A New Bound for the 2-Edge Connected Subgraph Problem

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Abstract. Given a complete undirected graph with non-negative costs on the edges, the *2-Edge Connected Subgraph Problem* consists in finding the minimum cost spanning 2-edge connected subgraph (where multi-edges are allowed in the solution). A lower bound for the minimum cost 2-edge connected subgraph is obtained by solving the *linear programming relaxation* for this problem, which coincides with the *subtour relaxation* of the traveling salesman problem when the costs satisfy the triangle inequality.

The simplest fractional solutions to the subtour relaxation are the $\frac{1}{2}$-*integral solutions* in which every edge variable has a value which is a multiple of $\frac{1}{2}$. We show that the minimum cost of a 2-edge connected subgraph is at most four-thirds the cost of the minimum cost $\frac{1}{2}$-integral solution of the subtour relaxation. This supports the long-standing $\frac{4}{3}$ *Conjecture* for the TSP, which states that there is a *Hamilton cycle* which is within $\frac{4}{3}$ times the cost of the optimal subtour relaxation solution when the costs satisfy the triangle inequality.

1 Introduction

The 2-Edge Connected Subgraph Problem is a fundamental problem in Survivable Network Design. This problem arises in the design of communication networks that are resilient to single-link failures and is an important special case in the design of survivable networks [11, 12, 14].

1.1 Formulation

An integer programming formulation for the 2-Edge Connected Subgraph Problem is as follows. Let $K_n = (V, E)$ be the complete graph of feasible links on which the 2-Edge Connected Subgraph Problem is formulated. We denote an

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edge of this graph whose endpoints are $i \in V$ and $j \in V$ by ij. For each vertex $v \in V$, let $\delta(v) \subset E$ denote the set of edges incident to v. For each subset of vertices $S \subset V$, let $\delta(S) \subset E$ denote the set of edges in the *cut* which has S as one of the *shores*, i.e. the set of edges having exactly one endpoint in S. Denote the edge variable for $e \in E$ by x_e, which is 0,1, or 2 depending on whether e is absent, occurs singly or doubly in the 2-edge connected subgraph. For $A \subset E$, let $x(A)$ denote the sum $\sum_{e \in A} x_e$. Let c_e denote the cost of edge e. We have the following integer programming formulation.

$$\begin{array}{lll} \text{minimize} & c \cdot x & \\ \text{subject to} & & \\ & x(\delta(v)) \geq 2 & \text{for all } v \in V, \\ & x(\delta(S)) \geq 2 & \text{for all } S \subset V, \\ & x_e \geq 0 & \text{for all } e \in E, \\ & x_e \ \text{integral.} & \end{array} \qquad (1)$$

The *LP relaxation* is obtained by dropping the integrality constraint in this formulation. This LP relaxation is almost the same as the *subtour relaxation* for the Traveling Salesman Problem (TSP). The Traveling Salesman Problem consists in finding the minimum cost *Hamilton cycle* in a graph (a Hamilton cycle is a cycle which goes through all the vertices). The subtour relaxation for the TSP is as follows.

$$\begin{array}{lll} \text{minimize} & c \cdot x & \\ \text{subject to} & & \\ & x(\delta(v)) = 2 & \text{for all } v \in V, \\ & x(\delta(S)) \geq 2 & \text{for all } S \subset V, \\ & x_e \geq 0 & \text{for all } e \in E. \end{array} \qquad (2)$$

The constraints of the subtour relaxation are called the *degree constraints*, the *subtour elimination constraints*, and the *non-negativity constraints* respectively.

If one has the relationship $c_{ij} \leq c_{ik} + c_{jk}$ for all distinct $i, j, k \in V$, then c is said to satisfy the *triangle inequality.* An interesting known result is that if the costs satisfy the triangle inequality, then there is an optimal solution to (1) which is also feasible and hence optimal for (2). This follows from a result of Cunningham [11] (A more general result called the *Parsimonious Property* is shown by Goemans and Bertsimas in [7]). We can show that this equivalence holds even when the costs do not satisfy the triangle inequality. In the latter case, we replace the given graph by its *metric completion*, namely, for every edge ij such that c_{ij} is greater than the cost of the shortest path between i and j in the given graph, we reset the cost to that of this shortest path. The intent is that if this edge is chosen in the solution of (1), we may replace it by the shortest cost path connecting i and j. Since multiedges are allowed in the 2-edge connected graph this transformation is valid. Hence without loss of generality, we can assume that the costs satisfy the triangle inequality.

1.2 Our result and its significance

Our main result is the following.

Theorem 1. *The minimum cost of a 2-edge connected subgraph is within $\frac{4}{3}$ times the cost of the optimal half-integral subtour solution for the TSP.*

This result is a first step towards proving the following conjecture we offer.

Conjecture 1. The minimum cost of a 2-edge connected subgraph is within $\frac{4}{3}$ times the cost of the optimal subtour solution for the TSP.

