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A Nearly Best-Possible Approximation Algorithm for Node-Weighted Steiner Trees*

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We give the first approximation algorithm for the node-weighted Steiner tree problem. Its performance guarantee is within a constant factor of the best possible unless $\tilde{P} \supseteq NP$. ($\tilde{P}$ stands for the complexity class deterministic quasi-polynomial time, or DTIME[$n^{\text{polylog } n}$].) Our algorithm generalizes to handle other network-design problems. © 1995 Academic Press, Inc.

1. INTRODUCTION

The Steiner problem in networks is a classic hard problem in combinatorial optimization. Much research has been devoted to heuristics for its solution [4, 7, 14, 17, 18, 24]. Despite a slew of new approximation algorithms for this problem and some of its variants, no approximation

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algorithm has been given for perhaps the most natural variant: the *node-weighted Steiner tree problem*, in which costs can be assigned to nodes as well as edges. Indeed, Winter's survey [22] on the network Steiner problem closes with the sentence, "Further investigation of the vertex-weighted SPN [Steiner problem in networks] is needed."

One reason for the dearth of results on the node-weighted variant may be that it is harder than the standard problem. Indeed, while constant-factor approximations are known for the standard problem [10, 15, 21, 25] and even some of its generalizations [1, 6], the node-weighted version cannot be approximated to within less than a logarithmic factor unless $\tilde{P} \supseteq NP$ [2, 13].[1]

In this paper, we give the first approximation algorithm for the node-weighted Steiner tree problem. The performance guarantee is logarithmic. Thus assuming $\tilde{P} \not\supseteq NP$, the accuracy of our approximation is within a constant factor of the best-possible approximation achievable in polynomial time.

The algorithm we propose is only a slight variant of a heuristic proposed by Rayward-Smith and Clare in 1986 [8, 17, 18] for the standard edge-weighted Steiner tree problem. The key to our analysis is a decomposition lemma for trees; this lemma may be useful in other contexts as well.

We also show how to generalize the algorithm and its analysis to handle more general connectivity requirements. Thus, we obtain approximation algorithms for node-weighted versions of, e.g., fixed and nonfixed point-to-point connection problems.

Preliminaries

Let G be a graph with nonnegative costs assigned to its nodes and edges. The cost of a subgraph of G is the sum of the costs of its nodes and edges. Let A be a subset of the nodes of G, called *terminals*. A *Steiner tree for A in G* is a connected subgraph of G containing all the nodes of A. (Note that an edge-minimal Steiner tree is indeed a tree.) The Steiner tree problem is to find a minimum-cost Steiner tree.

We introduce a problem that turns out to be closely related. Let B be a ground set, and let $S_1, \ldots, S_s$ be subsets of B with costs $c_1, \ldots, c_s$. A *set cover* is a collection of sets S_i whose union is B. The *set-cover problem* is to find a minimum-cost set cover.

Berman [2] showed that, in the presence of node-weights, approximating the minimum-cost Steiner tree is as hard as approximating set cover. More specifically, he showed that any instance of set cover can be formulated as

[1] Here we use $\tilde{P}$ to mean the complexity class deterministic quasi-polynomial time, or DTIME[$n^{\text{polylog } n}$].

an instance of the node-weighted Steiner tree problem. The reduction is illustrated in Fig. 1. Thus an approximation algorithm for minimum-cost Steiner tree could be used to achieve the same approximation for set cover.

It has recently been proved [13] that no polynomial-time approximation algorithm for set cover achieves an approximation factor smaller than $\frac{1}{4} \ln |B|$ (unless Deterministic Time $n^{\text{polylog } n}$ contains NP). By Berman's reduction, the same holds for the node-weighted Steiner tree problem. Thus we cannot expect to obtain an approximation algorithm that achieves a performance ratio better than logarithmic.

THEOREM 1.1. *There is a polynomial-time algorithm to approximate the node-weighted Steiner tree problem in networks. The performance ratio is* $2 \ln k$, *where* k *is the number of terminals.*

An example due to Chvátal [3] can be adapted to show that our analysis of the algorithm is nearly tight (See Fig. 2).

Our method can be easily applied to more general node-weighted network-design problems. For example, consider the following generalization of the Steiner problem: given a set of pairs of nodes (s_i, t_i), find a minimum-cost subgraph in which each s_i is connected to t_i. The edge-weighted version of this problem was addressed in [1]; the node-weighted version can be approximately solved using the method of this paper.

