A General Theory of Comparative Music Analysis

by

Richard R. Randall

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Supervised by
Professor Robert D. Morris
Department of Music Theory
Eastman School of Music

University of Rochester
Rochester, New York

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Curriculum Vitæ

Richard Randall was born in Washington, DC, in 1969. He received his undergraduate musical training at the New England Conservatory in Boston, MA. He was awarded a Bachelor of Music in Theoretical Studies with a Distinction in Performance in 1995 and, in 1997, received a Master of Arts degree in music theory from Queens College of the City University of New York. Mr. Randall studied classical guitar with Robert Paul Sullivan, jazz guitar with Frank Rumoro, Walter Johns, and Rick Whitehead, and solfege and conducting with F. John Adams. In 1997, Mr. Randall entered the Ph.D. program in music theory at the University of Rochester’s Eastman School of Music. His research at the Eastman School was advised by Robert Morris. He received teaching assistantships and graduate scholarships in 1998, 1999, 2000. He was adjunct instructor in music at Northeastern Illinois University from 2002 to 2003. Additionally, Mr. Randall taught music theory and guitar at the Old Town School of Folk Music in Chicago as a private guitar instructor from 2001 to 2003. In 2003, Mr. Randall was appointed Visiting Assistant Professor of Music in the Department of Music and Dance at the University of Massachusetts, Amherst.
Acknowledgments

TBA
Abstract

This dissertation establishes a kind of comparative music analysis such that what we understand as musical features in a particular case are dependent on the music analytic systems that provide the framework by which we can contemplate (and communicate our thoughts about) music. We value how different analytic systems allow us to create different perceptions of a piece of music and argue that the identity of a musical work is in fact dependent on the combination of a work and an analytic system. Comparing musical works, then, necessarily compares such combinations. Therefore we assert that comparing pieces ought to be reframed as comparing the interpretation of pieces under specific analytic systems.

Comparative analyses are carried out on local and global levels. Local music analytic systems map one set of musical events to another a set of musical events—the entirety of which comprises a piece of music. This is our description of what is normally thought of as analysis. Global music analytic systems analyze (or comment on) pieces of music determined by different types of local analyses.

The theory of comparative music analysis is divided into two parts. The
first part defines a geometry of global music analytic systems, by far the most non-traditional aspect of the research. The geometric model covers all global systems. The second part defines a comparison of two local music analytic systems. Specifically, these local systems are tonal models, one defined by Lerdahl and the other based on experimental data of Bharucha and Krumhansl. Comparing local systems requires the establishment of a contextual equivalence between the two systems. In the case presented below, the equivalence is the formal structure of the metric space. These two approaches (global and local) are not alternatives to each other. They are in fact comparisons of musical pieces on different, yet interrelated levels. Comparing local music analytic systems is complicated by arbitrary design choices inherent in each system. Like comparisons of local systems themselves, solutions to this “design choice problem” are contextual. One such solution is presented as is a discussion of a meta-analytic ramification of comparing local systems.
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Chapter 1

Preliminaries

This dissertation presents a general theory of comparative music analysis. We analyze pieces of music, we compare pieces of music, and we often do this to group or distinguish them.\(^1\) There is no doubt that in some quarters of the community of musical scholars there is a high degree of intersubjective agreement about what it means to “compare music analytically.” In fact, the second chapter reviews numerous projects that appear to put forth a rigorous treatment of the subject.

My goal is to establish a kind of comparative music analysis such that what we understand as musical features in a particular case are dependent on the music analytic systems that provide the framework by which we can contemplate (and communicate our thoughts about) music. We value how different analytic systems allow us to create different perceptions of a piece of music and argue that the identity of a musical work is in fact dependent

\(^{1}\)There are a variety of reasons for such groupings including historical, stylistic, and ethnographical—to name a few.
on the combination of a work and an analytic system. Comparing musical
works, then, necessarily compares such combinations. I assert, therefore,
that comparing pieces ought to be reframed as comparing the interpretation
of pieces under specific analytic systems.

Comparative analysis is carried out on local and global levels. Local
analytic systems map one set of musical events to another set of musical
events—the entirety of which comprises a piece of music. This is my de-
scription of what is ordinarily thought of as analysis. Global music analytic
systems analyze (or comment on) pieces of music determined by different
types of local analyses. In general, local music analytic systems imply a
“musicality” by describing the qualities a piece of music should have, or con-
ditions the piece should satisfy, in order to be considered coherent within
a local system. The more a piece has such qualities, the more “typically
musical” it is in the context of the analytic system. As musical thinkers,
we assign different weights to different qualities. This allows different pieces
to have different degrees of “typical musicality” under the same analytic
system or the same piece to have different degrees of typical musicality un-
der different analytic systems. Global music analytic systems model this
typical musicality. Every local music analytic system implies a concomitant
global music analytic system. The product of a global comparative analysis
is a geometric model of the degree of typical musicality of pieces defined
by different local systems. This chapter defines core terms and gives some
philosophical background.
1.1 Pieces of Music

I begin by clarifying what I mean by a piece of music. The purpose of this discussion is to frame the importance of including analysis in the comparative equation and to motivate the methodology—not to answer definitively the question of what is piece of music.

Trietler gives a brief, but useful, treatment of musical ontology. The musical work, he quotes Karl Popper, “is neither the score . . . the sum of total imagined musical experiences . . . Nor is it any of all performances . . . nor the class of all performances . . . [It is] a real ideal object which exists, but exists nowhere, and whose existence is somehow the potentiality of its being reinterpreted by human minds. So it is first the work of human minds, the product of human minds; and secondly it is endowed with the potentiality of being recaptured by human minds again” (1993, 483). Trietler notes that music, as Popper describes it, resembles what Roman Ingarden terms an “intentional object.”

The “intentional object” is, in fact, Edmund Husserl’s idea (Husserl, 1970). Ingarden, a student of Husserl’s from the Göttingen period, expanded this concept in an in-depth study of the ontology of the literary work and, later, the musical work (1986). It is no surprise that Trietler associates the idea with Ingarden, using it as a point of departure for his own discussion of musical ontology and history.
1.1.1 Music as an Intentional Object

Husserl’s phenomenology is based on the premise that reality consists of objects as they are perceived or understood in human consciousness and not of anything independent of human consciousness. However, not all objects exist in the same way. The objects “chair” and “$\sqrt{-1}$” are quite different as are the objects “Fitzgerald’s *The Great Gatsby*” and “the Empire State Building.” Different kinds of objects fall into different categories. Real objects are physical things like chairs and the Empire State Building, ideal objects such as “$\sqrt{-1}$” exist only in the human mind, and intentional objects, such as *The Great Gatsby*, exist in both. The literary work, although different from the musical work, is an important progenitor. Paul Armstrong writes,

> Unlike autonomous, fully determinate objects, literary works depend for their existence, [Husserl] argues, on the intentional acts of their creators and of their readers. But they are not simply private thoughts because they also have an intersubjective manifestation. Their ideal status as constructs of consciousness does not make them like triangles or other mathematical figures which are truly ideal objects . . . The “intentionality” of consciousness is its directedness toward objects, which it helps to constitute.\(^2\)

Objects are always grasped partially and incompletely, in “aspects” (*Abschattungen*) that are filled out and synthesized according to the attitudes, interests, and expectations of the per-

\(^2\)Clifton uses the term “constitution” to describe the process by which a person orients themselves to a particular object (1983, vii).
ceiver. Every perception includes a “horizon” of potentialities that the observer assumes, on the basis of past experiences with or beliefs about such entities, will be fulfilled by subsequent perceptions (1997).

The musical work is consciously reified in much the same way. By grasping a set of musical events that could be the sounds of a performance or notes in a score, “readers” (i.e., listeners) fill out these aspects by injecting into them a set of relations, values, or expectations. In doing so, one transforms a set of musical events—a piece of music—into a coherent musical piece.

Paul Armstrong expresses Roman Ingarden’s position:

...[the] musical work differs from the literary work because it is not reducible to the psychology of either the author or the reader. The existence of a work transcends any particular experience of it, even though it came into being and continues to exist only through various acts of consciousness. Roman Ingarden argues that the work has an “ontically heteronomous mode of existence,” because it is [paradoxically] neither autonomous of nor completely dependent on the consciousnesses of the composer and the listener (1997).

For Ingarden, the musical work is an intentional object, but one that differs from the literary work because its existence is dependent on the both the composer and listener. Once created, the work depends on the “recreation” of a listener to be truly realized. I argue that the emphasis on the composer as an equal partner in his ontology is overstated, since the source
of stimulus that can be considered music by a sympathetic listener may not be reducible to a partnering composer. We are compelled by our musical sensibilities to consider works such as Cage’s 4′33″ music. Cage defines a timeframe (a beginning and an end) in 4′33″, but does not contribute to the experienced events within that frame. In addition, we are compelled to include works that play with this boundary, such as Cage’s Cartridge Music, Stockhausen’s Zyklus, Boulez’s Third Piano Sonata. Indeed, music is not an ideal object like a triangle, nor is it a real object like a chair. But the bipartisan definition that includes composer intentionality is problematic for us because it seems to force a consideration of unknown or nonexistent forces. This position is similar to the “New Criticism” position of criticism based on close reading free of a consideration of an author’s intent. I choose, therefore, a somewhat radical step and disregard the composer, and solely examine the intentional acts of the listener to give music meaning. I propose the following:

**Definition 1.1.1.** A musical piece is an intentional object created by a listener’s conscious filling out of a set of (partially grasped) musical events according to specific relations, values and beliefs.

Intuitively, the set of grasped musical events (e.g., the score, the performance) is a kind of piece. Additionally, from a phenomenological point of view, the intentional object created by the interpretation of that piece is another kind of piece.\(^3\) Ingarden (1986) might claim that the latter is the only valid subject of critical investigation. We, however, value not only both

\(^3\)Pearsall (1999) expresses a similar concept of intentionality and music making, but quickly digresses into a discussion of Gerald Edelman’s “neuronal groups” and their relation to the “plasticity of the mind.”
CHAPTER 1. PRELIMINARIES

Figure 1.1: Pieces $p$ and $q$ as members of the set of all pieces $P(E)$.

Figure 1.2: Type I comparative analysis.
Figure 1.3: Type II comparative analysis.

Figure 1.4: Type III comparative analysis.
The objecthood of a musical work is determined by the points of view of different people. In as much as these views can be understood as being determined by certain analytic frameworks, we can get at a piece of music by considering what it “is” from the points of view of various analytic methodologies. Then it follows that to compare music pieces is also to compare analytic models or methods.

In the broadest sense, pieces of music are finite sequences of musical events. Let $E$ be a set of putative musical events. Whether they are musical in fact only occurs when they are shown to be associated with an analytic function. Given the set of all musical events $E$, a piece $P$ (in $E$) is a finite sequence of elements from $E$. The set of all pieces (in $E$) is denoted $P(E)$. It is clear that both “partially grasped musical events” and the events comprising the “intentional object” qualify as members of $P(E)$. Per my stated interest in the connection of one member of $P(E)$ with another, let $F(E)$ be the set of all functions that maps $P(E)$ to $P(E)$. I call $F(E)$ the set of analytic systems or methodologies over $P(E)$.

Let $p$ and $q$ be two pieces in $P(E)$. Define

$$f(p, q) = \{ f \mid f \subset F(E) \text{ and } f(p) = q \}$$

Intuitively, $f$ is the set of functions (which is a subset of $F$) that sends piece $p$ to piece $q$. In other words, given a piece $p$ (grasped musical events) and an analysis $q$, $f$ is the set of all functions that can map $p$ to $q$.

I identify three types of comparative analysis by asking the following leading questions.
I. Given \( f(p) = r \) and \( f(q) = s \), in what ways are resulting analyses \( r \) and \( s \) alike?

II. Given \( f(p) = r \) and \( g(p) = s \), in what ways are the resulting analyses \( r \) and \( s \) alike?

III. Given \( f(p) = r \) and \( g(q) = s \) in what ways are the resulting analyses \( r \) and \( s \) alike?

Comparisons described by types I and II are familiar terrain for many musicians. Comparing two pieces of music by comparing their analyses under a single analytic methodology (type I) is by far the most common means of comparison. It is often a fruitful endeavor. Given two pieces of music, \( p \) and \( q \), the function \( f \) sends them to two sequences of musical events, \( r \) and \( s \), respectively. We can use the event intersection \((i)\) of \( r \) and \( s \) to support statements about the similarity or dissimilarity of the pieces \( p \) and \( q \). The most common problem with such studies, however, is the failure to acknowledge the role of \( f \) in event generation.

Comparing the analyses of a single piece of music by two analytic systems, \( f \) and \( g \), (type II) as in Figure 1.3 is also possible. The event intersection can be difficult to interpret because of the possible conceptual incompatibility of the events in \( r \) and \( s \).

In order to make type III comparisons, as represented in Figure 1.4, we need to employ a new music analytic system whose explanatory scope covers musical works as syntheses of pieces and analytic systems. To achieve this, I employ a function \( h \) whose value represents the “typical musicality” of pieces \( r \) and \( s \) in terms of the analytic functions \( f \) and \( g \).
CHAPTER 1. PRELIMINARIES

Each analytic system suggests a musicality by supporting an analyst’s preferred set of relations, features, or values. One can easily imagine a local theory of tonal music where tonic/dominant relations are preferred or considered “more typically musical” than supertonic/mediant relations. The analytic system $h$, therefore, measures the degree of typical musicality of a particular local system that a piece exhibits.

In the discussion that follows, the function $h$ will take as its domain not an individual piece, but the set of all pieces $P(E)$. To reflect this, I refer to $h$ as a global analytic system. Furthermore, it is important to note that while the domain of the function $h$ is $P(E)$, the range of the function is $\mathbb{R}$. In other words, $h$ maps the set of all pieces to the set of all real numbers. Functions $f$ and $g$ are called local analytic systems because both the range and domain are $P(E)$.

1.2 Commentary

Comparative music analysis is an important part of music criticism with significant music-theoretic implications. However, comparing pieces of music in order to group and distinguish them has traditionally been the domain of musicology, not music theory. Consequently, comparative-analytic methodologies have been designed to meet musicology’s endemic needs and goals. There is considerable agreement as to the distinction between musicology and music theory (Brown and Dempster, 1989; Morris, 2001; Burkholder, 1993). Musicology studies the role and context of art music in Western culture. Music theory studies musical structure. Like the examination of pieces often does, comparing pieces from a musicological point of view versus that
of a music theoretical point of view needs to respect this distinction. Too often, comparative analytic programs (see for example Snyder (1990) and Youngblood (1958)) have blurred the lines between the two disciplines leading us to wonder “are we talking about music structure, or music history, or context?” The history of the idea of musical style has gone a long way toward exacerbating this problem.