By our remarks in the end of Section 1.1, it would follow from Conjecture 1 that the minimum cost of a 2-edge connected subgraph is also within $\frac{4}{3}$ times the cost of an optimal solution to the linear programming relaxation (1).

We formulated Conjecture 1 as an intermediate step in proving the following stronger "four-thirds conjecture" on the subtour relaxation for the TSP, which would directly imply Conjecture 1.

Conjecture 2. If the costs satisfy the triangle inequality, then the minimum cost of a Hamilton cycle is within $\frac{4}{3}$ times the cost of the optimal subtour solution for the TSP.

Note that Theorem 1 and Conjecture 1 imply similar relations between the fractional optimum of the subtour relaxation and a minimum-cost 2-vertex connected subgraph when the costs obey the triangle inequality. In particular, Theorem 1 implies that when the costs satisfy the triangle inequality, the minimum cost 2-vertex connected spanning subgraph is within $\frac{4}{3}$ times the cost of the optimal half-integral subtour solution for the TSP. This follows from the simple observation that from the minimum-cost 2-edge connected graph, we can short-cut "over" any cut vertices without increasing the cost by using the triangle inequality [5, 11].

1.3 Related work

A heuristic for finding a low cost Hamilton cycle was developed by Christofides in 1976 [4]. An analysis of this heuristic shows that the ratio is no worse than $\frac{3}{2}$ in both Conjecture 1 and Conjecture 2. This analysis was done by Wolsey in [16] and by Shmoys and Williamson in [15]. A modification of the Christofides heuristic to find a low cost 2-vertex connected subgraph when the costs obey the triangle inequality was done by Fredrickson and Ja Ja in [5]. The performance guarantee for this heuristic to find a 2-vertex connected subgraph is $\frac{3}{2}$. There has also been a spate of work on approximation algorithms for survivable network design problems generalizing the 2-edge connected subgraph problem [7–10, 13, 17]; however, the performance guarantee for the 2-edge connected subgraph problem from these methods is at best $\frac{3}{2}$ when the costs obey the triangle inequality (shown in [5, 7]) and at best 2 when they do not (shown in [9]).

Both Conjecture 2 and Conjecture 1 have remained open since Christofides developed his heuristic. In this paper, we suggest a line of attack for proving Conjecture 1.

2 Motivation

In this section we discuss two distinct motivations that led us to focus on half-integral extreme points and prove a version of Conjecture 1 for this special case. One follows from a particular strategy to prove Conjecture 1 and the other from examining subclasses of subtour extreme points that are sufficient to prove Conjectures 1 and 2.

2.1 A Strategy for Proving Conjecture 1

Let an arbitrary point x^* of the subtour polytope for K_n be given. Multiply this by $\frac{4}{3}$ to obtain the vector $\frac{4}{3}x^*$. Denote the edge incidence vector for a given 2-edge connected subgraph H in K_n by χ^H. Note that edge variables could be 0,1, or 2 in this incidence vector. Suppose we could express $\frac{4}{3}x^*$ as a convex combination of incidence vectors of 2-edge connected subgraphs H_i for $i = 1, 2, \ldots, k$. That is, suppose that

$$\frac{4}{3}x^* = \sum_{i=1}^{k} \lambda_i \chi^{H_i}, \tag{3}$$

where $\lambda_i \geq 0$ for $i = 1, 2, \ldots, k$ and

$$\sum_{i=1}^{k} \lambda_i = 1.$$

Then, taking dot products on both sides of (3) with the cost vector c yields

$$\frac{4}{3}c \cdot x^* = \sum_{i=1}^{k} \lambda_i c \cdot \chi^{H_i}. \tag{4}$$

Since the right hand side of (4) is a weighted average of the numbers $c \cdot \chi^{H_i}$, it follows that there exists a $j \in \{1, 2, \ldots, k\}$ such that

$$c \cdot \chi^{H_j} \leq \frac{4}{3}c \cdot x^*. \tag{5}$$

If we could establish (5) for any subtour point x^*, then it would in particular be valid for the optimal subtour point, which would prove Conjecture 1.

In an attempt at proving Conjecture 1, we aim at contradicting the idea of a minimal counterexample, that is, a subtour point x^* having the fewest number of vertices n' such that (3) can not hold for any set of 2-edge connected subgraphs. First we have the following observation.

Theorem 2. *At least one of the minimal counterexamples x^* to (3) holding (for some set of 2-edge connected subgraphs) is an extreme point of the subtour polytope.*

Proof. Suppose $x^* = \sum_l \mu_l x^l$, where each x^l is an extreme point which is not a minimal counterexample, and the μ_l's satisfy the usual constraints for a set of convex multipliers. Thus, for each l, we can find a set of 2-edge connected subgraphs H_i^l such that

$$\frac{4}{3}x^l = \sum_i \lambda_i^l \chi^{H_i^l},$$

where the λ_i^l's satisfy the usual constraints for a set of convex multipliers. Then

$$\frac{4}{3}x^* = \sum_l \frac{4}{3}\mu_l x^l = \sum_l \mu_l \sum_i \lambda_i^l \chi^{H_i^l}. \tag{6}$$

Since we have that

$$\sum_l \mu_l \cdot (\sum_i \lambda_i^l) = \sum_l \mu_l \cdot (1) = 1,$$

Equation (6) shows that $\frac{4}{3}x^*$ can be expressed as a convex combination of 2-edge connected subgraphs as well, from which this theorem follows.

