Uniformly distributed demand on a line

To illustrate the nature of the difficulty in analyzing this case, suppose first that $N = \{1, 2\}$, that is, Leader knows that the number of stages is either 1 or 2. In contrast with Theorem 2, the following theorem shows that, in the set-up of the original Hotelling game H, Leader can no longer ensure a payoff of $\frac{1}{2}$ across all possible outcomes.

Theorem 4 *Suppose Leader knows that $n \in N = \{1, 2\}$, and Follower knows n. Then, in the game $H(N)$, we have $\underline{r}_1(N) = \frac{5}{12}$.*

Proof: We first show that $\underline{r}_1(N) \geq \frac{5}{12}$. Consider the following strategy for Leader. She first locates at $s_1 = \frac{1}{2}$. If $n = 1$, Follower will also choose $t_1 = \frac{1}{2}$, so Leader earns exactly $\frac{1}{2}$ (that is, $\underline{r}_1(1) = \frac{1}{2}$).

Suppose $n = 2$. Without loss of generality (w.l.o.g.), Follower's first move is to $t_1 \leq s_1$. Firstly, if $t_1 = \frac{1}{2}$, then Leader chooses $s_2 = \frac{1}{4}$. It is easy to verify that in this case, Leader gets a revenue of at least $\frac{7}{16} \geq \frac{5}{12}$. If $\frac{1}{3} > t_1 > \frac{1}{2}$, Leader then chooses $s_2 = t_1 - \epsilon$, for some small $\epsilon > 0$. Now, regardless of Follower's second move, Follower obtains a payoff at most $\frac{1}{2} + (\frac{1}{2} - t_1)/2 \leq \frac{7}{12}$. By locating at $\frac{1}{2} + \epsilon$, for some small $\epsilon > 0$, Follower obtains a payoff that approximates (but is strictly less than) $\frac{7}{12}$.

On the other hand, if Follower first locates at $t_1 \leq \frac{1}{3}$, then Leader chooses $s_2 = \frac{5}{6}$. Now, if Follower chooses $t_2 > s_1$, he earns a payoff at most $\frac{7}{12}$. If $t_2 = s_1$, his payoff is at most $\frac{13}{24}$. For any other point $t_2 < s_1$, his payoff is at most $\frac{1}{2}$. Therefore, $\underline{r}_1(2) \geq \frac{5}{12}$, implying $\underline{r}_1(N) \geq \frac{5}{12}$.

Next we show that $\underline{r}_1(N) \leq \frac{5}{12}$. Suppose not. Then, Leader's first move must be to some point in $(\frac{5}{12}, \frac{7}{12})$ (else $\underline{r}_1(1) \leq \frac{5}{12}$). W.l.o.g, suppose Leader's first move is to $s_1 \in (\frac{5}{12}, \frac{1}{2}]$. Suppose $n = 2$, and consider the following sequence of play. Follower chooses $t_1 = \frac{2}{3}(1 - s_1) < s_1$. At the second stage, if Leader moves to $s_2 < s_1$, then Follower makes its second move to $t_2 = s_1 + \epsilon$ for some small $\epsilon > 0$. Otherwise, Follower moves to some $t_2 > s_1$ that obtains maximum payoff. The latter payoff is at least $\frac{1}{3}(1 - s_1)$. A simple calculation again shows that in either of these cases, Follower earns a payoff of at least $\frac{2}{3} - \frac{s_1}{6} \geq \frac{7}{12}$. ∎

4.2 General multi-stage location games

The above theorem shows that if H is played with the number of stages restricted to being no more than 2, then Leader's min-max payoff is lower than $\frac{1}{2}$. What if Leader has no information at all about the number of stages? The techniques used for the above theorem do not extend easily to larger n, since the number of cases increases rapidly as n increases. However, we show below that a simple strategy guarantees a payoff of $\frac{1}{4}$ to Leader irrespective of the number of rounds in the game.

We in fact show a more general theorem that applies to all sequential two-player location games, including H and those in $\mathcal{G}_d$. The theorem shows that in a multi-stage game, Leader must obtain at least $\frac{1}{2}$ of her payoff in the single-stage game, even when she has no knowledge of the number of stages (that is, the set of feasible stages, N, is the set of positive integers). We prove the theorem by exhibiting a particular strategy that earns this payoff: locate at the single-stage equilibrium location, then replicate each move of Follower.

Theorem 5 *Suppose that, in a Nash equilibrium of a single stage location game, Leader earns $r_1 = \rho$. Consider the multiple-stage game in which Leader only knows that $n \in Z_+$, but Follower knows n. In this game, $\underline{r}_1(Z_+) \geq \frac{\rho}{2}$.*

Proof: Consider the following strategy for Leader. At stage 1, she chooses a location s_1 that yields the payoff of a single-stage equilibrium, ρ. For $i > 1$, Leader replicates Follower's previous move, so that $s_i = t_{i-1}$. For any location $y \in S \cup T \setminus \{s_1\}$, we have $r(y, S \cup T \setminus \{y\}) \leq r(y, s_1) \leq 1 - \rho$.

Now, $r_1(S, T, n) \geq \sum_{i=2}^{n} r(s_i, S \cup T \setminus \{s_i\}) = \sum_{i=1}^{n-1} r(t_i, S \cup T \setminus \{t_i\}) = r(T, S) - r(t_n, S \cup T \setminus \{t_n\})$. This implies $2r_1(S, T, n) \geq 1 - r(t_n, S \cup T \setminus \{t_n\}) \geq \rho$. Thus, $\underline{r}_1(Z_+) \geq \min_n r_1(S, T, n) \geq \frac{\rho}{2}$. ∎

We get the following immediate implication:

Corollary 6 *Suppose Leader has no information about n, but Follower knows n.*
(i) for any location game $G_d(Z_+) \in \mathcal{G}_d$, we have $\underline{r}_1 \in [\frac{1}{2(d+1)}, \frac{1}{2}]$.
(ii) for the game $H(Z_+)$, we have $\underline{r}_1 \geq \frac{1}{4}$.

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5 Conclusion

We have shown that in a one move location game in $\mathbf{R}^d$, Leader can always guarantee at least $\frac{1}{d+1}$ of the total payoff. If Leader earns a payoff strictly less that $\frac{1}{d+1}$, this payoff could not have emerged from a Nash equilibrium of the location game in d-dimensional Euclidean space. Conversely, for every $x \in [\frac{1}{d+1}, \frac{1}{2}]$, there exists a location game such that Leader obtains a market share exactly x in equilibrium.

In the multiple-move game on a unit line, when both players know the number of moves, both obtain a payoff of $\frac{1}{2}$ in a Nash equilibrium. It would be interesting to generalize this result to games in higher dimensions.

The situation changes when Leader does not know the number of moves. Even if the number of moves is 1 or 2, in the game on a unit line, Leader obtains a payoff strictly less than $\frac{1}{2}$. However, we demonstrate a strategy for Leader, using which she can obtain at least half the payoff of the single-move game in a Nash equilibrium. An interesting open problem is to completely characterize this min-max payoff.

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