Boretz explains the kind of comparative analysis I am proposing: “...we compare individual pieces only to infer some terms whose interpreted transfer from one context to the other gives the attempt to “understand” a particular piece the benefit of discoveries and insights that have emerged in the course of “understanding” another ...” (1995, 242). Globally, we compare the typical musicality suggested by $F(E)$ (the set of all functions over $P(E)$). Locally, we establish a commonality between pieces produced by local analytic functions that enables us to meaningfully transfer terms from one context to another.

The theory of comparative music analysis is divided into two parts. First, I define a geometry of global music analytic systems addressing type III comparisons. This is by far the most non-traditional aspect of the research. The geometric model covers all global systems. Second, I define a comparison of two local music analytic systems addressing type II. Comparing local systems requires the establishment of a contextual equivalence between the two systems. In the case presented below, the equivalence is the formal structure of the metric space. These two approaches (global and local) are not alternatives to each other. They are in fact analyses of musical pieces on different, yet interrelated levels. Comparing local music analytic systems is complicated by arbitrary design choices inherent in each system. Like comparisons
of local systems themselves, solutions to this “design choice problem” are contextual. A solution to the local systems discussed is presented as is a discussion of a meta-analytic ramifications of comparing local systems. This later section addresses one concrete benefit from the comparison of local systems.
Chapter 2

History of Comparative Music Analysis

Now that the context for comparing pieces of music has been explained, it will be easy to see how existing comparative analysis methodologies fall short of what we would like them to do. All of the methodologies addressed below treat comparative analysis as the comparison of analyses of pieces. The difference between pieces, then, is the difference between the musical events present in the analyses produced by a single, often unheralded (and sometimes undefined) local analytic system. The unacknowledged local system becomes a suppressed premise which, once brought to light, undermines the analyst's argument. The analysts discussed below approach their projects with the belief that the way they look at pieces of music is the way music "is." However, there are many variables in analytic representation that can significantly change how music "looks." If the point of comparative analysis is to group pieces together, then changing these variables can change
how pieces are grouped. I begin with a general discussion of the origins of comparative analysis.

The origins of comparative analysis reach back to Guido Adler’s “scientific” casting of musicological functions (Adler, 1885). Adler divided musicology into historical and systematic components to aid in the gathering and interpreting of musical data. Figure 2.1 shows the issues concerning the historical musicologist. The historical musicologist classified music into geographies, epochs, schools. Figure 2.2 details systematic musicological categories. This discipline looked for non-historical features to corroborate historical classifications. The systematic musicologist uncovered the principles upon which the music of an epoch is founded, examining musical details in contrast to bibliographic details. Adler called these non-historical features “laws,” and it was systematic musicology’s job to establish them. Part of Adler’s systematic program was called comparative musicology (vergleichende Musikwissenschaft) and its purpose was the examination and comparison of music phenomena (“tonal products”) for ethnographic purposes. Adler writes: “Ein neues und sehr dankenswerthes Nebengebiet dieses systematischen Theiles is die Musikologie, d. i. die vergleichende Musikwissenschaft, die sich zur Aufgabe macht, die Tonprodukte, insbesondere die Volksgesänge verschiedener Völker, Länder und Territorien behufs ethnographischer Zwecke zu vergleichen und nach der Verschiedenheit ihrer Beschaffenheit zu gruppiren und sondern” (Adler, 1885, 14).¹

¹“A new and very rewarding neighboring field of study to the systematic subdivision is ‘musicology’, that is, comparative musicology. This takes as its task the comparing of tonal products, in particular the folk songs of various peoples, countries, and territories, with an ethnographic purpose in mind, grouping and ordering them according to the differences in their character.”
### TABLE 1

In Tabular Form, an Overview of the Entire Construction*** appears thus:

<table>
<thead>
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<th>MUSICOLOGY</th>
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<td><strong>I. HISTORICAL</strong></td>
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<tr>
<td>(History of music according to epochs, peoples, empires, nations, regions, cities, schools of art, artists).</td>
</tr>
<tr>
<td><strong>A. Musical paleography (notations).</strong></td>
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</table>


*** For purposes of comparison, the synoptic table according to Aristides Quintilianus, which contains the most comprehensive overview of the Greek system of musical didactics, is given.

<table>
<thead>
<tr>
<th>SYSTEM OF MUSIC</th>
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<tr>
<td><strong>I. ΘΕΩΡΗΣΙΚΟΝ</strong> (Theoretical or speculative section)</td>
</tr>
<tr>
<td><strong>A. φυσική (Physical-scientific)</strong></td>
</tr>
<tr>
<td>a. αριθμητική (Arithmetic)</td>
</tr>
</tbody>
</table>

Figure 2.1: Adler's *Historical Musicology* from Mugglestone (1981).
Figure 2.2: Adler's *Systematic Musicology* from Mugglestone (1981).
As the activities of comparative musicology became more consistently associated with the study of non-Western cultures and their music, the terms “comparative musicology” gave way in the 1950s to the more appropriate descriptor “ethnomusicology.” However, also in the 1950s, comparing pieces of written music to develop or support general categories of epoch, school, etc., evolved into a new area of study called “style analysis.” Style analysis is founded on the idea that if music can be considered as an arrangement of a set of objects, then pieces of music can be organized according to a composer’s choice of and disposition of those objects.

Numerous style studies (including many of those discussed below) are motivated by the desire to historically organize pieces according to such choices. In other words, if no historical evidence of a piece’s identity is available, then established taxonomic traits can be applied in reverse, top-down, to reveal the piece’s historical position. The analyst, therefore, as described by Eugene Narmour “serves under the imperatives of historiography” (1977, 171). However, the top-down-taxonomic approach can only work so well in the capacity of an historical “positioning system.” It is, in fact, an example of the fallacy of affirming the consequent. The modus ponens argument goes like this: If the piece is by Dittersdorf, then its taxonomic traits will be $X$, $Y$, and $Z$: Dittersdorf; Therefore $X$, $Y$, and $Z$. Affirming the consequent reverses the argument to say: If taxonomic traits $X$, $Y$, and $Z$, then the piece is by Dittersdorf. This is simply false and there is no suppressed premise to salvage the argument.

Boretz writes:

The most difficult area here, perhaps, is that associated with the
CHAPTER 2. HISTORY OF COMPARATIVE MUSIC ANALYSIS

notion of “style,” which is often treated as though it signified “the presentational-surface characteristics that are assertable of a given composition by regarding it as an instance of a fixed type of music taken \textit{a priori} as universally referential for all music, as they associate with the meanings such characteristics would be taken to have in a composition of the ‘referential’ type.”...However well this notion of “style” works for music of the referential kind, to regard its application as a universal implementation of the notion of “musical style” seems questionable at best (1995, 51).

Boretz highlights a problem found in a number of style studies. As style analysis became an autonomous “mode of inquiry,” it perpetuated the use of \textit{a priori} “universal” points of reference. In effect, it ossified its failure to recognize that taxonomy is contingent on examination methodology. Furthermore, the fact that this kind of comparative approach actually does work well for some sets of music (namely music of the “referential kind”) gives the illusion of general applicability.

Even though I do not propose a theory of musical style in the traditional historical-cultural sense, I find it useful to locate Boretz’s criticisms within the style analysis literature. This serves to reveal assumptions buried within this particular class of analytic activity.

...perhaps the notion of “style” might be explicated in a more general and perhaps conceptually deeper sense as ‘the particular relation on a given composition between the particulars of presentation and the syntactical functions inferred from them’
I agree with this assessment with the clarification that “particulars of presentation” and “inferred functions” are understood as intentional gestures. The problems with style studies are threefold and the studies examined below exhibit one or more of these problems. First, in varying degrees, they misjudge the object of examination, confusing the musical score for the musical work. Second, with the exception of one important study, they consistently fail to consider the bias of analytic systems. Third, confidence in the meaning of the results are undermined by ad hoc methodologies.

2.1 Leonard B. Meyer

Leonard B. Meyer is one of the most prolific writers on the topic of style analysis. His concept of style grows naturally and logically from his earliest writings to his latest. Meyer’s *Emotion and Meaning in Music* (1956) is the first large-scale incorporation of information-theoretic concepts into a theory of music interpretation and analysis. Meyer conceptualizes music as a communication channel between two people: a composer and a listener. (This idea resembles, albeit superficially, Husserl’s musical ontology.) His theory is founded on basic principles of information theory: given a finite number of possible outcomes for one musical event following another—and an expectation of the event sequence—the composer psychologically manipulates the listener by either meeting or thwarting those expectations.

In answering the question “what does music mean?”, Meyer incorporates the spirit of information theory, but his analyses do not result in quantitative
measures of entropy or information. Rather, Meyer is concerned with providing a framework within which information-theoretic ideas such as choice and constraint can be considered in an experiential fashion. For Meyer, in his “Style and Music: Theory, History, and Ideology,” the goal of style analysis is “to describe that patterning replicated in some group of works, to discover and formulate the rules and strategies that are the basis for such patternings, and to explain in the light of these constraints how the characteristics described are related to one another” (Meyer, 1989, 38). Meyer defines style as “a replication of patterning, whether in human behavior or in the artifacts produced by human behavior, that results from a series of choices made within some set of constraints” (1989, 3). The constraint concept, represented in Figure 2.3, is the most significant holdover from Meyer’s earlier work and the most important idea in his work on style. Constraints are divided into three hierarchically ordered classes: laws, rules, and strategies.

Laws, for Meyer, are trans-cultural constraints. Meyer gives gestalt-like examples such as “the proximity between stimuli or events tends to produce connection.” Meyer’s laws resemble the preference rules of Lerdahl and Jackendoff (1977) but are separate from specific musical contexts. Laws are divided into primary and secondary parameters with the former governed by “syntactic” constraints (the rules governing the organization of parameter elements) and the latter not governed by such constraints. Meyer identifies harmony in common-practice tonal music as a primary parameter and dynamics and textural density as secondary parameters.

Rules are intracultural and constitute the highest level of stylistic constraint (Meyer, 1989, 17). They specify “the permissible material means of
MEYER’S CONSTRAINT HIERARCHY (1989)

PRIMARY AND SECONDARY CONSTRAINTS

LAWS: Law-like, Perceptual Cognitive Constraints

RULES: Middle Ages, Renaissance, Classical

Dependency, Syntactical

STRATEGIES: Dialect, Idiom, Intraopus, Style

SUB-HIERARCHY:

Figure 2.3: Meyer’s constraint hierarchy.
a musical style” such as timbre, pitch collections, dynamics, and “durational divisions,” to name a few (1989, 17). It is the difference between rule sets that allows one to distinguish between the broad historical-style categories of medieval, classical, romantic, or modern. Rules are divided into three kinds: dependency rules, contextual rules, and syntactic rules. Meyer employs the following hypothesis to distinguish the differences between them: “On the highest level of style change [presumably Meyer is referring to the level of the epoch] the history of harmony can be interpreted as involving ever-greater autonomy and eventual syntactification” (1989, 17). In other words, moving progressively through historical epochs (Meyer starts with Notre Dame organum), harmony is first governed by dependency rules; in the Middle Ages (and Meyer gives Machaut as an example), it is governed by contextual rules; and finally reaches syntactical governance in the common-practice era. The most localized level in the constraint hierarchy is strategy. Strategies are compositional choices made between possibilities established by the rules of the style (1989, 20).

The remainder of Meyer’s theory is best thought of as a sub-hierarchy within the strategies category. This sub-hierarchy comprises (in order of decreasing coverage) dialect, idiom, and intraopus style. Dialect is what is common to works of different composers of the same style period. Idiom is what is common to different works of the same composer. Intraopus style is what is replicated within a single work. Meyer actually attaches this sub-hierarchy to both rules and strategies. However, for strategies to be differentiated by dialect, they must stem from the same rule set.

Meyer’s theory is compelling, but not without problems. First, it is not clear that pre-common-practice music is without syntax. Second, Meyer’s
theory is supposed to be a general theory, but it uses common-practice tonality (i.e., tonal syntax) as the singular defining term. Everything else is either without it, or about to get it. Meyer’s hierarchy does not form a closed system that can be replicated *in toto* in other musical contexts such as non-western or non-tonal. His ultimate goal is to show a causal relationship between cultural ideology and musical style as it is modeled by change and choice, but there is no way to verify this claim. Both Rink (1991) and Korsyn (1993) agree that Meyer fails to convince that a prevailing ideology leads to a change as opposed to merely allowing it. With no way to verify causality, Meyer’s theory of style is relegated to a heuristic or even metaphorical status.

### 2.2 Jan LaRue

Aside from philosophical and methodological differences, LaRue distinguishes himself from Meyer by setting forth a specific comparative analytic methodology. This methodology appears in its first form in his 1962 article “On Style Analysis” (1962) and culminates in his *Guidelines for Style Analysis* (1970). In his 1962 article, LaRue walks a fine line between his belief in the importance of objective analysis and an approach that is phenomenological in nature. LaRue begins, “In all thinking, writing, and teaching about music, if we are to progress beyond a catalogue of personal reactions we must employ some *objective framework* for our reflecting” (1962, 91). LaRue’s intention is clear, but his understanding of what an objective framework is and to what end it may be used is not. LaRue goes on to embrace a pseudo-phenomenological music-interpretive approach by describing how
“our apparently intuitive responses are mostly learned . . .” (1962, 93).

LaRue believes that an exhaustive list of features is the best way to represent a piece analytically. To this end, he creates a template that is filled in according to an individual analyst’s tools, techniques, and taste. The goal was to create a way an analyst could record and organize his or her musical experiences. LaRue developed families of typologies grouped into a single global schema termed “Sound, Harmony, Melody, Rhythm, Growth” or “SHMeRG” for short and represented in Figure 2.4. Each typology comprises a group of related terms used to describe one of these musical parameters at different levels of detail.

In his review of Guidelines, Jackson (1971) identifies two shortcomings with LaRue’s approach. The first is LaRue’s tendency to view all music from the vantage point of the Classic period. Jackson writes, “Although the book purports to be a method applicable to all styles, periods, and composers . . . it is actually colored by the author’s eighteenth-century background and preferences.” The second is LaRue’s tendency to regard analysis as a mere compiling of data rather than as a result of “penetrating insight.”

While I agree wholeheartedly with Jackson’s first criticism (it, in fact, echoes the criticism I levied against Meyer’s reliance on tonal syntax), the second deserves scrutiny. Jackson is correct in his interpretation of analysis. The interpretation of data with penetrating insight is certainly what we expect from analysis. But compiling data is hardly simple or unbiased.