Thus we need to focus only on minimal counterexamples x^* in $K_{n'}$ which are extreme points. To carry out the proof, we wish to find a *substantial tight cut* $\delta(H)$ for x^*, i.e. an $H \subset V$ such that $3 \leq |H| \leq n' - 3$ and

$$x^*(\delta(H)) = 2.$$

We can then split x^* into 2 smaller subtour solutions x^1 and x^2 in the following way. Take the vertices of $V \setminus H$ in x^* and contract them to a single vertex to obtain x^1. Likewise, take the vertices of H in x^* and contract them to a single vertex to obtain x^2. An example of this is shown in Figure 1.

Since x^1 and x^2 are not counterexamples to our conjecture, we would be able to decompose $\frac{4}{3}x^1$ and $\frac{4}{3}x^2$ into combinations of 2-edge connected subgraphs, which we may then attempt to glue together to form a similar combination for $\frac{4}{3}x^*$, thereby showing that x^* is not a counterexample (We show how this can be accomplished for the case of half-integral extreme points in Case 1 in the Proof of Theorem 6).

What if there are no tight substantial cuts however? The following proposition which is shown in [1] shows us what we need to do.

Proposition 1. *If x^* is an extreme point of the subtour polytope and has no substantial tight cuts, then x^* is a 1/2-integer solution.*

This led us to focus on 1/2-integral solutions x^*, and we were able to complete the proof for this special case. In the next section, we show our main result that if x^* is a 1/2-integer subtour solution, then (3) can always be satisfied.

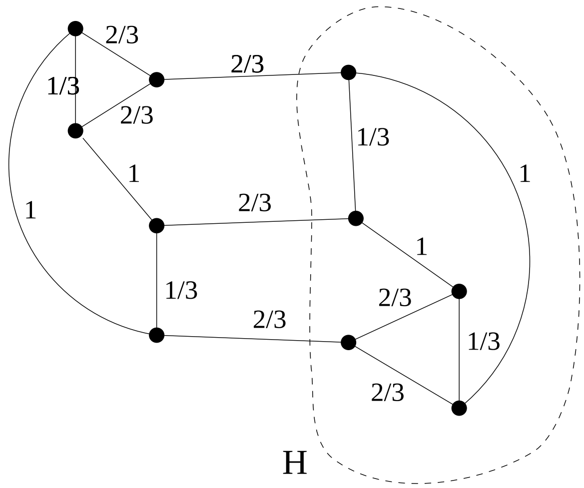

Fig. 1. An idea for splitting a minimal counterexample into two smaller instances. Note that H defines a substantial tight cut, i.e., both H and $V \setminus H$ have at least three vertices and $x(\delta(H)) = 2$.

2.2 The Important Extreme Points

Consider any extreme point x^*. We wish to express $\frac{4}{3}x^*$ as a convex combination of 2-edge connected graphs for Conjecture 1 or a convex combination of Eulerian graphs for Conjecture 2. An important question is what features of x^* make it difficult to do this? In an effort to answer this question, we try to transform x^* into another extreme point $\overline{x}^*$ on a larger graph so that $\overline{x}^*$ belongs to a subclass of the extreme points, but $\frac{4}{3}\overline{x}^*$ is at least as hard to express as a convex combination of 2-edge connected graphs (or Eulerian graphs) as $\frac{4}{3}x^*$ is. The idea then is that we only have to be able to express $\frac{4}{3}\overline{x}$ as a convex combination of 2-edge connected graphs (or Eulerian graphs) for all extreme points $\overline{x}$ belonging to this particular subclass in order to prove Conjecture 1 (or Conjecture 2). If we have a subclass S of extreme points $\overline{x}$ such that being able to express $\frac{4}{3}\overline{x}$ as a convex combination of 2-edge connected graphs for all extreme points $\overline{x}$ belonging to this particular subclass is sufficient to prove Conjecture 1, then we say that S is *sufficient* to prove Conjecture 1. Likewise, a subclass S can be sufficient to prove Conjecture 2.

We have found two different subclasses of extreme points which are sufficient to prove both Conjecture 1 and Conjecture 2. In some sense, the extreme points in such a subclass are the hardest extreme points to deal with when proving Conjecture 1 or Conjecture 2. One class, termed *fundamental extreme points*, can be found in [2].