We use a framework due to Goemans and Williamson [6] to formulate problems like that described above. Many network-design problems can be formulated as finding a subgraph of minimum cost that covers a family of

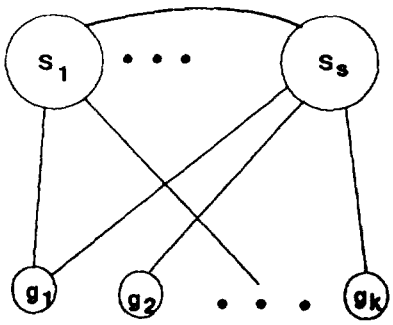

FIG. 1. Let G be a graph with k nodes, one for each ground element, plus s nodes, one for each set S_i. There is an edge between a ground element node and a set node if the set contains the element. In addition, all the set nodes are pairwise adjacent. The cost of each set node is that of the set it represents. All other nodes and all edges have zero cost. It is easy to check that a Steiner tree for the k nodes corresponds to a set cover of the same cost, and vice versa.

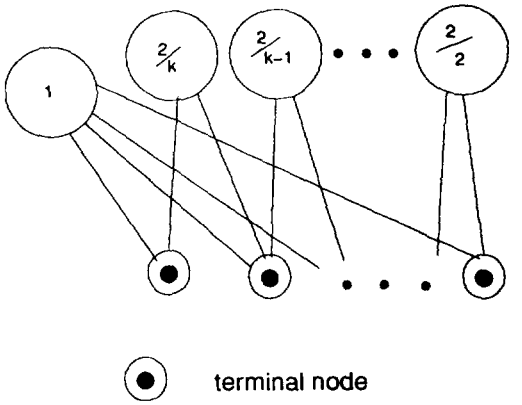

FIG. 2. In the example above, the minimum-node-cost Steiner tree consists of the node of unit cost with edges to all the terminals. The greedy algorithm may choose each of the other nodes from left to right successively to produce a Steiner tree of cost $2(H(k) - 1)$, where $H(k)$ denotes the kth harmonic number.

cuts in the graph (the particular family depends on the problem). For certain families of cuts, the edge-weighted problem can be approximated to within a factor of two [6]. We show in Section 5 that the node-weighted variants of these network-design problems can also be approximated; the performance is as in Theorem 1.1.

2. THE ALGORITHM

In this section we describe the algorithm. Note that since any Steiner tree must include all the terminals, we can assume without loss of generality that the terminals have zero cost.

The algorithm maintains a node-disjoint set of trees containing all the terminals. Initially, each terminal is in a tree by itself.

The algorithm uses a greedy strategy to iteratively merge the trees into larger trees until there is only one tree. In each iteration, it selects a node v and a nonsingleton subset of the current trees so as to minimize the ratio

$$\frac{\text{cost of the node } v \text{ plus sum of distances to the trees}}{\text{number of trees}} \tag{1}$$

Here the distance along a path does not include the costs of its endpoints. Thus the choice minimizes the average node-to-tree distance. The algo-

rithm uses the shortest paths between the node and the selected trees to merge the trees into one.

It is easy to implement an iteration. For each node v, define the *quotient cost* of v to be the minimum value of (1) taken over all nonsingleton subsets of the current trees. To find the quotient cost of v, compute the distances d_i from v to each of the trees T_i; assume without loss of generality that the trees are numbered so that $d_1 \leq d_2 \leq \cdots \leq d_k$. In computing the quotient cost of v, it is sufficient to consider subsets of the form $\{T_1, T_2, \ldots, T_i\}$. Thus the quotient cost for a given vertex can be computed in polynomial time; by computing the quotient cost of all the vertices, we can determine the minimum quotient cost and thus carry out an iteration in polynomial time.

In Fig. 2, we adapt an example due to Chvátal [3] to show that the performance ratio in Theorem 1.1 is nearly tight. In this example, the Steiner tree of minimum node cost is the single node of unit cost adjacent to all the terminals. However the greedy algorithm may choose the node of cost $2/k$ in the first iteration to merge its two neighbors into a tree since this node has the minimum quotient cost of $1/k$ for this iteration. In each subsequent iteration i, the greedy algorithm may choose the node of cost $2/(k + i - 1)$ which achieves the minimum quotient cost of $1/(k + i - 1)$ for this iteration. Thus the tree produced by the greedy algorithm has cost $2(H(k) - 1)$ where $H(k)$ denotes the kth harmonic number.