LaRue’s method is based on an individual’s interaction with a piece. In addition, the flexibility of his approach allows different people to interact with those pieces in different ways. However, the approach is hardly sys-
BACKGROUND
The Frame of Reference
Significant Observation: Selection rather than multiplication of evidence

OBSERVATION
The Three Standard Dimensions of Analysis: LARGE, MIDDLE, SMALL.
The Four Contributing Elements: SOUND, HARMONY, MELODY, RHYTHM (SHMR)
The Fifth, Combining and Resultant Element: GROWTH
1. Sources of MOVEMENT: Varying Degrees and Frequency of Change
   a. General States of Change: Stability, Local Activity, Directional Motion
   b. Specific Types of Change: Structural, Ornamental (Secondary)
2. Sources of SHAPE
   a. Articulation
   b. The Four Options for Continuation
      Recurrence: Repetition, Return after Change
      Development (Interrelationship): Variation, Mutation
      Response (Interdependence): S—Forte vs. piano, tutti vs. solo, etc.
      H—Tonic vs. dominant, major vs. minor, etc.
      M—Rising vs. falling, stepwise vs. skips or leaps, etc.
      R—Stability vs. activity or direction, balance between modules (4 bars answered by 4), etc.
   c. Degrees of Control: connection, correlation, concinnity
   d. Conventional Forms
The Style-Analytical Routine: Typology—Movement—Shape

EVALUATION
Achievement of Growth (Movement, Shape, Control)
Balance of Unity and Variety
Originality and Richness of Imagination
External Considerations: novelty, popularity, timeliness, etc.

Figure 2.4: LaRue’s SHMeRG schema (2001).
tematic, and its extreme subjectivity undermines comparisons. Much of the terminology is *ad hoc*. Analyses get stuck in phrases such as: “Short modulation to D minor gives fresh harmonic color to second half,” or “Impressive harmonic stabilization at the end shows large-dimension concern.” Simply put, comparison is jeopardized by incorrigible statements\(^2\) and an inability to stably represent analytic information so as to apply inferences learned in one context to another.

### 2.3 Eugene Narmour

Eugene Narmor is best known for his development of the implication-realization theory (*IR* theory) of melodic expectancy (Narmour, 1992). This theory is intended to provide accounts of the processes experienced by a listener as a piece progresses. It focuses on the ways a listener’s expectations can be explained by reference to features of melodic structure as they unfold in time. His article “Hierarchical Expectation and Musical Style” effectively applies this idea to style analysis (1999). For Narmour, the perception of replicated patterns determines style (1999, 441). Accurate style-structure mappings, he writes, “take place only after long-term memory recognizes the pitches, intervals, and timbres typifying tonal diatonic music” (1999, 442).

In this variation of Meyer’s definition of style, Narmour separates himself from historical issues. This view does raise some questions about bottom-up processing and how schemas develop, but Namour asks us to accept the existence of schema in order to strengthen implicative claims.

\(^2\)See Babbitt (1965) and Guck (1994).
between percept and memory and thus to map learned, top-down expectations. Matching an emerging implicative pattern to the learned continuation of a previously stored schema tends to be automatic ... (1999, 441).

The activation of schema is usually automatic as Narmour points out, and this process has been described formally in terms of computational neural-net models (Bharucha, 1987; Gjerdingen, 1988) and less formally in terms of archetypes (Meyer, 1973) and schematic clustering (Gjerdingen, 1988).

Narmour describes listeners mapping top-down from an environmentally acquired schema onto what he calls “incoming foreground variations.” He argues that the style distinctions are a result of the difference in hierarchic depth of the complex of style structures formed when listeners map patterns to memory. The style structure is the characterization of a musical parameter in terms of the IR theory.

Narmour’s basic premise is laid out by looking at a single melody. By simplifying the melody in specific ways, he reduces the depth of the analysis. The levels of reduction, shown in Figure 2.5, are characterized by durational spans of IR units. Narmour hypothesizes that these spans are coded into different neural locations. The stylistic differences between pieces, then, would be the differences between patterns of neural activation.

Eerola et al. (2002) showed Narmour’s IR theory to be an accurate predictor of some aspects of tonal music perception. What’s interesting about his “Style” article is that he is using the IR theory as model of perceptual complexity. What is problematic, in my view, is his insistence of the IR
Figure 2.5: Four reductive levels from Narmour (1999).
theory’s “inevitability” due to its claimed psychological and cognitive provenance. IR theory is a hybrid of suggestive (or proscriptive) and descriptive models.\(^3\) While the selection of a single local analytic system for comparing multiple pieces of music is an appropriate tack (see, for example, my type I approach defined in section 1.1.), the assertion (usually tacitly—by exclusion of other systems) that a chosen local system is categorically “better” than all others is highly suspect, even when scientifically motivated.

### 2.4 Alan Lomax

In 1968, Alan Lomax published his *Folk Song Style and Culture (FSSC)* and formally introduced a culturally comprehensive comparative system called *Cantometrics*. Downey (1970) both hails the system as a “major achievement in ethnomusicology and cross-cultural method” and criticizes details of the methodology. The book is the collection of reports delivered by the staff members of the Cantometrics Project at the Washington, DC, meeting of the American Association for the Advancement of Science in 1966. In an attempt to find patterns of distribution and correlations between singing styles and social institutions among world cultures, the Project created a model of what Driver (1970) calls “interdisciplinary cooperation.” Cantometrics combines analytic approaches from “musicology, ethnomusicology, linguistics and ethnolinguistics, psychology, communications theory, sociology, cultural anthropology ... statistical method and computer mathematics” (Lomax, 1968, 302).

\(^3\)“Suggestive,” “descriptive,” and “hybrid” are three classes of analytic systems defined by David Temperley (2001).
CHAPTER 2. HISTORY OF COMPARATIVE MUSIC ANALYSIS

The project’s principal contribution was providing evidence for the musicological intuition that music “symbolizes and reinforces certain important aspects of social structure” (1968, vii). To wit, Lomax and his staff proposed a theory of musical activity and cultural relationships and a system of musical analysis and description.

Song is defined as “the use, by the human voice, of discrete pitches or regular rhythmic patterns or both” (1968, 36). Chapter 3 of FSSC enumerates 37 variables used to code and classify each song. Each variable is identified as a “line,” and each line is given up to 13 coding conditions. Table 2.1 and Table 2.2 show, for example, a sample of Lines and the 13 conditions for Line 1, respectively.

Each line’s “symbolic field is quickly memorized and it is here that the rater can swiftly record his perceptions of the song he is listening to” (1968, 37). It is clear that the analytic approach for rating is strongly impressionistic. Analysts are encouraged to not dwell on the samples and to pay as little attention as possible to characteristics not directly addressed by the line topic. The coded analyses of all songs are compared and grouped together according to shared traits and then cross-referenced against geographical ethnic units defined in George P. Murdock’s (1967) Ethnographic Atlas.

Lomax asserts two corollaries to the principal finding (stated above):

First, the geography of song styles traces the main paths of human migration and maps the known historical distribution of culture. Second, some traits of song performance show a powerful relationship to features of social structure that regulate interaction in all cultures (1968, 3).
Table 2.1: Sample of Lines from Lomax (1968).

| Line 1. | The vocal group |
| Line 2. | The relationship between the accompanying orchestra and the vocal part |
| Line 3. | The instrumental group |
| Line 4. | Basic musical organization of the voice part |
| Line 5. | Tonal blend of the vocal group |
| Line 6. | Rhythmic blend of the vocal group |
| Line 7. | The basic musical organization of the orchestra |
| Line 8. | Tonal blend of the orchestra |
| Line 9. | Rhythmic blend of the orchestra |
| Line 10. | Words to nonsense |
| ... | ... |
| Line 34. | Nasalization |
| Line 35. | Raspiness |
| Line 36. | Accent |
| Line 37. | Enunciation of consonants |
Table 2.2: *Line 1: The Vocal Group* from Lomax (1968).

<table>
<thead>
<tr>
<th></th>
<th></th>
<th>No singers</th>
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</thead>
<tbody>
<tr>
<td>1</td>
<td>$\phi$</td>
<td>No singers</td>
</tr>
<tr>
<td>2</td>
<td>$\frac{L}{N}$</td>
<td>One singer</td>
</tr>
<tr>
<td>3</td>
<td>$\frac{L}{N/\text{NA}}$</td>
<td>One singer with an audience</td>
</tr>
<tr>
<td>4</td>
<td>$-L$</td>
<td>One solo singer after another</td>
</tr>
<tr>
<td>5</td>
<td>$L/N$</td>
<td>Social unison with a dominant leader</td>
</tr>
<tr>
<td>6</td>
<td>$N/L$</td>
<td>Social unison with the group dominant</td>
</tr>
<tr>
<td>7</td>
<td>$\frac{L/N/\text{N/L}}{N/\text{L}}$</td>
<td>Heterogeneous group</td>
</tr>
<tr>
<td>8</td>
<td>$L + N$</td>
<td>Simple alternation: leader-chorus</td>
</tr>
<tr>
<td>9</td>
<td>$N + N$</td>
<td>Simple alternation: chorus-chorus</td>
</tr>
<tr>
<td>10</td>
<td>$L(N$</td>
<td>Overlapping alternation: leader-chorus</td>
</tr>
<tr>
<td>11</td>
<td>$N(L$</td>
<td>Overlapping alternation: chorus-leader</td>
</tr>
<tr>
<td>12</td>
<td>$N(N$</td>
<td>Overlapping alternation: chorus-chorus</td>
</tr>
<tr>
<td>13</td>
<td>$W$</td>
<td>Interlocking</td>
</tr>
</tbody>
</table>
These are powerful claims. And the project’s data support the principal finding and the corollaries. However, according to Driver (1970) the problems with the study are threefold:

1. The classification of 233 ethnic units was determined not by music analytic data, but by adopting the area units of Murdock’s (1967) *Ethnographic Atlas*.

2. The sample size for each unit was limited to 10 songs.

3. Inverse weighted averaging of analytic variables questionably produces a large number of song styles and unexpectedly groups together geographically disparate peoples.

Making claims about the relation between song style and specific cultural entities requires that such entities be demarcated. However, it is often the case that cultural geographical divisions do not exactly match cultural trait divisions. According to Driver, Lomax made a serious mistake in “trying to fit his musical styles to the Procrustean bed of Murdock’s general cultural areas” (1970, 58). Lomax might have been in better standing had he, instead, allowed his style taxonomies to determine cultural areas.

Observing a somewhat different set of problems, Downey (1970) believes that musicologists will question three steps in the process by which the principal finding and corollaries were developed:

1. The authenticity of the individual samples of music;

2. The descriptive techniques employed by the Cantometrics staff to define patterns of similarity;
3. The cultural factors which are related to each of the variables measured in the music sample.

The critique of the descriptive techniques is the most relevant to my research. But I admire the lengths to which the Cantometrics staff has gone to develop a rather well-defined list of song qualities. It is clear that while no methodology can possibly quantify every sonic parameter of a song, enough parameters have been accurately defined so as to produce convincing representative analyses. The main problem I see is reflected in both Downey’s step 3 and Driver’s problem 2. The association of song qualities with cultural mores is a significant and realistic subject of investigation. However, the extraordinarily small sample size combined with the arbitrary inverse weighting scheme undermines one’s confidence in the associations made between song styles and political-cultural attributes.

Even with these problems, the methodology seems well appreciated. While the results don’t inspire a great deal of confidence, the methodology and aims are thought-provoking and perhaps do more to illustrate the issues and challenges for ethnomusicology than to answer any questions definitively. The scope of the project (a world-style map) is really quite unrealistic given the requirements of a confidence-inspiring scientific model.

2.5 The Information Theorists

The codification of the mathematical theory of communication by Shannon (1948) launched a revolution, not only in the sciences, but also in the humanities. Shannon proved that for any amount of noise present in a com-
munication channel, a coded message can be successfully transmitted if it is proportionally redundant. The ideological foundations of this theory codified that the more redundant a message, the less information it contains. Conversely, ripping a page from the thermo-dynamics text book, the less redundant a message is, the more random or entropic it is. To make clear a point that is often misunderstood: entropy = randomness = information. The information content of a message composed using an alphabet $A$ is determined by the probability of the occurrence of specific members of $A$.

Less technically speaking, the idea that information and its transmission could be measured was appealing to scholars of all disciplines. As discussed above, Meyer used this as a heuristic basis for discerning meaning in music. For a small group of musicologists, the confluence of information theory and the advent of the digital computer signaled a new era for systematic musicology. Harvard Professor Joel Cohen begins his article “Information Theory and Music” with a critique of extraordinary prescience:

In the last decade, information theory has been applied in at least a dozen different fields. In some extensions, the use of the calculus of information theory was carefully justified or the calculus was modified according the the requirements of the field of study. In other extensions, however, “experiments” were performed without regard to the validity or significance. This was usually done by appealing to the reader’s intuition with amorphous generalities, then leapfrogging to the $H$ formula . . . for information-content and inserting some numbers. Of this trick, extensions into music theory have been particularly guilty. Mu-
sicians with a taste for mathematics have applied the $H$ formula without regard to the assumptions they were making. Technicians have applied informational measurements without regard to aesthetic considerations. To the degree that these interfield minglings have stimulated objectivity in examining musical “intangibles,” they have been helpful; but when validity is claimed for their results, inquiry should be made into the means of obtaining them (1962).

And while John Synder, who almost 30 years later begins his article with the same quote, doubts that scholars have really been that negligent (1990, 122), I argue that, in fact, they have. The following section outlines the major theses and arguments made in the incorporation of information theory (IT) into comparative analysis. One question that has never been asked in the literature is why the application of IT became so connected with comparative analysis? It appears that IT did not appeal to the practitioners of music criticism in the 1950s. Cohen’s observation that informational measurements have been applied without regard to aesthetic considerations is true not only for the literature that was available to him in 1962, but for every informational study thereafter. Musicology had a hoary systematic component that never had been adequately treated, and it seemed that IT had the capability to articulate in concreta the interopus laws underlying historical musicology’s epochal divisions. Confusion between musical qualia as something that inherently contains information and musical qualia as something that is capable of being assigned information content would ultimately undo IT applications in music analysis. In other words, nobody could figure out to what, as Cohen put it, the “physical sign-vehicles” were
really referring. Throughout the 1950s, 60s and 70s, there were hundreds of papers written that applied information theory to music analysis. The following section restricts itself to analysts who use the theory for the explicit purpose of comparative analysis. This excludes, therefore, Cohen’s (1962) and Pinkerton’s (1956) seminal analytic applications.

### 2.5.1 Joseph Youngblood

Joseph Youngblood received his Ph.D. in music theory from Indiana University in 1960 with a dissertation entitled *Music and Language: Some Related Analytical Techniques* and taught at the University of Miami until his retirement in 1996. Youngblood’s most important contribution to comparative analysis is his 1958 article “Style as Information,” the purpose of which “is to explore the usefulness of information theory as a method of identifying musical style” (1958, 24). This study differentiates melodies from 20 major-mode vocal works by Schubert, Mendelssohn and Schumann (grouped eight, six and six respectively).

Youngblood calculates three values for each composer: entropy \( H \), relative entropy \( H_r \), and redundancy \( R \) ((1958, 28). Entropy \( H \) is defined

\[
H(X) = -\sum_{x \in X} p(x) \log p(x).
\]

It is the multiplication of the \( \log_2 \) of an event probability by its probability and summing the results for all events.