Definition 1. *A fundamental extreme point is an extreme point for the subtour relaxation satisfying the following conditions.*

(i) The support graph is 3-regular,

(ii) There is a 1-edge incident to each vertex,
(iii) The fractional edges form disjoint cycles of length 4.

A second class of such sufficient extreme points is described below. We will restrict our attention to showing that the subclass described below is sufficient to prove Conjecture 1, although showing that it is also sufficient to prove Conjecture 2 requires only minor modifications in our arguments.

Consider any extreme point x^*. Pick the smallest integer k such that x^*_e is a multiple of $\frac{1}{k}$ for every edge $e \in E$. Then form a $2k$-regular $2k$-edge connected multigraph $G_k = (V, E_k)$ as follows. For every edge $e = uv \in E$, put l edges between u and v, where $l := kx^*_e$. Then showing that $\frac{4}{3}E_k$ can be expressed as a convex combination of 2-edge connected graphs is equivalent to showing that $\frac{4}{3}x^*$ can be so expressed. But suppose every vertex in G_k is replaced by a circle of $2k$ nodes, each node with one edge from E_k, and $2k-1$ new edges linking this node to its two neighboring nodes in the circle, all in such a way that the resulting graph $\overline{G}_k = (\overline{V}, \overline{E}_k)$ is still $2k$-regular and $2k$-edge connected. Note that loosely speaking, we have $E_k \subset \overline{E}_k$. We seek to then show that if we can express $\frac{4}{3}\overline{E}_k$ as a convex combination of 2-edge connected graphs, then we can do so for $\frac{4}{3}E_k$ as well. The graph $\overline{G}_k$ will turn out to corresponds to a subtour extreme point $\overline{x}^*$ (in the same way that G_k corresponds to x^*). It is more convenient to define this subtour extreme point $\overline{x}^*$ than to define $\overline{G}_k$.

Let us now define $\overline{x}^*$.

Definition 2. *Expand each vertex in V into a circle of $2k$ nodes, with an edge of E_k leaving each such node, as described in the previous paragraph. Take the equivalent of an Eulerian tour through all the edges of E_k by alternately traversing these edges and jumping from one node to another node in the same circle until you have traversed all of the edges in E_k and have come back to the edge in E_k you started with. When you jump from node u to node v in the same circle in this Eulerian tour, define $\overline{x}^*_{uv} := \frac{k-1}{k}$. For every edge $e \in E_k$, we naturally define $\overline{x}^*_e := \frac{1}{k}$. For each circle C_v of nodes corresponding to the vertex $v \in V$, we pick an arbitrary perfect matching M_v on the nodes in C_v, including in M_v only edges e which have not yet been used in the definition of $\overline{x}^*$. We then define $\overline{x}^*_e := 1$ for all $e \in M_v$.*

We have the following:

Theorem 3. *$\overline{x}^*$ in Definition 2 is a subtour extreme point.*

Proof. The support graph of $\overline{x}^*$ is 3-regular, with the fractional edges in $\overline{x}^*$ forming a Hamilton cycle on the vertices $\overline{V}$. Call the edges in $\overline{x}^*$'s support graph $\overline{E}_k$.

We first show that $\overline{x}^*$ is a feasible subtour point. If it were not, there would have to be a cut in the graph $\overline{G}_k = (\overline{V}, \overline{E}_k)$ of value less than 2. Clearly, such a cut $\mathcal{C}$ would have to go through some circle C_v of nodes since G_k is $2k$-edge connected. But the contribution of the edges from the circle C_v to any cut crossing it is at least 1 since the edges in the circle C_v each have a value greater than or equal to $1/2$. Hence, the contribution from the non-circle edges in the cut $\mathcal{C}$ is less than 1. But this is not possible because when v is ripped out of x^*, the minimum cut in the remaining solution is greater than or equal to 1. Hence, $\overline{x}^*$ is a feasible subtour point.

We show that $\overline{x}^*$ is an extreme point by showing that it can not be expressed as $\frac{1}{2}x^1 + \frac{1}{2}x^2$, where x^1 and x^2 are distinct subtour points. Suppose $\overline{x}^*$ could be so expressed. Then the support graphs of x^1 and x^2 would coincide with or be subgraphs of the support graph $\overline{E}_k$ of $\overline{x}^*$. Because of the structure of the support graph, setting the value of just one fractional edge determines the entire solution due to the degree constraints. Hence, all the edges $e \in \overline{E}_k$ such that $x_e = \frac{1}{k}$ would have to say be smaller than $\frac{1}{k}$ in x^1 and larger than $\frac{1}{k}$ in x^2. But, then a cut separating any circle of nodes C_v from the rest of the vertices in x^1 would have a value less than 2, which contradicts x^1 being a subtour point.