In Section 4, we show that the cost of all nodes and edges selected by the algorithm is not much more than the minimum cost of a Steiner tree.

3. Spider Decomposition

The proof of the performance guarantee of the algorithm involves showing the existence of a node with low quotient cost relative to the minimum cost of a Steiner tree. To show this, we prove that a minimum Steiner tree can be decomposed into subtrees we call *spiders*. It follows that one of these spiders has low quotient cost relative to the cost of the minimum Steiner tree. It then follows that the cost of an iteration of the greedy algorithm is small.

Definition. A *spider* is a tree with at most one node of degree greater than two. A *center* of a spider is a node from which there are edge-disjoint paths to the leaves of the spider. Note that if a spider has at least three leaves, its center is unique. A *foot* of a spider is a leaf, or, if the spider has at least three leaves, the spider's center. Note that every spider contains disjoint paths from each of its centers to all of its feet. A *nontrivial spider* is one with at least two leaves.

DEFINITION. Let G be a graph, and let M be a subset of its nodes. A *spider decomposition* of M in G is a set of node-disjoint nontrivial spiders in G such that the union of the feet of the spiders in the decomposition contains M.

THEOREM 3.1. *Let G be a connected graph, and let M be a subset of its nodes such that $|M| \geq 2$. Then G contains a spider decomposition of M.*

Proof. The proof we provide here is due to Wolsey [23]. The depth of a node in the tree is defined as the distance of the node from an arbitrarily chosen root.

We prove the theorem by induction on $|M|$. Choose a node v of maximum depth in the tree such that the subtree rooted at v contains at least two nodes in M. Ties are broken arbitrarily. By choice of v, all the paths from the nodes in M to the node v in the subtree of v are node-disjoint, and together form a nontrivial spider centered at v.

We now delete the subtree rooted at v including the node v from the tree. If no node in M remains in the tree, we are done. It the tree contains two or more nodes in M, then we can find a spider decomposition of these nodes by the induction hypothesis. Otherwise, there is exactly one node in M remaining in the tree. In this case, we add the path in the tree from this node in M to the spider centered at v. This leaves a spider centered at v and we are done. ■

4. THE PERFORMANCE GUARANTEE

Now we prove the performance guarantee for the greedy algorithm. Let ϕ_i denote the number of trees in the solution just after iteration i. Thus, for instance, ϕ_0 is the number of terminals in G. Let the number of trees merged at iteration i be h_i. Then we have

$$\phi_i = \phi_{i-1} - (h_i - 1). \tag{2}$$

Let C_i denote the cost of the subgraph added by the algorithm in iteration i. Let OPT denote the cost of a minimum-cost Steiner tree spanning the terminals. The key ingredient of our proof is the following lemma, which depends on our spider-decomposition theorem.

LEMMA 4.1. *At any iteration i of the algorithm,*

$$h_i \geq \frac{C_i \cdot \phi_{i-1}}{OPT}. \tag{3}$$

The above lemma can be intuitively interpreted as follows. At the beginning of iteration i, any optimal tree of total cost OPT can be used to connect the ϕ_{i-1} remaining trees into one. This tree can be decomposed

using Theorem 3.1 into a spider decomposition of the remaining trees. By averaging, one of these spiders has quotient cost better than the "average quotient cost" of the whole tree, i.e., $OPT/(\phi_{i-1})$, and thus the best quotient cost for this iteration is at most this quantity. We proceed with the formal proof of the lemma below.

Proof. Let T^* be a minimum-cost Steiner tree. Let $T_1, \cdots, T_{\phi_{i-1}}$ be the current trees at the beginning of iteration i of the algorithm. Let T_i^* be the graph obtained from T^* contracting each T_j to a supernode of zero cost. Namely, beginning with T^*, for each T_j, we contract all the nodes in $T_j \cap T^*$ into a single supernode in T_i^* and assign this supernode zero cost. The nodes in T^* that do not participate in any contraction in forming T_i^* retain their original cost. Note that T_i^* is connected and contains all supernodes. Let M be the set of supernodes. We apply Theorem 3.1 to the graph T_i^* to obtain a spider decomposition of M. Furthermore, the total cost of spiders in the decomposition is at most that of T_i^* which is in turn at most OPT.