\( H_r \) is the difference between the real entropy of a source and the hypothetical entropy if all probabilities are equal. Since the relative entropy will
either be equal to or less than the entropy for a given action, it is expressed as a real number between 1 and zero. The difference between $H_r$ and 1 is the redundancy ($R$) and it is expressed as a percentage (1958, 28).

A Markov chain is a sequence of symbols whose probability of occurrence is affected by preceding events. A first-order Markov chain accounts for this effect by weighing the probability of a symbol occurring by what immediately preceded it. For example, in a sequence of letters that comprise a word in the English language, a first-order Markov chain would inform us that the probability of the occurrence of the letter U is quite high in a sequence involving the letter Q.

The aforementioned melodies were encoded into pitch-classes whose zeroeth element was oriented to what the author determined to be the key of the excerpt. In this encoding system, there is enharmonic equivalence so that sharp scale-degree 1 is the same as flat scale-degree 2 and there was no accounting for modulation. Youngblood calculates the first-order transition probabilities for each pitch class according to composer. Figure 2.6 reproduces Youngblood’s Table II.

Do the fluctuations that Youngblood finds represent individual identities within a certain, undefined range? Or do the small variations represent group membership in a single taxonomic class? Youngblood does not answer these questions. He can not. As he points out, many more pieces would need to be analyzed in order to define a range (1958, 30). However, Youngblood does move towards range definition by analyzing the information content of four selections of Gregorian chants to compare against the Romantic composers.
### Figure 2.6: First-order transition probabilities from Youngblood (1958).

Matrix of First-order Transition Probabilities for Schubert (St), Mendelssohn (Mn), and Schumann (Sn).

<table>
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<tr>
<th>Tone</th>
<th>I</th>
<th>II</th>
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<th>IV</th>
<th>V</th>
<th>VI</th>
<th>VII</th>
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<th>IX</th>
<th>X</th>
<th>XI</th>
<th>XII</th>
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<td>Mn</td>
<td>St</td>
<td>Mn</td>
<td>St</td>
<td>Mn</td>
<td>St</td>
<td>Mn</td>
<td>St</td>
<td>Mn</td>
<td>St</td>
<td>Mn</td>
</tr>
<tr>
<td>I</td>
<td>29.0</td>
<td>0.27</td>
<td>5.25</td>
<td>3.33</td>
<td>2.51</td>
<td>1.41</td>
<td>0.17</td>
<td>0.18</td>
<td>32.0</td>
<td>0.27</td>
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<td>3.33</td>
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<td>0.14</td>
<td>0.25</td>
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<td>0.25</td>
</tr>
</tbody>
</table>

Average Mn = 44.3
Average Sn = 39

P_{ij}^{St}

\[
P_{ij}^{St} = \frac{35.7}{50}
\]

P_{ij}^{Mn}

\[
P_{ij}^{Mn} = \frac{5.37}{50}
\]

P_{ij}^{Sn}

\[
P_{ij}^{Sn} = \frac{5.32}{50}
\]
CHAPTER 2. HISTORY OF COMPARATIVE MUSIC ANALYSIS

The analysis of chant excerpts shows a high degree of relative entropy. The excerpts were analyzed in a seven pitch-class diatonic context. Youngblood also analyzes the excerpts in a twelve pitch-class context and finds that the redundancy is higher than the average redundancy of the Romantic composers.

The writer does not feel that this is completely unjustified, since modern listeners are prepared to respond to twelve divisions of the octave, and consequently, maximum uncertainty is for them represented by \( \log_2 12 \) and not by \( \log_2 7 \) (Youngblood, 1958).

What Youngblood means by “prepared to respond” is far from clear. I believe his intention is to recognize that there are different ways of understanding music and that a modern listener understands music more in the context of a universe of twelve pitch-classes than a universe of seven. However, this recognition is too weak and vague to be of any use.

Youngblood concludes his study by critiquing its limited scope of included detail. More detail, he decides, such as that pertaining to rhythm, word painting, meter, and harmony, would need to be included “before this system could be accurate even for melody alone” (1958, 31). This is not entirely true, and his critique is founded more in insecurity of thesis than in lack of accuracy. Youngblood accomplished exactly what he said he would do, but it would seem that, in the final conclusion, he does not believe that his findings are particularly meaningful in a musical way. I agree.
2.5.2 Barry S. Brooks

Brooks, a musicologist teaching at Queens College, CUNY, was instrumental in organizing three symposia dedicated to exploring musicological-computer applications. The first symposium, *Musicology and the Computer I*, was held in 1965, as was the second, *Input Languages to Represent Music*; the third, *Musicology 1966-2000: A Practical Program*, was held in 1966. Proceedings from the three symposia were published in *Musicology and the Computer-Musicology 1966-2000: A Practical Program* (1970).

Brooks’s (1969) article “Style and Content Analysis in Music: The Simplified Plaine and Easie Code,” while ultimately introducing a method of music encoding for computer manipulation, outlines some of the perceived problems of the era. The concept “content analysis” is meant to characterize the implicit methodology of “systematic and objective quantification (applied) under the name of “style analysis” (1969, 287). Brooks is responding to the perversion of comparative analysis by a “multitude of sinfully subjective descriptions and unsubstantiated conclusions” (1969, 287). He is most likely criticizing Meyer and LaRue. Like many of his time, Brooks believes that “objective content analysis is not only feasible but is essential if real progress is to be made in defining musical style and understanding how stylistic evolution occurs” (1969, 288). The problems that Brooks encounters are common to this hard line. The conclusion that an automated analysis is an objective analysis is, of course, false. Brooks seems to understand this at a basic level (he does acknowledge that “perhaps style analysis is the application of educated intuition and hypothesis to (an) analysis of content” (1969, 287), but not at the level of application. Intuition and hypothesis
play important roles at every level, not just at the “analysis of content” level. The encoding of data is entirely prejudicial for we must decide what details to encode as content. How we interpret and analyze the content is also governed by choice. However, choice is not the problem. Rather, it is the pretense that choice has been eliminated or greatly minimized by automation that is the problem.

2.5.3 Frederick Crane and Judith Fiehler

The problem that Brooks (1969) encounters is not uncommon for the era. Crane and Fiehler (1970) in their article “Numerical methods of comparing musical style” are similarly wooed by the sirens of computation. “It is natural that analysts should look to the computer for assistance,” they write, “as music lends itself well to alphanumeric notation, and because so much of analysis (counting chords and the like) is mechanical and tedious” (209 1970, emphasis added). This quote shows how Crane and Fiehler believe, as does Brooks, that objective content analysis is a desirable and attainable goal.

I can concede that the notation (i.e., the score) of much western classical music can be easily represented in alternate formats. However, the real question is to what extent does the score, either in an alphanumeric or a more standard clef/staff notation, represent the music? As I discussed in Chapter 1, the score is only part of a musical work’s identity. From an analytic point of view, the score is indeed a valuable surrogate for all instances of physical manifestations of the music. It is not until we contemplate the data represented (however inefficiently) in that score as music that a musical work
begins to exist.

2.5.4 James Gabura

James Gabura’s (1970) article “Style analysis by computer” does a fair job of identifying the pitfalls found in Brooks (1969) and Crane and Fiehler (1970). The advantage promised by the computer revolution to comparative analysts was the ability to count and sort huge amounts of data quickly. But the question of how to code the data and what, exactly, to count and sort were undefined variables. For example, using a punch card system, as Gabura does, assumes that the information that will lead to the desired findings can be coded onto such a card. The statistical approach of Gabura shows that, like all other automated analysis approaches, analyst-made choices extend right up to the point of the “final” analysis. What to code, how to code it, what to look for, and how to interpret findings are all choices made by the analyst.

2.5.5 Leon Knopoff and William Hutchinson

Leon Knopoff and William Hutchinson’s principal contribution to comparative analysis is their 1983 Journal of Music Theory article “Entropy as a measure of style: the influence of sample length.” Like all comparative studies using information theory, this one is founded on the belief that the act of musical composition is a selection of elements from several musical parameters. “These choices,” write Knopoff and Hutchinson, “will bring about distributional characteristics that may belong to a ‘style’ ” (1983, 75). Their article begins with a discussion of information theory in general.
The authors take Youngblood’s 1958 article as a point of departure. While information theory is present in music research between these two papers, the focus has not been specifically on its use in a comparative program. Following Knopoff and Hutchinson’s critique of Youngblood, “qualitative estimates of entropy were taken to be descriptions of musical style, and the entropies were also used for comparative purposes; that is, specific assessments of entropy were related, not only to a maximum potential entropy” (1983, 76). While the comparison of entropy values is not problematic in and of itself, Knopoff and Hutchinson argue that sample size should be factored into the equation.

Whether or not entropy values can represent a piece of music is not the question. To be more specific than Youngblood as to the definition of style, Knopoff and Hutchinson define the entropy of the infinite sample pool as the style for that pool. The question the authors address then is how well the finite sample reflects the infinite sample.

We will show that entropies can indeed be used for comparative purposes, and for both discrete and continuous alphabets. If a value of entropy is derived from a finite musical sample, the analyst must, however, be prepared to calculate the likelihood that there may be a difference between the value calculated and that of the parent musical style it purports to represent (1983, 77).

Therefore, to the computation of entropy, Knopoff and Hutchinson have added a second calculation that is the “determination of the extent to which the length of a given sample may be judged to represent safely a homoge-
A measure of stylistic entropy is in direct relation to what we recognize musically as comparative and is potentially valuable to the formal analysis of music and our general theoretical structuring of music. Thus the calculation of equation (A.2) is relevant to a central problem in the study of music: the identification of stylistic properties and our capacity, through objective analysis, to distinguish these properties for comparative purposes (1983, 81).

The authors limit their discussion to entropies, omitting consideration of redundancies because they “do not believe the reporting of redundancy values contributes additional information beyond that already imparted by the entropy values, at least if the maximum entropies are identical” (1983, 81).

2.5.6 John L. Snyder

For Snyder (1990) in his article “Entropy as a Measure of Musical Style: The Influence of a Priori Assumptions,” meaningful statistical studies of music and musical style depend on a clearly defined problem and of “an accurate assessment of the nature of the materials to be studied” (1990, 121). Snyder is motivated by the claim that previous application of information theory to the study of style—namely studies by Youngblood (1958) and Knopoff and Hutchinson (1983)—failed to define “the problem” correctly. Snyder’s solution resembles aspects of my thesis insofar as he recognizes
CHAPTER 2. HISTORY OF COMPARATIVE MUSIC ANALYSIS

Figure 2.7: Snyder’s Table 3 comparing results under rpc/ksd and sd/st (1990).

that how data is coded affects resulting analyses of that data. Information theory measures the probability of event occurrence, requiring that events be grouped into a single alphabet. A priori assumptions about the nature of that alphabet, and about the relationships between alphabet members, can have a significant effect on the measure of entropy.

Figure 2.7 shows the varying results achieved using two sets of a priori assumptions. Snyder takes as his point of departure the work of Knopoff and Hutchinson discussed above. The underlying assumptions found in their article are (1) that the pitch universe consists of 12 tones (i.e., pitch classes), (2) that these tones are related to a central tone, and (3) that printed key sig-

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Table 3. Comparison of Results Obtained under Two Sets of A Priori Assumptions

<table>
<thead>
<tr>
<th>notes</th>
<th>rpc/ksd</th>
<th>sd/st</th>
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<tbody>
<tr>
<td></td>
<td>H</td>
<td></td>
</tr>
<tr>
<td>3150</td>
<td>3.039 0.060*</td>
<td>Hasse, all arias</td>
</tr>
<tr>
<td>5057</td>
<td>3.009 0.048*</td>
<td>Mozart, all songs/arias</td>
</tr>
<tr>
<td>176</td>
<td>3.308 0.286*</td>
<td>Mozart, “Das Veilchen”</td>
</tr>
<tr>
<td>[5448]</td>
<td>2.970 0.049***</td>
<td>Schubert, <em>Die schöne Müllerin</em></td>
</tr>
<tr>
<td>[143]</td>
<td>2.689 0.211***</td>
<td>Schubert, “Halt!”</td>
</tr>
<tr>
<td>[4022]</td>
<td>3.295 0.058***</td>
<td>Schubert, <em>Die Winterreise</em></td>
</tr>
<tr>
<td>[3088]</td>
<td>3.273 0.066***</td>
<td>Schubert, <em>Schwanengesang</em></td>
</tr>
<tr>
<td>[146]</td>
<td>2.769 0.231***</td>
<td>Schumann, “Widmung”</td>
</tr>
<tr>
<td>1220</td>
<td>3.397 0.104*</td>
<td>R. Strauss, early songs</td>
</tr>
<tr>
<td>[206]</td>
<td>2.947 0.244***</td>
<td>R. Strauss, “Heimliche Aufforderung”</td>
</tr>
<tr>
<td>88</td>
<td>3.399 0.414***</td>
<td>R. Strauss, “Mein Auge”</td>
</tr>
</tbody>
</table>

*from Knopoff and Hutchinson, “Entropy”
** calculated from data given by Knopoff and Hutchinson
*** from data tabulated according to rpc/ksd

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4The title of Snyder’s article is a parody of Knopoff and Hutchinson’s “Entropy as a measure of musical style: the influence of sample length.”
CHAPTER 2. HISTORY OF COMPARATIVE MUSIC ANALYSIS

natures are the principle basis for determining key. These assumptions are referred to as relative-pitch-class \((rpc)\) based and key-signature-dependent \((ksd)\) based. Snyder points out a “strange” result obtained under \(rpc\) and \(ksd\) assumptions. Namely, Synder is concerned about the lack of monotonicity in entropy values \((H\) values in the leftmost \(H\) column) when they are mapped to a time line. Of course, these results are not strange at all. The results follow the methodology and the discomfort experienced by Synder is psychological. He does not like the results obtained by Knopoff and Hutchinson because they do not conform to his intuition and expectation.

By following only notated key signatures \((ksd)\), analysts miss any internal “accidentalized” modulations. By encoding pitch information in terms of pitch class numbers \((rpc)\), analysts assume enharmonic equivalence—and thus lose spelling information. Snyder proposes a new set of \textit{a priori} assumptions that are scale-degree \((sd)\) based and structural-tonic \((st)\) oriented. This means that musical excerpts that switch between parallel modes would retain the same scale-degree number for tonic, and distinction between enharmonic equivalents (e.g., \(C\#/Db\)) are maintained. The \(st\) orientation insures that internal modulations are accounted for, thus refining the explanation for the presence of chromaticism.