We now have the following:

Theorem 4. *If $\frac{4}{3}\overline{x}^*$ can be expressed as a convex combination of 2-edge connected graphs spanning $\overline{V}$, then $\frac{4}{3}x^*$ can be expressed as a convex combination of 2-edge connected graphs spanning V.*

Proof. Suppose $\frac{4}{3}\overline{x}^*$ can be expressed as a convex combination

$$\frac{4}{3}\overline{x}^* = \sum_i \lambda_i \chi^{\overline{H}_i}, \tag{7}$$

where the $\overline{H}_i$'s are 2-edge connected graphs spanning $\overline{V}$. For each i, contract each circle of nodes C_v back to the vertex $v \in V$ in $\overline{H}_i$. Call the resulting graph H_i. Since contraction preserves edge connectivity, H_i is a 2-edge connected graph spanning V. When one performs this contraction on $\overline{x}^*$, one gets x^*. As a result, we obtain that

$$\frac{4}{3}x^* = \sum_i \lambda_i \chi^{H_i}, \tag{8}$$

which proves our theorem.

We can now define the subclass of *important extreme points.*

Definition 3. *An important extreme point is an extreme point for the subtour relaxation satisfying the following conditions.*

(i) The support graph is 3-regular,
(ii) There is a 1-edge incident to each vertex,
(iii) The fractional edges form a Hamilton cycle.

We are now ready for the culminating theorem of this section.

Theorem 5. *The subclass of important extreme points is sufficient to prove Conjecture 1.*

Proof. If there is an extreme point x^* such that $\frac{4}{3}x^*$ cannot be expressed as a convex combination of 2-edge connected graphs, then by Theorem 4, the important extreme point $\frac{4}{3}\overline{x}^*$ cannot be expressed as a convex combination of 2-edge connected graphs either. Hence, our theorem follows.

The analogous theorem for the class of fundamental extreme points can be found in [2].

3 The Proof of Theorem 1

Let x^* be a 1/2-integer subtour solution on $K_n = (V, E)$. Denote the edges of the *support graph* of x^* (the set of edges $e \in E$ such that $x_e^* > 0$) by $\hat{E}(x^*)$. Construct the multigraph $G(x^*) = (V, E(x^*))$, where $E(x^*) \supset \hat{E}(x^*)$ and differs from $\hat{E}(x^*)$ only in that there are two copies in $E(x^*)$ of every edge $e \in \hat{E}(x^*)$ for which $x_e^* = 1$. Note that the parsimonious property [7] implies that there are no edges e with $x_e > 1$ in the optimal fractional solution.

Because of the constraints of the subtour relaxation, it follows that $G(x^*)$ is a 4-regular 4-edge connected multigraph. Similarly, corresponding to every 4-regular 4-edge connected multigraph is a 1/2-integer subtour solution, although this solution may not be an extreme point.

Showing (3) for some choice of 2-edge connected subgraphs H_i for every 1/2-integer subtour solution x^* would prove Conjecture 1 whenever the optimal subtour solution was 1/2-integer, as was discussed in the last section. So, equivalently to showing (3) for some choice of 2-edge connected subgraphs H_i for every 1/2-integer subtour solution x^*, we could show

$$\frac{2}{3}\chi^{E(G)} = \sum_i \lambda_i \chi^{H_i}, \tag{9}$$

where this expression is a convex combination of some chosen set of 2-edge connected subgraphs H_i, for every 4-regular 4-edge connected multigraph $G = (V, E(G))$. These are equivalent because of the remarks in the previous paragraph and the observation that $G(x^*)$ behaves like $2x^*$.

It turns out that (9) is very difficult to show directly, but the following slight strengthening of it makes the task easier. Consider any 4-regular 4-edge connected multigraph $G = (V, E(G))$ and any edge $e \in E(G)$. Then, we prove instead that

$$\frac{2}{3}\chi^{E(G)\setminus\{e\}} = \sum_i \lambda_i \chi^{H_i} \tag{10}$$

where this expression is a convex combination of some chosen set of 2-edge connected subgraphs H_i.

For technical reasons, we will prove (10) with the additional restriction that none of the H_i's may use more than one copy of any edge in $E(G)$. Note however that G may itself have multiedges so H may also have multiedges. In the latter case, we think of two parallel multiedges in H as being copies of two distinct multiedges in G.

For any 4-regular 4-edge connected graph G and any edge $e \in E(G)$, we define $P(G, e)$ to be the following statement.

Statement 1 $P(G, e) \Leftrightarrow$ *For some finite set of 2-edge connected subgraphs H_i, we have (10), where $\lambda_i \geq 0$ for all i and $\sum_i \lambda_i = 1$, and none of the H_i's may use more than one copy of any edge in $E(G)$.*

As noted above, Statement 1 does not rule out the possibility of doubled edges in the H_i's because there may be doubled edges in G.

We define a *tight cut* for a 4-edge connected graph G to be a cut which has exactly 4-edges in it. We define a *non-trivial cut* for such a graph to be a cut where both shores have at least 2 vertices each. We have the following lemma.