Let $c_1, \ldots, c_r$ be the centers of the spiders in the decomposition. For a spider with only two leaves, i.e., a path, pick any node in the path as its center. Let $m_1, \ldots, m_r$ denote the number of nodes of M in each of these spiders, respectively. Since every spider in the decomposition is nontrivial, each m_j is at least two. Let the cost of the spider centered at c_j be $Cost_j$. Moreover, a spider with center c_j induces a subset of the current trees, namely the m_j trees whose supernodes belong to this spider. The cost of c_j plus the sum of distances to these trees is exactly $Cost_j$. Hence the quotient cost of c_j is at most

$$\frac{Cost_j}{m_j}.$$

Since the algorithm chooses a vertex of minimum quotient cost, for each spider in the decomposition we have

$$\frac{C_i}{h_i} \leq \frac{Cost_j}{m_j}.$$

Summing over all the spiders in the decomposition yields

$$\frac{C_i}{h_i} \sum_{j=1}^{r} m_j \leq \sum_{j=1}^{r} Cost_j.$$

By node-disjointness, the sum $\sum_{j=1}^{r} m_j$ of the number of nodes of M in each of the spiders is exactly the number ϕ_{i-1} of current trees. Also

$\Sigma_{j=1}^{r}\text{Cost}_j$ is exactly the cost of the spider decomposition, which is at most OPT. Substituting in the above equation and simplifying yields

$$\frac{C_i \phi_{i-1}}{OPT} \leq h_i.$$

■

We now use the above lemma in conjunction with an analysis technique due to Leighton and Rao [11] to complete the proof of the performance guarantee.

Proof. Substituting Eq. (3) into (2) and using the inequality $h_i \leq 2(h_i - 1)$, we get

$$\phi_i \leq \phi_{i-1}\left(1 - \frac{C_i}{2 \cdot OPT}\right). \tag{4}$$

If the total number of iterations of the algorithm is p, then $\phi_p = 1$. Simplifying (4), we obtain

$$\phi_p \leq \phi_0 \prod_{j=1}^{p}\left(1 - \frac{C_j}{2 \cdot OPT}\right)$$

Taking natural logarithms on both sides and simplifying using the approximation $\ln(1 + x) \leq x$, we obtain

$$\sum_{j=1}^{p} C_j \leq 2 \ln\left(\frac{\phi_0}{\phi_p}\right) \cdot OPT.$$

Since we assumed that all the terminals have zero cost, note that the cost of the final solution is the exactly the sum $\Sigma_{j=1}^{p} C_j$. Noting that $\phi_0 = k$ and $\phi_p = 1$ we finally have

$$\sum_{j=1}^{p} C_j \leq 2 \ln k \cdot OPT. \tag{5}$$

This proves the performance guarantee in Theorem 1.1. ■

5. General Node-Weighted Network-Design Problems

In this section, we generalize our algorithm to handle more general network-design problems. As discussed in the Introduction, many such

problems can be formulated as cut-covering problems: for a certain family of cuts in a graph, find a minimum-cost subgraph intersecting all the cuts in the family. Fix a graph G with node and edge weights. For any node subset S, there is a corresponding cut $\Gamma(S)$ in G, namely that consisting of edges with exactly one endpoint in S. Thus we can use a 0–1 function f on the set of node subsets to define a family of cuts: $f(S) = 1$ whenever $\Gamma(S)$ is in the family. Goemans and Williamson [6] proposed a class of cut families and showed that they are useful in modeling network-design problems such as the point-to-point connection problem. They called a function *f proper* if it obeys the following properties: (symmetry property) $f(S) = f(V - S)$ for all $S \subseteq V$; and (disjointness property) if A and B are disjoint, the $f(A) = f(B) = 0$ implies that $f(A \cup B) = 0$.

Given a function f, the *terminals* are the nodes v of G such that $f(\{v\}) = 1$. As before, every solution subgraph must contain all the terminals. Hence we can assume without loss of generality that the terminals have zero cost.

Goemans and Williamson gave a 2-approximation algorithm for edge-weighted cut-cover problems where the cut family corresponds to a proper function f. In this paper we give a complementary result for node-weighted problems.