As shown in Figure 2.7 in the rightmost \(H\) column, modification of the coding conditions has its intended effect—namely that Hasse’s music is less entropic than Mozart’s, which is is less entropic than Schubert’s, which is less entropic than Strauss’s. While not completely monotonic, the ordering of composers by their maximum \(H\) values is, with the exception of Schumann, coordinates with their position on a time line.
CHAPTER 2. HISTORY OF COMPARATIVE MUSIC ANALYSIS

This study is important mainly for recognizing that changing how one chooses to look at music changes how music looks. As simple as that may seem, the studies by Meyer, LaRue, Narmour, and the gallery of information theorists fail to incorporate this idea in a meaningful way. Whether one agrees or disagrees with Snyder’s use of information theory or the results he gets, it is a matter of fact that he was able to modify the analytic techniques of IT to achieve a result that better matched his own intuitions.
Chapter 3

Global Music Analytic Systems

3.1 Mathematical Underpinnings

3.1.1 Introduction

This chapter outlines the conditions and materials necessary for geometric comparisons of global music analytic systems. In order to promote clarity, it will be beneficial to present the formal details of the theory in their entirety first and then show examples and applications. The theory utilizes three principal postulates:

Postulate 1. There is an encoding scheme for representing musical events as natural numbers.

Postulate 2. There is an ordered sequence of natural numbers that we will
CHAPTER 3. GLOBAL MUSIC ANALYTIC SYSTEMS

51

call a piece.

Postulate 3. There is a global music analytic system which is a function on a piece and whose value represents the typical musicality of the piece within the system.

Recall that in Chapter 1, a local music analytic systems was defined as a map from one piece to another piece. Global music analytic systems were defined as a map from a piece to the real numbers. A single real number represents the typical musicality of a piece as determined by the global analytic system. If a global analytic system analyzes ten pieces, then it produces a sequence of ten real numbers. This sequence stands in for or represents the global analytic system and, unless defined identically, different global analytic systems will be represented as different sequences of real numbers.

Using the techniques from the mathematical discipline of functional analysis, such sequences can be reinterpreted as “positions” in a vector space. Distances can be measured from these points and, given three or more points, angles can also be measured. These distances and angles are interpreted as “degrees of opinions of typical musicality” and “agreement of such opinion” respectively.

More specifically, I will show that if two global analytic functions, $f$ and $g$, are members of a vector space with norms (a formal way of saying “distances”) and have an inner product (a concept defined below), then $f$ and $g$ satisfy the parallelogram law. The vector space is therefore imbued

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A vector space $F$ is a set that is closed under finite vector addition and scalar multiplication. I will define these terms in greater detail.
with a geometry allowing us to draw geometric inferences about the relation between \( f \) and \( g \).

### 3.1.2 Pieces of Music

Each *musical event type* is encoded by a natural number taken from the set of natural numbers \( \mathbb{N} = \{0, 1, \ldots \} \). The number 0, by convention, will denote the “null” event type.

A *musical event sequence* is a one-way\(^2\) infinite ordered sequence

\[
\bar{x} = (x_0, x_1, \ldots, x_i, \ldots)
\]

where each \( x_i \) is a musical event type from the set \( \mathbb{N} \). Recall that in Chapter 1 I defined a “piece of music” as a finite sequence of elements from \( E \). Here also a musical event sequence is called a *piece* if it has “finite support,” that is, if \( x_i = 0 \) for all but finitely many \( i \in \mathbb{N} \). In other words, a musical event sequence is a piece if it converges after finitely many terms to the constant sequence consisting of only events of the null type. The “piece” (written \( \bar{x} \)) presently defined differs from the piece in Chapter 1 in that it comprises not musical events, but *natural numbers* that represent musical events.\(^3\)

The length of a musical piece \( \bar{x} = (x_0, x_1, \ldots, x_i, \ldots) \) is denoted \(|\bar{x}|\) and is defined to be the least integer \( k \) such that \( x_i = 0 \) for all \( i > k \).\(^4\)

---

\(^2\)By “one-way” I mean that the sequence is read left to right.

\(^3\)Morris (1987) uses a similar notation to indicate complimentary set relations.

\(^4\)The notation \(|\cdot|\)” conventionally indicates “length” or “size” and will be used as such throughout this dissertation. In functional analysis, \(|f|\) indicates the “length of \( f \).” Also in one important discussion, two vertical bars indicate the “absolute value” of the formula contained inside.
The set of all musical pieces is denoted \( \mathbb{N}^{<\infty} \). Note that \( \mathbb{N}^{<\infty} \) is easily shown to be countably infinite, and hence its constituent members can be enumerated. Given an enumeration of \( \mathbb{N}^{<\infty} \)

\[
\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_j, \ldots (j \in \mathbb{N}),
\]

(where each \( \bar{x}_j \) is in \( \mathbb{N}^{<\infty} \)) I define an enumeration function \( \eta : \mathbb{N} \to \mathbb{N}^{<\infty} \) by taking \( \eta(j) = \bar{x}_j \). Note that \( \eta \) is a bijection, and hence \( \eta^{-1} : \mathbb{N}^{<\infty} \to \mathbb{N} \) is also a function.

The enumeration \( \eta \) is length-monotone if \( i \leq j \) implies \( |\bar{x}_i| \leq |\bar{x}_j| \). Therefore, a length monotone enumeration of \( \mathbb{N}^{<\infty} \) lists shorter pieces before listing longer pieces. Clearly, there are many enumerations of \( \mathbb{N}^{<\infty} \) which are length monotone.

Hereafter, fix \( \eta \) to be any such length-monotone enumeration of \( \mathbb{N}^{<\infty} \).

### 3.1.3 Global Music Analytic Systems

A global music analytic system is a function \( f \) which maps \( \mathbb{N}^{<\infty} \) into the set of all real numbers, \( \mathbb{R} \). In other words, \( f \) takes the set of all pieces of music and assigns each piece in the set a real number. Given a piece \( \bar{x} \), the value of \( f(\bar{x}) \) is interpreted as the certainty with which \( \bar{x} \) is classified as being “typically music” within the framework of the music analytic system.
Larger positive values for $f(\bar{x})$ signify that $\bar{x}$ is more typically musical within the analytic system; larger negative values for $f(\bar{x})$ signify that the $\bar{x}$ is more less typically musical within the analytic system. If $f(\bar{x}) = 0$, then the music analytic system $f$ makes no judgment about the piece $\bar{x}$.

Note that given the fixed enumeration $\eta : \mathbb{N} \rightarrow \mathbb{N}^{<\infty}$ of all pieces (which assigns each piece a natural number), and a music analytic system $f : \mathbb{N}^{<\infty} \rightarrow \mathbb{R}$ (which assigns each piece a real number), the two may be composed together to yield a function $f\eta : \mathbb{N} \rightarrow \mathbb{R}$. (Of course, the function $f\eta$ is not bijective.) Thus, a music analytic system $f$, within the context of the fixed enumeration $\eta$ of all pieces, is simply an infinite sequence of real numbers.\(^7\) Conversely, consider any infinite sequence of real numbers \(\alpha_0, \alpha_1, \ldots, \alpha_i, \ldots\) \((i \in \mathbb{N})\). Then, having fixed an enumeration $\eta$ of all pieces, the above sequence of real numbers completely defines a music analytic system $f_\alpha : \mathbb{N}^{<\infty} \rightarrow \mathbb{R}$. In the analytic system $f_\alpha$ each piece $\bar{x} \in \mathbb{N}^{<\infty}$ is assigned a value according to the rule

$$f_\alpha(\bar{x}) = \alpha_{\eta^{-1}(\bar{x})}.$$  

This observation shows that (once an enumeration $\eta$ of all pieces is fixed) it is natural to view each music analytic system as an infinite sequence of real numbers.

\(^7\)It should be clear by now that I am asserting that there exists a countably infinite set of all pieces of music (which is called $\mathbb{N}^{<\infty}$). Since a global music analytic system $(f)$ assigns each piece in that set a real number, $f$ can be represented by the resulting infinite sequence of real numbers. The requirement for $\mathbb{N}^{<\infty}$ as, at its core, mathematical.
The set of all music analytic systems is denoted

\[ F = \{ f \mid f : \mathbb{N}^{<\infty} \to \mathbb{R} \}. \]

In light of the previous remark, \( F \) may (when convenient) be considered as the set of all infinite sequences of real numbers. The set of all infinite sequence of real numbers is a well-developed subject of study, and is one aspect of the mathematical discipline of functional analysis. In what follows, I reinterpret the classical theory of functional analysis within the context of our present study concerning the properties of the set of all music analytic systems.

### 3.1.4 A Vector Space

The following discussion explains four key points. First, it explains how the set of all global analytic systems (which is called \( F \)) meets the formal requirements of a vector space. Second, I introduce an important subset of \( F \) called \( l_p(F) \). Third, if an analytic system \( f \) can be shown to be in \( l_p(F) \), then a norm or length of the system \( f \) can be defined. Fourth, being more restrictive and defining \( p = 2 \), if an analytic system \( f \) can be shown to be in \( l_2(F) \), then all of the necessary conditions have been met for making geometric inferences about the relation between \( f \) and any other system in \( l_2(F) \).

I begin by defining addition on \( F \). Given two music analytic systems \( f, g \in F \) I define the music analytic system \( (f + g) \) by making it act on a piece \( \bar{x} \in \mathbb{N}^{<\infty} \) as follows:

\[ (f + g)(\bar{x}) = f(\bar{x}) + g(\bar{x}). \]
Now it follows from results in functional analysis that \((F, +)\) is a vector space over \(\mathbb{R}\), which is to say:

**Closure.** For all \(f, g \in F\), their sum \((f + g) \in F\).

**Identity.** Let \(I \in F\) be the function that is identically 0 on all of \(\mathbb{N}^{<\infty}\). Then for all \(f \in F\), \(f + I = I + f = f\). Moreover, \(I\) is the only element of \(F\) having this property.

**Commutative.** Since the standard \(+\) is commutative on \(\mathbb{R}\), it follows that the defined \(+\) operation is commutative on \(F\).

**Associative.** Since the standard \(+\) is associative on \(\mathbb{R}\), it follows that the defined \(+\) operation is associative on \(F\).

**Inverses.** Given \(f \in F\), let \(f^{-1} \in F\) be the function that is defined by \((f^{-1})(\bar{x}) = -f(\bar{x})\) for each \(\bar{x} \in \mathbb{N}^{<\infty}\). Then \(f + f^{-1} = f^{-1} + f = I\). Moreover, for any given \(f\), the prescribed \(f^{-1}\) is the only element of \(F\) having this property.

**Scalars.** Given \(f \in F\), and \(c \in \mathbb{R}\), let \(cf\) to be the function defined by \((cf)(\bar{x}) = cf(\bar{x})\) for each \(\bar{x} \in \mathbb{N}^{<\infty}\). Then \(cf \in F\).

**Distributivity.** Given \(c \in \mathbb{R}\), and \(f, g \in F\), \(c(f + g) = (cf) + (cg)\).

Since \(F\) is an additive abelian group under \(+\), I will often denote \(f + g^{-1}\) as simply \(f - g\). Of paramount importance to the following discussion is the identity function \(I\). This is the analytic system that has no opinion about any piece. In the discussion that follows, \(I\) will be the unique point in \(F\) to which all other analytic systems are compared. I will introduce the idea of a “distance” from \(I\) and an angle formed at \(I\).
3.1.5 \( l_p(F) \)

For each integer \( p > 0 \), define \( l_p(F) \subset F \) to be the set of all music analytic systems \( f \) for which

\[
\sum_{i \in \mathbb{N}} |f(\eta(i))|^p
\]

is finite. In other words, viewing \( f \) as a sequence of real numbers, \( f \) is in \( l_p(F) \) if and only if the summation of the \( p^{th} \) powers of elements of the sequence converges.\(^8\)

3.1.6 \( l_p \)-norms

Given a vector space \((F, +)\) over \(\mathbb{R}\), for each integer \( p > 0 \) define the \( p \)-norm on the subset \( l_p(F) \) as a function \( l_p : l_p(F) \to \mathbb{R}^+ \) by taking

\[
l_p(f) = \left( \sum_{i \in \mathbb{N}} |f(\eta(i))|^p \right)^{1/p}.
\]

I shall hereafter denote the value \( l_p(f) \) as \( |f|_p \).

\( |f|_p \) is read as the \( p \)-norm of \( f \) and describes the function that defines the length of \( f \) in \( l_p(F) \).

Now it follows from results in functional analysis that the length function \( |\cdot|_p \) (of which \( |f|_p \) is one example) is a norm on \( l_p(F) \), which is to say that

1. For all \( f \in l_p(F) \), \( |f|_p \geq 0 \) and \( |f|_p = 0 \) if and only if \( f = I \).

It is easy to see that \( |I|_p = 0 \). In the reverse direction,

\(^8\)\( l_p \) spaces are spaces of \( p \)-power integrable functions. The integral test for convergence is a method used to test infinite series of nonnegative terms for convergence.
consider a function \( f \in l_p(F) \) for which \(|f|_p = 0\). Then by definition of the \( p \)-norm, it is known that

\[
\sum_{i \in \mathbb{N}} |f_\eta(i)|^p = 0
\]

and so it must be that for each \( i \in \mathbb{N} \), \( f_\eta(i) = 0 \). Hence \( f = I \). This shows that the unique element of \( l_p(F) \) having \( p \)-norm equal to 0 is the music analytic system which makes no judgment about any piece.

2. \( |cf|_p = c|f|_p \).

3. \( |f + g|_p \leq |f|_p + |g|_p \) for all \( f, g \in l_p(F) \).

### 3.1.7 \( l_2 \)-norms

Arguably the most important \( l_p \) norm is what is often called the Euclidean norm where \( p = 2 \). In general, the \( p \)-norm of a music analytic system \( f \) is a quantitative measure of the extent to which \( f \) decides the coherence of pieces in \( \mathbb{N} \). Intuitively, the \( p \)-norm of \( f \) is a measure of how “far” \( f \) is from \( I \), the analytic system which passes no judgment about any piece.

Given that the length function \(| \cdot |_p \) is a norm on \( l_p(F) \), \( l_p(F) \) can be immediately converted into a metric space by defining a binary operation (i.e., a pairwise function) \( d_p : l_p(F) \times l_p(F) \to \mathbb{R} \) as

\[
d_p(f, g) = |f - g|_p
\]

for each \( f, g \in l_p(F) \). The metric function \( d_p \) then permits us to measure.

\footnote{There is no relation between the observation that Euclidean space is \textit{two} dimensional and \( p = 2 \).}
the distances between the music analytic systems in \( l_p(F) \) in a consistent manner.

Let us define another binary operation on \( l_p(F) \) as follows. Given \( f, g \in l_p(F) \), let
\[
\langle f, g \rangle = \sum_{i \in \mathbb{N}} f(\eta(i)) \cdot g(\eta(i))
\]
It follows from results in functional analysis that when \( p = 2 \), the operation \( \langle \cdot \rangle \) is a real inner product on \( l_2(F) \), which is to say that

1. For all \( c, c' \in \mathbb{R} \), \( f, f', g \in l_2(F) \), we have that \( \langle cf + c'f', g \rangle = c\langle f, g \rangle + c'\langle f', g \rangle \).