Lemma 1. *Let $G = (V, E)$ be a 4-regular 4-edge connected graph which has no tight non-trivial cut which includes an edge $e = uv \in E$. Let the other 3 (not necessarily distinct) neighbors of v be x, y, and z. Then either ux or yz is a loop or $G' = G - v + ux + yz$ is 4-regular and 4-edge connected, and likewise for the other combinations.*

Proof. Let $G = (V, E)$ and $e = uv \in E$ be given, where the neighbors of v are as stated. First, note that any cut in G containing all four edges incident on v has size at least 8, since the cut formed by moving v to the opposite side of the cut must have size at least 4 since G is 4-edge connected.

Suppose neither ux or yz is a loop. Then clearly, G' is a 4-regular connected graph. Since it is 4-regular, every cut has an even number of edges in it. By our earlier observation, there can be no cuts $\delta(H)$ in G' of cardinality zero. Suppose G' has a non-trivial cut $\delta(H)$ with only 2 edges in it. Consider $\hat{G} = G + ux + yz$ with vertex v back in. The two non-trivial cuts $\delta(H \cup \{v\})$ and $\delta((V \setminus H) \cup \{v\})$ can each have at most 3 more edges each (for a total of 5 edges each) since as observed earlier, these cuts could not have all 4 edges incident to v in them. But, $G = \hat{G} - ux - yz$ has only cuts with an even number of edges in them since it is 4-regular. Hence the cuts $\delta(H \cup \{v\})$ and $\delta((V \setminus H) \cup \{v\})$ in G have at most 4 edges in them. One of these two cuts is a tight non-trivial cut which contains e, which yields the lemma.

We are now ready for our main theorem.

Theorem 6. *Let x^* be a minimum cost 1/2-integer subtour solution. Then there exists a 2-edge connected subgraph H such that $c \cdot \chi^H \leq \frac{4}{3} c \cdot x^*$.*

Proof. As remarked in the discussion before this theorem, it is sufficient to prove $P(G, e)$ for all 4-regular 4-edge connected multigraphs G and for all $e \in E(G)$. To prove this, we show that a minimal counterexample to $P(G, e)$ can not happen.

Let $G = (V, E(G))$ be a 4-regular 4-edge connected multigraph and $e \in E(G)$ which has the minimum number of vertices such that $P(G, e)$ does not hold. Since by inspection, we can verify that $P(G, e)$ holds when G has 3 vertices, we can assume that $|V| > 3$. We now consider the cases where G has a tight non-trivial cut which includes edge e and where G has no tight non-trivial cut which includes e.

Case 1: G has a tight non-trivial cut which includes edge e.

Choose such a tight non-trivial cut and denote the edges other than e in this cut by a, b, and c. As before, consider contracting one of the shores of this cut to a single vertex v_1. Denote the edges incident to v_1, which corresponded to e, a, b, and c, by e_1, a_1, b_1, and c_1 respectively. This resulting graph $G_1 = (V_1, E_1)$ can be seen to be 4-regular and 4-edge connected. (To see this, suppose there was a cut of cardinality less than four in G_1 and let H_1 be the shore of this cut not containing v_1. Then the cut $\delta(H_1)$ in G shows that G is not 4-edge-connected, a contradiction.) Since (G, e) was a minimal counterexample to $P(G, e)$, we have $P(G_1, e_1)$. By contracting the other shore, we can get a 4-regular 4-edge connected graph G_2, and we know that $P(G_2, e_2)$ also holds.

By $P(G_1, e_1)$ we have

$$\frac{2}{3}\chi^{E(G_1)\setminus\{e_1\}} = \sum_i \lambda_i \chi^{H_i^1}, \tag{11}$$

and by $P(G_2, e_2)$ we have

$$\frac{2}{3}\chi^{E(G_2)\setminus\{e_2\}} = \sum_i \mu_i \chi^{H_i^2}. \tag{12}$$

In (11), consider the edges incident to v_1 in each of the H_i^1's. There are clearly at least 2 such edges for every H_i^1. The values of edges a_1, b_1, c_1, and e_1 in $\frac{2}{3}\chi^{E(G_1)\setminus\{e_1\}}$ are $\frac{2}{3}$, $\frac{2}{3}$, $\frac{2}{3}$, and 0 respectively. This adds up to 2. Hence, since we are dealing with convex combinations, which are weighted averages, when the weights are taken into account, the H_i^1's have on average 2 edges incident to v_1 each. But since every H_i^1 has at least 2 such edges, it follows that every H_i^1 has exactly 2 edges incident to v_1 in it.