THEOREM 5.1. *Let G be a graph with node and edge weights. Let f be a proper function on the node subsets of G. There is a polynomial-time approximation algorithm for finding a minimum-cost subgraph covering all cuts in the family defined by f. The performance guarantee is* $2 \ln k$, *where k is the number of terminals.*

Proof of Theorem 5.1. The algorithm in the theorem follows the outline of the algorithm in Section 2 very closely. As before, the algorithm maintains a set of node-disjoint trees. Initially each terminal is in a tree by itself; the algorithm merges them iteratively. An important difference is that only some of the trees are candidates for merging.

We designate a tree to be *active* if $f(\{\text{nodes in the tree}\}) = 1$. At each iteration, the algorithm proceeds as before but considers only active trees in computing the quotient costs of nodes. That is, the algorithm selects a node and a set of at least two active trees so as to minimize

$$\frac{\text{cost of the node plus sum of distances to the active trees}}{\text{number of active trees}}. \qquad (6)$$

Then the algorithm uses the paths from the node to the selected trees and merges them into a single tree.

As long as there is at least one active tree in the network, it follows by the symmetry and disjointness properties of f that there are at least two.

The algorithm reduces the number of active trees in the network by at least one in each iteration; thus it eventually terminates and outputs a set of inactive trees as the final solution. Thus each connected component of the solution is inactive. It follows [6] that the solution is in fact a cut-cover.

The proof of the performance guarantee proceeds as before, except that we define the value of the potential function ϕ_i to be the number of active trees in the current solution, rather than the number of trees. To prove the analogue of Lemma 4.1, we consider an optimal cut cover and contract the current trees. This may result in a subgraph with possibly many inactive connected components. We then apply Theorem 3.1 to each of these connected components to obtain a spider decomposition of the contracted active trees. We use the result in the inequality

$$\phi_i \leq \phi_{i-1} - (h_i - 1), \tag{7}$$

where h_i is the number of active trees merged in the ith iteration. Note that with our new definition of the potential function ϕ_i we have inequality in (7) instead of equality as in (2) because the tree resulting from the merge may be inactive. The rest of the analysis is identical. This completes the proof of Theorem 5.1. ■

6. Conclusions and Open Problems

We have described approximation techniques for a variety of one-connected network-design problems. These are the first approximation algorithms that can handle costs on the nodes. Recently our techniques have been generalized [16] to handle node-weighted network-design problems with an additional constraint on the maximum degree of any node in the tree.

Two other natural generalizations of the Steiner tree problem are at least as hard to approximate as the set-cover problem, the directed Steiner problem, and the group Steiner problem. In the directed Steiner problem, we are given a directed arc-weighted graph, a distinguished node, and a set of terminals. The goal is to find a minimum-cost arborescence rooted at the distinguished node and spanning the terminals. A transformation of the node Steiner problem to the directed version was proposed by Segev [20].

The group Steiner problem, proposed by Reich and Widmayer [19], arises in VLSI design. In this problem, we are given an undirected edge-weighted graph and a collection of node subsets, called *groups*. The goal is to find a minimum-cost connected subgraph containing at least one node from each group.

We now show how to reduce the set-cover problem to a group Steiner problem: construct a graph with an auxiliary node and one node for each subset in the set-cover problem. Each subset node has an edge to the auxiliary node of cost equal to the cost of the subset. In formulating the group Steiner problem on this graph, define a group for each ground set element in the set-cover problem: namely, the set of nodes corresponding to the subsets that contain the ground element. It is easy to check that a group Steiner tree in this graph corresponds to a set cover of the same cost and vice versa.

Thus both the group Steiner problem and the directed Steiner problem are candidates for logarithmic-factor approximation algorithms.

Indeed, we also observe that an approximation algorithm for the latter would yield one for the former. An instance of the group Steiner problem can be transformed to an instance of the directed Steiner problem as follows: replace each edge in the input graph with a pair of antiparallel arcs of the same cost. For each group $g = \{u_1, \ldots, u_t\}$ of vertices, introduce a new vertex v_g and add zero-cost arcs from each $u_i \in g$ to v_g. It is easy to verify that an outward-directed Steiner tree originating from a node in any group in this transformed graph can be converted to a group Steiner tree of the same edge cost in the original graph and vice versa.

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