2. For all \( f, g \in l_2(F) \), \( \langle f, g \rangle = \langle g, f \rangle \).

3. For all \( f \in l_2(F) \), \( \langle f, g \rangle = 0 \) if and only if \( f = I \).

The inner product \( \langle \cdot \rangle \) imbues \( l_2(F) \) into a vector space with a geometry. This permits us to conduct geometric inference within the space of all music analytic systems. I give a few illustrative examples of such inferences.

Given two music analytic systems \( f, g \) in \( l_p(F) \) one can compute the angle \( \theta_{f,g} \) that is made at system \( I \) by the two segments spanning \( I \) to \( f \), written \( \overline{IF} \), and \( I \) to \( g \), written \( \overline{IG} \). This is angle is given by the relation
\[
\cos(\theta_{f,g}) = \frac{\langle f, g \rangle}{|f|_2|g|_2}.
\]
Systems \( f \) and \( g \) satisfy the parallelogram law:
\[
(|f + g|_2)^2 + (|f - g|_2)^2 = (2|f|_2)^2 + (2|g|_2)^2.
\]
A third music analytic system \( h \) lies on the segment \( \overline{IF} \) if \( \theta_{h,g} = \theta_{fg} \) and \( |h|_2 \leq |f|_2 \). If \( h \) indeed lies on \( \overline{IF} \), then \( |f - h|_2 + |h - I|_2 = |f|_2 \).
3.2 Representations in $l_2(F)$

In this section I discuss strategies for encoding global music analytic functions by elements in $l_2(F)$. Having described the necessary conditions for a geometry on $F$, I now give examples that will clarify and expand upon specific formal statements. Figure 3.1 describes the set of all music analytic systems $F$. The space $l_2$ exists as a subset $F$. The following example explains what is meant by representing $f$ and $g$ as points within $l_2(F)$ and the importance of their relation to $I$.

I define musical event types as some kind of musical data. Event types are encoded by natural numbers and our music analytic systems are functions that act on sequences of these encoded events. I postulate that any and every musical event type can be represented by $\mathbb{N}$. In this example I define an alphabet of three event types $\mathcal{A} = 0, 1, 2$ where $0 = a$ null event, $1 = a$
CHAPTER 3. GLOBAL MUSIC ANALYTIC SYSTEMS

silent event, and \(2 = \) a sound event. These events are composed together into musical event sequences that represent all of the pieces that can be composed using \(A\). Formally, sequences are only pieces of music if they have finite support. More informally, musical event sequences are pieces because they end or converge to the constant sequence of null events. The null event must therefore always be included in the set of coded events from which all pieces are composed. In Table 3.1, I enumerate all of the possible pieces composed from \(A\) and list them in order of increasing size (length monotone enumeration). For notational convenience, I have omitted null events since the length of pieces is defined up to, and not beyond, the beginning of the sequence of null events.\(^{10}\)

For the sake of example, I define an analytic system \(j\) that examines every piece in \(\mathbb{N}^{<\infty}\) (the set of all pieces) and returns a real number for each piece that represents the analysis of that piece’s status as music as determined by \(j\)’s design. System \(j\) examines each piece and returns 1 for every sound event in a piece and otherwise returns 0. In other words, if a piece has no sound, then \(j\) has “no opinion” on its musicality. If a piece has three sounds, then \(j\) returns 3 and “considers” such a piece more typically musical than the piece whose analysis is 1.

I define a second analytic system \(k\) that returns a value of -1 if two like-events are consecutive and otherwise returns 1. For example, under \(k\), the piece “121” would return 1 and the piece 112 would return -1. In other words, under \(k\) we understand “121” to be music and “112” to be not music.

\(^{10}\)Null events are not included “within” a piece—they are used to identify the end of a piece.
Table 3.1: Analysis of $\mathbb{N}$ under $j$ and $k$.

<table>
<thead>
<tr>
<th>$\mathbb{N}$</th>
<th>$\eta$</th>
<th>$j$</th>
<th>$k$</th>
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<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>0.0</td>
<td>1.0</td>
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<tr>
<td>1</td>
<td>2</td>
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<td>2</td>
<td>11</td>
<td>0.0</td>
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<tr>
<td>6</td>
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<td>0.0</td>
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<td>7</td>
<td>112</td>
<td>1.0</td>
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<tr>
<td>8</td>
<td>121</td>
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<td>211</td>
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<td>10</td>
<td>212</td>
<td>2.0</td>
<td>1.0</td>
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<tr>
<td>11</td>
<td>221</td>
<td>2.0</td>
<td>-1.0</td>
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<tr>
<td>12</td>
<td>222</td>
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</tbody>
</table>
Postulate 4. Two musical event sequences related by rotation, translation, or scaling are not equivalent.

Table 3.1 shows the analyses of the first 12 possible pieces. We see that $j(22) = 2.0$, while $k(22) = -1.0$. In the simple case presented here, outputs of $j$ and $k$ are in fact real numbers. Recall that $j$ and $k$ are members of the set of all analytic systems $F$. Having defined addition on $F$, $j$ and $k$ exist within the vector space over $\mathbb{R}$ called $(F, +)$ where $F$ is an additive abelian group.

In pursuit of our geometry, it must be determined whether or not $j$ and $k$ are members of $l_2(F)$ and are thereby imbued with a norm $|j|_2$ and $|k|_2$ respectively. As discussed earlier, if $j$ and $k$ are members of $l_2(F)$ then there is a real inner product, permitting us to conduct geometric inference between $j$ and $k$ within $F$. Geometric inference is the basis for meta-analysis. The geometric relation enjoyed by members $l_2(F)$ facilitates unique and tangible metaphors for comparing music analytic systems.

In order for $j$ and $k$ to be in $l_2(F)$ the summation of the squares of the elements of each sequence must converge. We can easily see that they do not. They in fact diverge. As $\eta$ increases, so do the sums of $j^2$ and $k^2$. We will return to convergent series, but for now I can show a simple solution to this problem. In order to get $j$ and $k$ into $l_2(F)$ a limit to the length of pieces that $j$ and $k$ analyze is set. In this example, pieces longer than length 3 are not considered. This makes $j$ and $k$ finite, forcing a convergence, and thus members of $l_2(F)$. Truncation in this case says that $j$ and $k$ return zero and thus have no opinion on any pieces longer than length 3. Table 3.2 expands Table 3.1 to include sums, differences and the inner products of $j$
and $k$.

Putting truncation into a practical context, many analytic systems have a limit as to what they consider to be music. Imagine listening to a piece of music on the radio. Via some music analytic system, we may consider some auditory stimulus emitted from the radio as music. The interpretation of that auditory input as music, under that particular analytic system, however, ceases when we turn off the radio. The interpretation of sound as music does not continue to include “ambient” sounds of birds, cars, vacuum cleaners, etc. That is not to say that those sounds cannot be understood as being music. The question is whether or not those sounds are the same music or are included in the same musical experience as the sounds emitted from the radio. For the purpose of this example, I will assert that they are not. Depending on the analytic system used, there can be very clear beginnings and terminations to musical moments. The concept of truncation is not only practical for admitting non-converging analytic functions into $l_2(F)$, but it is also “musical” in a familiar sense. This leads to the following

**Postulate 5.** Music analytic systems have finite support and no system will have opinions about infinitely many things.

Functions *can be* defined so that their sums do converge. A sequence $x_0, x_1, x_2, \ldots$ in a metric space $(X, d)$ is a convergent sequence if there exists a point $x \in X$ such that, for every real number $\epsilon > 0$, there exists a natural number $N$ such that $d(x, x_n) < \epsilon$ for all $n > N$. The point $x$, if it exists, is

---

11The very distinction between “ambient” or “distant” sounds and “close” sounds is a function of an analytic system that metaphorically allows (or brings) sounds into a frame of reference or keeps them out.
unique, and is called the limit point or limit of the sequence. One can also say that the sequence \(x_0, x_1, x_2, \ldots\) converges to \(x\).

For example, we can invent a function that converges to a limit. Referring to pieces composed from our event alphabet \(\mathcal{A} = 0, 1, 2\), let \(i\) consider as musical only pieces that contain 2s exclusively. It has no opinion on pieces that contain 1. For each piece, \(i\) scores \((\frac{1}{2})^n\) where \(n = \text{the number of 2s in the piece}\). Naturally, this models the well-understood geometric series that converges at a limit 1.

### 3.2.1 Norms

Recall that the length of a function \(|\cdot|_2\) is defined
\[
|f|_2 = \left( \sum_{i \in \mathbb{N}} |f \eta(i)|^2 \right)^{1/2}.
\]

Returning to our example, \(|j| = \sqrt{27} = \text{ and } |k| = \sqrt{13}\). The lengths of \(\overline{Ij}\) and \(\overline{Ik}\), or simply \(j\) and \(k\), are indicative of the degree to which each analytic system understands pieces in \(\mathbb{N}\) to be typically musical. The norms are the distances of \(j\) and \(k\) from \(I\), the unique point in \(l_2(F)\) of no opinion. In addition, \(I\) is the generation point of two line segments \(\overline{Ij}\) and \(\overline{Ik}\) between which is an angle \(\theta_{j,k}\) which is determined by
\[
\cos(\theta_{\sqrt{27}, \sqrt{13}}) = \frac{\langle \sqrt{27}, \sqrt{13} \rangle}{\sqrt{27} * \sqrt{13}}.
\]

The reader can further verify that \(j\) and \(k\) satisfy the parallelogram law:
\[
(\sqrt{27} + \sqrt{13})^2 + (\sqrt{27} - \sqrt{13})^2 = 2(\sqrt{27})^2 + 2(\sqrt{13})^2
\]
The comparison of music analytic systems $j$ and $k$ is expressed by two relations: length of line segments and angles between line segments.

### 3.2.2 Comparison of two functions

It’s clear that $j$ and $k$ are related to $I$ independently from each other. On the one hand, we may favor music analytic systems that interpret pieces as more typically musical. If this is the case, then $j$ proves to be “better.” On the other hand we may favor analytic systems that have the greatest coverage. In this case, $k$ proves “better” because it “has an opinion” on every piece, whereas $j$ does not. Agreement between $j$ and $k$ as to whether or not a piece is or is not music is a function of the angle $\theta_{j,k}$. Since norms are always positive distances from $I$, it is the angle formed at $I$ that determines,
in effect, the degree of agreement.

For example, as shown in Figure 3.3 (and here I return briefly to the “general functions” $f$ and $g$), for opinions on disjoint collections $< f, g >= 0$ and therefore $\cos \theta = < f, g >= 0 = 90^\circ$. In other words, if $f$ “thinks” only pieces by Babbitt are or are not music and it has no opinion about other pieces, and if $g$ “thinks” only pieces by Josquin are or are not music and has no opinion about others, then $f$ and $g$ will be disposed at a $90^\circ$ angle. The degree to which $f$ and $g$ think their respective pieces are or are not music does not change the angle. It only changes the lengths of $f$ and $g$.

Figure 3.4 shows the arc of agreement between two music analytic systems $f$ and $g$ when the value $\theta_{f,g}$ is varied from 0 to 180. The value of $\theta_{f,g}$ clearly comments on two parameters: (1) the kind of opinion each function has on a single set of pieces and (2) the amount of intersection between the set on which $f$ has some opinion and the set on which $g$ has some opinion. As the angle approaches $0^\circ$ or $180^\circ$ we know that the size of the set on which both $f$ and $g$ have opinions approaches the number of pieces in $\mathbb{N}^{<\infty}$. As
Figure 3.4: The arc of agreement for $f$ and $g$

an angle moves past $90^\circ$, mutual opinions begin to either increasingly agree ($< 90^\circ$) or disagree ($> 90^\circ$).

In the above example, collections are not disjoint. Systems $j$ and $k$ have opinions on the majority of pieces. That the angle is just past $90^\circ$ indicates that systems are in mild disagreement.

3.2.3 Implications of Concatenated Sequences

Since any piece $\bar{x}$ is a member of $\mathbb{N}^{<\infty}$ (the set of all pieces), then any subset of $\bar{x}$ is also in $\mathbb{N}^{<\infty}$ and is also a piece. This says that taken as a whole, members of $\mathbb{N}^{<\infty}$ can be considered as segments of a larger and possibly “complete” piece. Thinking of the $\eta$ ordering induced on $\mathbb{N}^{<\infty}$ as an ordering of segments of a larger piece has a familiar resonance. The ordering function $\eta$ is therefore considered a member of $\Omega$, the set of all ordering functions. We are free to induce different orderings on $\mathbb{N}^{<\infty}$ for different heuristic effects.

In other words, $\eta$ is a monotonic ordering function—it lists shorter pieces before longer pieces. We can, however, order $\mathbb{N}^{<\infty}$ any way we like. For example, it might be of analytic import to order $\mathbb{N}^{<\infty}$ such that pieces with
Table 3.2: Sums, differences, and inner product of $j$ and $k$.

<table>
<thead>
<tr>
<th>N</th>
<th>$\eta$</th>
<th>$j$</th>
<th>$k$</th>
<th>$j + k$</th>
<th>$j - k$</th>
<th>$\langle j, k \rangle$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>0.0</td>
<td>1.0</td>
<td>1.0</td>
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<td>0.0</td>
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<tr>
<td>1</td>
<td>2</td>
<td>1.0</td>
<td>1.0</td>
<td>2.0</td>
<td>0.0</td>
<td>1.0</td>
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<tr>
<td>2</td>
<td>11</td>
<td>0.0</td>
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<td>0.0</td>
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<tr>
<td>6</td>
<td>111</td>
<td>0.0</td>
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<td>1.0</td>
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<td>7</td>
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<td>9</td>
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<td>1.0</td>
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<td>-1.0</td>
<td>1.0</td>
<td>3.0</td>
<td>-2.0</td>
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<tr>
<td>12</td>
<td>222</td>
<td>3.0</td>
<td>-1.0</td>
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<td>4.0</td>
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</tbody>
</table>
sound events as their first event are listed before pieces with silent events as their first event. Different orderings of $\mathbb{N}^{<\infty}$ do not affect the norms of music analytic systems. They may, however, support or inhibit particular conceptualizations of $\mathbb{N}^{<\infty}$.

### 3.2.4 Comparing Subsets of $\mathbb{N}^{<\infty}$

Comparing analyses of two pieces of music using a single music analytic system is a natural extension of the geometry on $F$ and approximates type I comparisons. As before, let $\mathcal{A} = 0, 1, 2$ where $0 =$a null event, $1 =$a silent event, and $2 =$a sound event. These events are composed together into musical event sequences that represent all of the pieces that can be composed using $\mathcal{A}$. Consider the sequences pieces to have finite support. Table 3.3 gives an $\eta$ enumeration of the pieces composed from $\mathcal{A}$.