For each 2-edge connected subgraph H_i^1 which has edges a_1 and b_1, denote the corresponding convex multiplier by λ_i^{ab}. Define λ_i^{ac} and λ_i^{bc} similarly. One can see that the only way for the variable values of edges a_1, b_1, and c_1 to end up all being $\frac{2}{3}$ in $\frac{2}{3}\chi^{E(G_1)\setminus\{e_1\}}$ is for the following to hold:

$$\sum_i \lambda_i^{ab} = \sum_i \lambda_i^{ac} = \sum_i \lambda_i^{bc} = \frac{1}{3}. \tag{13}$$

Similarly, we must have

$$\sum_i \mu_i^{ab} = \sum_i \mu_i^{ac} = \sum_i \mu_i^{bc} = \frac{1}{3}. \tag{14}$$

Call the three types of 2-edge connected graphs H_i^j as ab-graphs, ac-graphs, and bc-graphs. Our strategy is to combine say each ab-graph H_i^1 of G_1 with an ab-graph H_j^2 of G_2 to form an ab-graph H_{ij}^{ab} of G which is also 2-edge connected. So, we define

$$H_{ij}^{ab} := (H_i^1 - v_1) + (H_j^2 - v_2) + a + b, \tag{15}$$

where H_i^1 and H_j^2 are ab-graphs. Since $H_i^1 - v_1$ and $H_j^2 - v_2$ are both connected, it follows that H_{ij}^{ab} is 2-edge connected. Similarly define H_{ij}^{ac} and H_{ij}^{bc}.

Now consider the following expression:

$$\sum_{i,j} 3\lambda_i^{ab}\mu_j^{ab}H_{ij}^{ab} + \sum_{i,j} 3\lambda_i^{ac}\mu_j^{ac}H_{ij}^{ac} + \sum_{i,j} 3\lambda_i^{bc}\mu_j^{bc}H_{ij}^{bc}. \tag{16}$$

One can verify that this is in fact a convex combination. Any edge f in say $G_1 - v_1$ occurs in (16) with a weight of

$$\sum_{\{i \mid f \in H_i^1\}} (\lambda_i^{ab} \cdot (3 \cdot \sum_j \mu_j^{ab}) + \lambda_i^{ac} \cdot (3 \cdot \sum_j \mu_j^{ac}) + \lambda_i^{bc} \cdot (3 \cdot \sum_j \mu_j^{bc})). \tag{17}$$

In light of (14) we have that (17) evaluates to

$$\sum_{\{i \mid f \in H_i^1\}} \lambda_i = \frac{2}{3}. \tag{18}$$

We have a similar identity when f is in $G_2 - v_2$ and we also have that edges a, b, and c each occur in (16) with a weight of $\frac{2}{3}$ as well. Therefore we have

$$\sum_{i,j} 3\lambda_i^{ab}\mu_j^{ab}H_{ij}^{ab} + \sum_{i,j} 3\lambda_i^{ac}\mu_j^{ac}H_{ij}^{ac} + \sum_{i,j} 3\lambda_i^{bc}\mu_j^{bc}H_{ij}^{bc} = \frac{2}{3}\chi^{E(G)\setminus\{e\}}, \tag{19}$$

which contradicts (G, e) being a minimal counterexample.

Case 2: G has no tight non-trivial cut which includes edge e.

Denote the endpoints of e by $u \in V$ and $v \in V$, and denote the other 3 not necessarily distinct neighbors of v in G by $x, y, z \in V$. Because e is in no tight non-trivial cut, we have that $x \neq y \neq z$. (If any two of the neighbors x, y and z are the same, say $x = y$, then the cut $\delta(\{v, x\})$ will be a tight non-trivial cut). Thus, without loss of generality, if any two neighbors are the same vertex, we can assume that they are u and z. Hence, $u \neq x$ and $u \neq y$.

Define the graph $G_1 = (V_1, E_1)$ by

$$G_1 = G - v + ux + yz, \tag{20}$$

and define $e_1 = ux$. We know by Lemma 1 that G_1 is 4-regular and 4-edge connected. Since (G, e) is a minimal counterexample, we therefore know that $P(G_1, e_1)$ holds. Similarly, define the graph $G_2 = (V_2, E_2)$ by

$$G_2 = G - v + uy + xz, \tag{21}$$

and define $e_2 = uy$. As before, we know that $P(G_2, e_2)$ holds as well.

So, we can form the following convex combinations of 2-edge connected graphs:

$$\frac{2}{3}\chi^{E_1\setminus\{e_1\}} = \sum_i \lambda_i \chi^{H_i^1}, \tag{22}$$

and

$$\frac{2}{3}\chi^{E_2\setminus\{e_2\}} = \sum_i \mu_i \chi^{H_i^2}. \tag{23}$$

Define $\hat{H}_i^1$ by

$$\hat{H}_i^1 = \begin{cases} H_i^1 - yz + yv + zv & \text{for } yz \in H_i^1, \\ H_i^1 + yv + xv & \text{for } yz \notin H_i^1. \end{cases} \tag{24}$$

Likewise, define $\hat{H}_i^2$ by

$$\hat{H}_i^2 = \begin{cases} H_i^2 - xz + xv + zv & \text{for } xz \in H_i^2, \\ H_i^2 + yv + xv & \text{for } xz \notin H_i^2. \end{cases} \tag{25}$$

Consider the convex combination of 2-edge connected subgraphs

$$\frac{1}{2}\sum_i \lambda_i \chi^{\hat{H}_i^1} + \frac{1}{2}\sum_i \mu_i \chi^{\hat{H}_i^2}. \tag{26}$$

Every edge in $f \in E\setminus\delta(v)$ occurs with a total weight of $\frac{2}{3}$ in (26) since f occured with that weight in both (22) and (23). Since yz occurs with a total weight of $\frac{2}{3}$ in (22) and xz occurs with a total weight of $\frac{2}{3}$ in (23), one can verify that xv, yv, and zv each occur with a total weight of $\frac{2}{3}$ in (26) as well. Therefore, we have

$$\frac{2}{3}\chi^{E\setminus\{e\}} = \frac{1}{2}\sum_i \lambda_i \chi^{\hat{H}_i^1} + \frac{1}{2}\sum_i \mu_i \chi^{\hat{H}_i^2}, \tag{27}$$

which contradicts G, e being a minimal counterexample.

4 Concluding Remarks

An obvious open problem arising from our work is to extend our strategy and settle Conjecture 1. In another direction, it would be interesting to apply our ideas to design a $\frac{4}{3}$-approximation algorithm for the minimum cost 2-edge- and 2-vertex-connected subgraph problems.

Another interesting question is the tightness of the bound proven in Theorem 1. The examples we have been able to construct seem to demonstrate an asymptotic ratio of $\frac{6}{5}$ between the cost of a minimum cost 2-edge connected subgraph and that of an optimal half-integral subtour solution. Finding instances with a worse ratio or improving our bound in Theorem 1 are open problems.

References

1. M. Balinski, On recent developments in integer programming, in: H.W. Kuhn, ed., *Proceedings of the Princeton Symposium on Mathematical Programming* (Princeton University Press, N.J., 1970, 267-302).
2. S. Boyd and R. Carr, "Finding low cost TSP and 2-matching solutions using certain half-integer subtour vertices," Manuscript, March 1998.

3. S. Boyd and R. Carr, "A new bound for the 2-matching problem," Report TR-96-07, Department of Computer Science, University of Ottawa, Ottawa (1996).
4. N. Christofides, Worst case analysis of a new heuristic for the traveling salesman problem, Report 388, Graduate School of Industrial Administration, Carnegie Mellon University, Pittsburgh (1976).
5. G. N. Fredrickson and J. Ja Ja, "On the relationship between the biconnectivity augmentation and traveling salesman problems," *Theoretical Computer Science*, 19, 189-201 (1982).
6. M. X. Goemans, "Worst-case comparison of valid inequalities for the TSP," *Math. Programming* 69, 335-349 (1995).
7. M. X. Goemans and D. J. Bertsimas, "Survivable networks, linear programming relaxations and the parsimonious property," *Math. Programming*, 60, 145-166 (1993).
8. M. X. Goemans, A. Goldberg, S. Plotkin, D. Shmoys, E. Tardos, D. P. Willamson, "Approximation algorithms for network design problems," Proceedings of the *Fifth Annual ACM-SIAM Symposium on Discrete Algorithms (SODA '94)* (1994), 223-232.
9. S. Khuller and U. Vishkin, "Biconnectivity approximations and graph carvings," *J. Assoc. Comput. Mach.*, 41 (2), 214-235 (1994).
10. P. Klein and R. Ravi, "When cycles collapse: A general approximation technique for constrained two-connectivity problems," Proceedings of the *Conference on Integer Programming and Combinatorial Optimization (IPCO '93)* (1993), 39-56.
11. C. L. Monma, B. S. Munson and W. R. Pulleyblank, "Minimum-weight two-connected spanning networks," *Math. Programming*, 46, 153-171 (1990).
12. C. L. Monma and D. F. Shallcross, " Methods for designing communication networks with certain two-connectivity survivability constraints," *Oper. Res.* 37, 531-541 (1989).
13. H. Saran, V. Vazirani, and N Young, "A primal-dual approach to approximation algorithms for network Steiner problems," *Proc. of the Indo-US workshop on Cooperative research in Computer Science* Bangalore, India, 166-168 (1992).
14. K. Steiglitz, P. Weiner and D. J. Kleitman, "The design of minimum-cost survivable networks," *IEEE Trans. on Circuit Theory* CT-16, 4, 455-460 (1969).
15. D.B. Shmoys and D.P. Williamson, Analyzing the Held-Karp TSP bound: A monotonicity property with application, *Inf. Process. Lett. 35* (1990) 281-285.
16. L.A. Wolsey, Heuristic analysis, linear programming and branch and bound, *Math. Program. Study 13* (1980) 121-134.
17. D. P. Williamson, M. X. Goemans, M. Mihail and V. Vazirani, "A primal-dual approximation algorithm for generalized Steiner network problems." *Combinatorica* 15, 435-454 (1995).