In order to preserve geometric inference, the single analytic system $j$ is treated as two systems, $j_1$ and $j_2$, each acting on different parts of $\mathbb{N}^{<\infty}$. How $j_n$ looks at individual pieces in $\mathbb{N}^{<\infty}$ will be the same for any $n$ with the exception of some differentiating argument or arguments that distinguish $j_1$ from $j_2$. Different “pieces” then are actually variations of a single function designed to look at separate parts of $\mathbb{N}^{<\infty}$.

Let $j_n$ examine each piece and return 1 for every sound event in a piece and otherwise return 0. In other words, if a piece has no sound, then $j_n$ has no opinion on its musicality. If a piece has three sounds, then $j_n$ returns 3.0 and “considers” such a piece more typically musical than the piece whose analysis is 1.0.

I continue by extending the definition of an analytic system $j_1$ so that
Table 3.3: Analysis of $N$ under $j_1$ and $j_2$.

<table>
<thead>
<tr>
<th>N</th>
<th>$\eta$</th>
<th>$j_1$</th>
<th>$j_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>0.0</td>
<td>-1.0</td>
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<tr>
<td>1</td>
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<td>4</td>
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<td>1.0</td>
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<td>5</td>
<td>22</td>
<td>2.0</td>
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<td>112</td>
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it examines only pieces in $\mathbb{N}^{<\infty}$ whose length is $\leq 2$. For any piece whose length is shorter than 1 or greater than 2, $j_1$ returns -1.0. The definition of an analytic system $j_2$ is also extended so that it examines every piece in $\mathbb{N}$ whose length is less than 3, $j_2$ returns -1.0.

In this example $|j_1| = \sqrt{14}$ and $|j_2| = \sqrt{26}$. The lengths of $j_1$ and $j_2$, are indicative of the degree to which the analytic system $j_n$ understands associated subsets of $\mathbb{N}^{<\infty}$ to be typically musical. Again, the norms are the distances of $j_1$ and $j_2$ from $I$, the unique point in $l_2(F)$ of no opinion. In addition, $I$ is the generation point of two line segments $\overline{IJ_1}$ and $\overline{IJ_2}$ between which is an angle $\theta_{j_1,j_2}$ which, as determined by

$$\cos(\theta_{\sqrt{14},\sqrt{26}}) = \frac{\langle \sqrt{14}, \sqrt{26} \rangle}{\sqrt{14} \ast \sqrt{26}},$$

says that $|j_1|$ and $|j_2|$ are disposed at a $141.83^\circ$ angle. This relation is shown in Figure 3.5.
Chapter 4

Local Music Analytic Systems

4.1 Comparing the Tonal Models of Lerdahl and Bharucha, et. al.

The preceding chapter discussed what a geometry of *global music analytic systems* looks like and how it works. In what follows, we turn our attention to *local music analytic systems* and develop a contextual comparative methodology for pieces of music interpreted by such systems. Local systems have different structures than the global systems (they say different things) and comparison is carried out in a different fashion. Recall that a local music analytic system $(t)$ is defined $t(p) = q$, where $p$ is a sequence of music event types with finite support (formally a *piece of music*) and $q$ is another sequence of musical events—often descriptors intended to comment
on p. In what is presented below, I give a detailed example of a comparison of two local analytic systems. Comparing local systems is different from comparing global systems principally in that there is no single methodology that can compare all local systems (as is the case with global systems). I assert that local systems can be compared only if they can be modeled by the same formal structure (e.g., groups or metric spaces). Local systems can be informally associational or formally transformational. Tonal models (Lerdahl, 2001), neo-Riemannian hexatonic systems (Cohn, 1996), semiotic models (Tarasti, 1994), Schenker theory (Schenker, 1979), atonal set-theory approaches (Forte, 1973; Morris, 1987), and generalized interval systems (Lewin, 1987) are some examples of local music analytic systems.

I have imbued the comparison of global systems with the metaphor of distance, and a number of local music analytic systems employ the same metaphor as an integral part of their design (Lewin, 1987; Lerdahl, 2001). These systems are “geometrically” oriented insofar as they assign extrasyystematic meaning to “distances” (or “intervals,” or “spans”), not between two systems as I have done in Chapter 3, but to descriptors within a single system. In this chapter I focus on approaches to representations of tonal hierarchy and its constituent harmonic relations. Such approaches involve the collection and modelling of data from experiments in music cognition. The multidimensional-scaling models of Bharucha and Krumhansl (1983), and Deutsch (1999), for example, seek to encode cognitive relationships between chords within a single key area (region) as euclidean distances. The work of Heinichen (1728), Kellner (1737), and Weber (1824), when formalized, is more speculative and develops geometric representations of relationships between different key areas (regions) through their placement.
within a multidimensional space.

This chapter presents a comparative analysis of two local music analytic systems. In practice, these two systems are “intra-regional” (within a single key) tonal models. What a global music analytic system (as defined above) does and what a local music analytic system does are theoretically two completely different things—an important distinction to remember as we transition from the discussion of comparing global analytic systems to comparing local analytic systems. A global system decides the degree to which pieces are typically musical and, as I have shown, has the capacity to address the typical musicality of large sets of pieces at one time. A local system, on the other hand, makes a commentary on individual pieces of music.

To say “a piece of music is composed using only tonic and dominant harmonies,” is a local statement about music using the language of rudimentary harmonic theory. To say “alternating occurrences of tonic and dominant harmonies in pieces is more typically musical than alternating occurrences of tonic and mediant harmonies,” is a global interpretation of pieces of music made by a music analytic system in conjunction with the concepts and language of a theoretical model. That said, each local analytic system that comments on single pieces has the capacity to be correlated with a global analytic system that comments on every and all pieces. Let us proceed by examining two local systems both of which are tonal models: Lerdahl’s tonal pitch space (L) model and a similar model (insofar as it models the same set of objects) based on the multidimensional-scalings of Bharucha and Krumhansl (BK). This chapter presents a theory of comparison and does not critique their representation of tonality in terms of
adequacy or completeness.

What is significant about the following comparative analytic approach is that it is systematically based. I am presenting a solution to the problem of type II comparisons (discussed in Chapter 1). In order to be able to compare different local analytic systems, a higher-level formal equivalence must be defined. As it happens, both systems can be modeled by the formal structure of a metric space.

In order to make the interpretation and subsequent comparison of $L$ and $BK$ cognitively tangible, I ask “are $L$ and $BK$ ‘considering’ the same kinds of things in the same way?” The comparative treatment of analytic systems as others have developed them, such as $L$ and $BK$, raises questions about the equivalence of such systems. Pre-developed analytic systems are not designed *per se* to be compared with each other. In order to achieve a conceptually relevant comparison, the measure of unit distance for one analytic system should be the same as for the other. The specific question of unit-measure equivalence will be dealt with in the following chapter.

### 4.1.1 Rationale for Comparing L and BK

Fred Lerdahl’s tonal pitch space ($L$) model (2001) approximates the cognitive perceptual relation between chords by providing a combinatorial procedure for computing the distance value between two arbitrary chords. The procedure employed by the $L$ model is informed by experimental data and plausible hypotheses about how we perceive tonal relations. The $L$ model is able to describe relations both between chords within a region (e.g., Bharucha and Krumhansl) and between regions themselves (e.g., Heinichen,
et al.); $L$ thus bridges these two representational classes.

Because of the influence of experimental data on the $L$ model, we would expect a high correlation between experimental data and analyses of intra-regional chord progressions generated by the $L$ model. The value of such a comparison is clear. If the $L$ model posits a hypothesized model of perception, then it would be interesting to know if and by how much it differs from the experimental data it claims to approximate. I shall focus on the intra-regional relation descriptions of $L$ and their counterparts in $BK$.

4.2 Graphical Models and Associated Metric Models

Local analytic systems have their own structures that can have formal properties. For example $L$ and $BK$ can be represented as graphical models of key areas. These graphical models can be extended to finite metric spaces with metrics on the same set of chords—e.g., the seven diatonic triads of a major key. Let us begin with a general discussion of graphical models and their “associated” metric models.

Given a key area (or “key region”) $R$, a graphical intra-regional model (within a single key) $G_R = (V, E, d)$ is graph $(G)$ whose vertex set $V$ consists of every chord in $R$. $E$ is the set of all edges. Two chords (and by “chords” I mean triads from the key area $R$) $c, c' \in V$ are said to be connected by an edge if $(c, c') \in E$, and in this event, the weight of this edge is determined by the positive-valued function $d : E \to \mathbb{R}^{>0}$. This makes $d$ the function that associates every edge in $G_R$ with a positive real number.
I say that $G_R$ is loop-free meaning that $(c, c') \in E$ implies $c \neq c'$. In other words, a chord cannot be connected to itself. I say that $G_R$ is simple asserting that $E$ is indeed a set (without multiplicities). Finally, $G_R$ is undirected implying that $(c, c') \in E$ if and only if $(c', c) \in E$, and in this case $d(c, c') = d(c', c)$.

Given a graphical intra-regional model $G_R = (V, E, d)$, there is a natural extension of the function $d$ to a closely related “global distance function” $d^*$. Recall that the edge weights in $G_R$ are determined by the positive-valued function $d : E \to \mathbb{R}^{>0}$. The global distance function $d^*$ assigns a non-negative real number to each pair of vertices. Using $d^*$, I can convert $G_R$ into a metric space by defining $d^*$ as follows: Given two vertices $c, c'$ (representing two chords) in the set $V$, let $P_{c,c'}$ be the set of all (non self-intersecting) paths connecting $c$ to $c'$ in $G_R$. Each path $p$ in $P_{c,c'}$ can be viewed as a sequence of edges $p = (e_1, e_2, \ldots, e_{|p|})$, where $e_i \in E$ ($i = 1, \ldots, |p|$). I define $d(p) = \sum_{i=1}^{|p|} d(e_i)$ (and adopt the convention that $d(p) = 0$ for paths of length 0). In this manner, I extend the definition of $d$ to all of $P_{c,c'}$. Now define

$$d^*(c, c') \overset{def}{=} \min_{p \in P_{c,c'}} d(p).$$

In other words, $d^*(c, c')$ is the length of the shortest (what I later call the minimal separation) path $(d(p))$ from one chord to another in $G_R$. Because a graphical intra-regional model is required to be a connected graph, $d^*$ assigns

\footnote{By definition, a metric space is a set $M$ with a global distance function (the metric $d^*$) that, for every two points $x, y$ in $M$, gives the distance between them as a non-negative real number $d^*(x, y)$.}
a finite non-negative value to every pair of vertices in $V$. The definition of $d^*$ assures that every pair of chords is labeled by the shortest path between the two chords it connects.

The above definition of $d^*$ permits us to convert a graphical intra-regional model $G_R = (V, E, d)$ into a metric space $M_R = (V, d^*)$. This is referred to as the associated metric intra-regional model of $R$.

The reader may verify that the metric axioms are satisfied by $M_R$, since given any $x, y, z \in V$,

1. REFLEXIVITY: $d^*(x, y) = 0$ IF AND ONLY IF $x = y$,
2. SYMMETRY: $d^*(x, y) = d^*(y, x)$,
3. TRIANGLE INEQUALITY: $d^*(x, y) + d^*(y, z) \geq d^*(x, z)$.

It is important to note that by the construction above, every graphical model of $R$ gives rise to a unique metric model. The correspondence is not bijective, however, since several graphical models may give rise to the same metric model. The simplest example of this is to consider a triangle with edges weighted 2, 1, and 1, and a chain of three vertices with two edges weighted 1 and 1. Both graphs, though different, give rise to the same metric model.

In light of the previous remark, note that the set of all metric models of a region $R$ is no larger than the set of all graphical models. Initially, then, the discussion will be restricted to metric models over $R$. Accordingly, let $M_R$ be the set consisting of all $M_R$ (metric models of $R$) derived from $G_R$ (graphical models of $R$).

The following notation is used as it will be useful in later arguments
(both in this chapter and in the following chapter). Given a key region $R$ and a model $M_R = (V, d^*)$ from $\mathcal{M}_R$ I denote the minimal (resp. maximal) separation as $\sigma^\text{min}_R(M_R)$ (resp. $\sigma^\text{max}_R(M_R)$) and define these quantities by

$$\sigma^\text{min}_R(M_R) \overset{\text{def}}{=} \min_{v_1, v_2 \in V} d^*(v_1, v_2), \quad v_1 \neq v_2$$

$$\sigma^\text{max}_R(M_R) \overset{\text{def}}{=} \max_{v_1, v_2 \in V} d^*(v_1, v_2), \quad v_1 \neq v_2$$

In other words, $\sigma^\text{min}_R(M_R)$ is the smallest distance ($d^*$) between two chords (written as $v_1, v_2$) in $R$ where $v_1$ is not the same as $v_2$. For $\sigma^\text{max}_R(M_R)$, $d^*$ is the largest distance between two chords in $R$.

### 4.2.1 $L$ and $BK$ as Intra-Regional Metric Models

In the following section I discuss the properties and design of the $L$ and $BK$ as intra-regional metric models. While both models assign values to the relationships between pairs of chords ($d^*(v_1, v_2)$ in the preceding discussion) in a single key, both do so in different ways and, as I mentioned above, for different reasons. I examine the original designs of $L$ and $BK$ and, in the case of $BK$, discuss the issue of “reformatting” data in order to represent it as a metric model.
Lerdahl Tonal Pitch Space ($L$)

Using a large body of empirical evidence, Lerdahl (2001) created an algebraic model for quantifying the distance between any two chords. He dwells mainly on triads, but considers other sonorities as well. Lerdahl’s $L$ model assigns a number to each chord pair in a single key or region. That number is the intra-regional chord-pair “distance” as defined by the chord distance rule. Following Lerdahl:

**CHORD DISTANCE RULE:** $\delta(x \rightarrow y) = j + k$, where $\delta(x \rightarrow y)$ is the distance between chord $x$ and chord $y$; $j$ is the number of applications of the chordal circle-of-fifths rule needed to shift $x$ into $y$; and $k$ is the number of distinctive pcs in the basic space of $y$ compared to those in the basic space of $x$.(2001)

Figure 4.1 shows the basic space and the relation between I and V. The basic space is for the V chord and the x’s below the space are the positions of the source chord I.

First, a region is determined by affixing a major scale to the universal chromatic space. Second, triadic structures are overlaid on the diatonic space in a weighted fashion, reflecting the perceptual hierarchy of root, fifth, and third. This is the “basic space” of a triad. Finally, triad structures are shifted to different positions on the diatonic space. The distance ($\delta$) between two triads $X$ and $Y$ is number of diatonic fifths moved plus the number of pcs ($p$) in the basic space of a chord $X$ that are unique to $X$ plus those that intersect with $Y$: if a $p \in X$ and $Y$ (as in the case of 7), then only the “highest” occurrence is counted (shown by the underscore). If a $p \in X$ and
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Figure 4.1: Lerdahl’s L model (2001).

\[ Y \], then every occurrence of \( p \) is counted (2001, 55-58).

For a single region, the pairwise chord distances are given in Table 4.1.

**Bharucha-Krumhansl Space (BK)**

Table 4.2 shows the results of the 1983 experiment by Bharucha and Krumhansl in which all possible pairs of diatonic triads from a single major key were rated by subjects in terms of their perceived relatedness (1983). The higher the number between two chords, the more strongly they are associated. Bharucha and Krumhansl’s experiments considered ordered sequences of chords.

The symmetrical regularity of Lerdahl’s L model contrasts with the irregularity of the findings of Bharucha and Krumhansl. Whereas the L model produces symmetrical relations between chord pairs, Bharucha and Krumhansl found a significant ordering effect. In Lerdahl’s basic intra-regional model, interval-cycle 7 plays a central organizing role. A problem
is created when perceptually important relations are generalized and used to model perceptually less-important relations. For example, is the relationship between vii and IV the same as that between ii and V? There are some significant differences between the Bharucha and Krumhansl (BK) space and the Lerdahl (L) space. For instance, in BK space, in any order, the progression I to ii is perceptually stronger (i.e. I is “closer” to ii) than the progression I to iii.

We must keep in mind that both models are products of different lines of questioning. The relations they describe are the result of different methodologies designed for different reasons and therefore they show different kinds of information. Nevertheless, they describe relations between the same set of objects in similar ways. Furthermore, Lerdahl claims that there is a correlation between the relations his model describes and the relations that have been (and could be) described by experimental results. The idea of perceptual “closeness” is extended to our idea of “typical musicality.” If L or BK claims that chords \(x\) and \(y\) are more strongly or closely related than \(x\) to \(z\), then the progression of \(x\) to \(y\) (and vice versa) is said to be more typically musical than the progression of \(x\) to \(z\).

Comparing these two models requires formal similitude. We can see that L, as a graph of the chords of a key, can be easily converted into an associated metric space. However, it is clear from the above discussion of BK and from looking at the data in 4.2 that the distance \(d\) between chord pairs in the BK model is quite different from the undirected, shortest-path value of \(d^*\) needed to represent BK as a metric model. Furthermore, in terms of meaning, the values in BK are inversely related to the corresponding values in L.
Table 4.1: Theoretical Harmonic Relations from Lerdahl (2001).

<table>
<thead>
<tr>
<th>First Chord</th>
<th>Second Chord</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>0</td>
</tr>
<tr>
<td>II</td>
<td>8 0</td>
</tr>
<tr>
<td>III</td>
<td>7 8 0</td>
</tr>
<tr>
<td>IV</td>
<td>5 7 8 0</td>
</tr>
<tr>
<td>V</td>
<td>5 5 7 8 0</td>
</tr>
<tr>
<td>VI</td>
<td>7 5 5 7 8 0</td>
</tr>
<tr>
<td>VII</td>
<td>8 7 5 5 7 8 0</td>
</tr>
</tbody>
</table>

Table 4.2: Perceived Harmonic Relations from Bharucha-Krumhansl (1983).

<table>
<thead>
<tr>
<th>First Chord</th>
<th>Second Chord</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>0 5.1 4.78 5.91 5.94 5.26 4.57</td>
</tr>
<tr>
<td>II</td>
<td>5.69 0 4.0 4.76 6.1 4.97 5.14</td>
</tr>
<tr>
<td>III</td>
<td>5.38 4.47 0 4.63 5.03 4.6 4.47</td>
</tr>
<tr>
<td>IV</td>
<td>5.94 5.0 4.22 0 6.0 4.35 4.79</td>
</tr>
<tr>
<td>V</td>
<td>6.19 4.79 4.47 5.51 0 5.19 4.85</td>
</tr>
<tr>
<td>VI</td>
<td>5.04 5.44 4.72 5.07 5.56 0 4.5</td>
</tr>
<tr>
<td>VII</td>
<td>5.85 4.16 4.16 4.53 5.16 4.19 0</td>
</tr>
</tbody>
</table>
In order to formally compare \( L \) and \( BK \), unordered harmonic relationships in \( BK \) are \textit{approximated} by considering the symmetrized magnitudes of the relationships they reported. To wit, the \((i,j)\) entry in Table 4.3 is obtained by averaging the \((i,j)\) and \((j,i)\) entries of Table 4.2. The values are proportionally inverted and symmetrized (averaged) so, like \( L \), the cognitively closest pair is represented by the smallest distance.

Table 4.3: Symmetrized Harmonic Relations derived from Bharucha-Krumhansl.

<table>
<thead>
<tr>
<th>Second Chord</th>
<th>I</th>
<th>II</th>
<th>III</th>
<th>IV</th>
<th>V</th>
<th>VI</th>
<th>VII</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>II</td>
<td>0.185</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>III</td>
<td>0.197</td>
<td>0.236</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>IV</td>
<td>0.165</td>
<td>0.205</td>
<td>0.226</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>V</td>
<td>0.165</td>
<td>0.184</td>
<td>0.211</td>
<td>0.174</td>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>VI</td>
<td>0.194</td>
<td>0.192</td>
<td>0.215</td>
<td>0.212</td>
<td>0.186</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>VII</td>
<td>0.192</td>
<td>0.215</td>
<td>0.232</td>
<td>0.215</td>
<td>0.200</td>
<td>0.230</td>
<td>0</td>
</tr>
</tbody>
</table>
4.3 A General Theory for Distance Measures Between Models

This section presents a methodology for measuring the similarity between two intra-regional models of a given region $R$. This of course will be applied to $BK$ and $L$, but the theory is presented generally first. This comparative methodology is applicable to any set of metric models. In order to encourage a broad conception of the distance measure, I will couch the discussion in context of metric model of a region $R$, but do not limit the discussion to a specific “intra-regional model” over the seven diatonic triads.

The first step is to define a distance measure from one metric model to another. Fix a region $R$ consisting of a set of chords $V$, where the cardinality of $V \geq 2$. Recall that $\mathcal{M}_R$ is the set of all metric models derived from graphical models of $R$. Let $M^1_R = (V, d^*_1)$ and $M^2_R = (V, d^*_2)$ be any two metric models in $\mathcal{M}_R$. Define:

\begin{align*}
\mu_R(M^1_R, M^2_R) & \overset{\text{def}}{=} \max_{v_1, v_2 \in V, \text{where } v_1 \neq v_2} \left\{ \frac{d^*_2(v_1, v_2)}{d^*_1(v_1, v_2)} \right\}, \\
\epsilon_R(M^1_R, M^2_R) & \overset{\text{def}}{=} |\log[\mu_R(M^1_R, M^2_R)]|^2.
\end{align*}

Define the $\mu$ of two metric models as the maximum separation of one model over the maximum separation of the other model. Define the $\epsilon$ of those same two models as the absolute value of the log of $\mu$. Clearly, $\epsilon_R : \mathcal{M}_R \times \mathcal{M}_R \to \mathbb{R}^{\geq 0}$. Moreover, $\epsilon_R(M^1_R, M^2_R)$ measures the maximum distortion of pairwise distances that would be experienced by switching from $M^1_R$ to $M^2_R$. 
In representing the distance between two models a single value might appear inadequate. It is a narrow view of the relation between two models. However, a single distance measure between pairs of models allows us later to define a special kind of relationship between sets of metric spaces.

Consider two degenerate examples: (i) Suppose $M_2^R$ simply reduces every pairwise distance in $M_1^R$ by a factor of $k$. Then $\mu_R(M_1^R, M_2^R) = 1/k$, so $\epsilon_R(M_1^R, M_2^R) = |\log(1/k)| = |\log 1 - \log k| = \log k$. (ii) Suppose $M_2^R$ simply expands every pairwise distance in $M_1^R$ by a factor of $k$. Then $\mu_R(M_1^R, M_2^R) = k$, so $\epsilon_R(M_1^R, M_2^R) = |\log(k)| = \log k$.

In the case where $k = 2$, for example, we can easily see that for either expansion or reduction by a factor of 2, $\epsilon = \log 2$. The examples show that by incorporating both $\log$ and absolute values into the definition, the measure $\epsilon_R$ is insensitive to whether distortion is expansive or contractive. Furthermore, because $\epsilon_R$ grows logarithmically with the extent of the distortion, it exhibits greater measurement sensitivity in situations where the distortion is low. In Chapter 5, I will describe equivalence classes between models. In particular, I define the cases when distortion between two models results from specific functions on the distances of those models.

It is clear from our definition of $\epsilon$ that unless the maximum separation is the same for both models, $\epsilon_R(M_1^R, M_2^R) \neq \epsilon_R(M_2^R, M_1^R)$. Having defined $\epsilon_R$ we now define the distance between models $M_1^R$ and $M_2^R$ to be

$$\delta_R(M_1^R, M_2^R) \overset{\text{def}}{=} \epsilon_R(M_1^R, M_2^R) + \epsilon_R(M_2^R, M_1^R).$$

Intuitively, $\epsilon_R(M_1^R, M_2^R)$ interprets distance between two models as the sum of two values: (i) the log of the maximum distortion of pairwise distances that would be experienced by switching from $M_1^R$ to $M_2^R$ and (ii) the
log of the maximum distortion of pairwise distances that would be experienced by switching from $M^2_R$ to $M^1_R$. Clearly, $\delta_R : \mathcal{M}_R \times \mathcal{M}_R \rightarrow \mathbb{R}_{\geq 0}$.

The next result provides a compelling argument for our choice of $\delta_R$ as a similarity measure between intra-regional models of a given region $R$.

**Theorem 4.3.1.** $(\mathcal{M}_R, \delta_R)$ is a metric space.

The theorem follows from the following Propositions (4.3.2, 4.3.3, and 4.3.4), which verifies that each of the three defining properties for a metric space hold in $(\mathcal{M}_R, \delta_R)$.

**Proposition 4.3.2.** (Reflexivity) Let $M^1_R = (V, d^*_1)$ and $M^2_R = (V, d^*_2)$ be any two metric models in $\mathcal{M}_R$. Then

$$\delta(M^1_R, M^2_R) = 0 \iff M^1_R = M^2_R.$$

**Proof.** If $M^1_R = M^2_R$ then $d^*_1 \equiv d^*_2$ as functions. Hence for all $v_1, v_2 \in V$, $d^*_1(v_1, v_2) = d^*_2(v_1, v_2)$. Hence $\epsilon_R(M^1_R, M^2_R) = |\log 1| = 0$. By a symmetric argument, $\epsilon_R(M^2_R, M^1_R) = 0$. Hence $\delta(M^1_R, M^2_R) = 0$.

Suppose $\delta(M^1_R, M^2_R) = 0$. Since $\delta_R(M^1_R, M^2_R)$ is the sum of two non-negative quantities, it follows that $|\log[\mu_R(M^2_R, M^1_R)]| = |\log[\mu_R(M^1_R, M^2_R)]| = 0$. Hence $\mu_R(M^2_R, M^1_R) = \mu_R(M^1_R, M^2_R) = 1$. It follows that for each $v_1, v_2 \in V$, $d^*_1(v_1, v_2) = d^*_2(v_1, v_2)$, and so $M^1_R = M^2_R$. \hfill $\Box$

**Proposition 4.3.3.** (Symmetry) Let $M^1_R = (V, d^*_1)$ and $M^2_R = (V, d^*_2)$ be any two metric models in $\mathcal{M}_R$. Then

$$\delta(M^1_R, M^2_R) = \delta(M^2_R, M^1_R).$$
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Proof. Immediate, since

\[ \delta_R(M^1_R, M^2_R) = \epsilon_R(M^1_R, M^2_R) + \epsilon_R(M^2_R, M^1_R) \]

\[ = \epsilon_R(M^2_R, M^1_R) + \epsilon_R(M^1_R, M^2_R) \]

\[ = \delta_R(M^2_R, M^1_R) \]

i.e. \( \delta_R \) is symmetric in its arguments. \( \Box \)

**Proposition 4.3.4.** (Triangle inequality) Let \( M^1_R = (V, d^1_R) \), \( M^2_R = (V, d^2_R) \), and \( M^3_R = (V, d^3_R) \) be any three metric models in \( \mathcal{M}_R \). Then

\[ \delta_R(M^1_R, M^3_R) \leq \delta_R(M^1_R, M^2_R) + \delta_R(M^2_R, M^3_R). \]

Proof. Let \( v_1 \) and \( v_2 \) be vertices. Then

\[ \frac{d^1_R(v_1, v_2)}{d^3_R(v_1, v_2)} = \left[ \frac{d^1_R(v_1, v_2)}{d^2_R(v_1, v_2)} \right] \cdot \left[ \frac{d^2_R(v_1, v_2)}{d^3_R(v_1, v_2)} \right] \leq \mu_R(M^1_R, M^2_R) \cdot \mu_R(M^2_R, M^3_R). \]

Since \( \mu_R(M^1_R, M^3_R) \) is the maximum value of \( \frac{d^1_R(v_1, v_2)}{d^3_R(v_1, v_2)} \) (maximized over all distinct \( v_1, v_2 \) in \( V \)), we see that

\[ \mu_R(M^1_R, M^3_R) \leq \mu_R(M^1_R, M^2_R) \cdot \mu_R(M^2_R, M^3_R). \]

Taking logarithms and appealing to convexity of absolute values, it follows that

\[ |\log [\mu_R(M^1_R, M^3_R)]| \leq |\log [\mu_R(M^1_R, M^2_R)] + \log [\mu_R(M^2_R, M^3_R)]| \]

\[ \leq |\log [\mu_R(M^1_R, M^2_R)]| + |\log [\mu_R(M^2_R, M^3_R)]|. \]

It follows, by the definition of \( \epsilon_R \), that

\[ \epsilon_R(M^1_R, M^3_R) \leq \epsilon_R(M^1_R, M^2_R) + \epsilon_R(M^2_R, M^3_R). \quad (4.1) \]
A symmetric argument yields that
\[
\epsilon_R(M^3_R, M^1_R) \leq \epsilon_R(M^3_R, M^2_R) + \epsilon_R(M^2_R, M^1_R). \tag{4.2}
\]
By combining corresponding sides in expressions (4.1) and (4.2), we conclude that
\[
\delta_R(M^1_R, M^3_R) \leq \delta_R(M^1_R, M^2_R) + \delta_R(M^2_R, M^3_R),
\]
as claimed. \qed

4.3.1 Results of \( \delta(L, BK) \)

We are now in a position to formally compare \( L \) and \( BK \) using \( \delta \). Taking the sum of the absolute value of the log of the \( \max \) separation experienced switching between \( L \) and \( BK \) and \( BK \) and \( L \) generates \( \epsilon \) values. As shown in Table 4.4, the two \( \epsilon \) values are summed resulting in the \( \delta \) or “distance” between the two metric models. The distance measure quantifies our basic comparative procedure. That said, as I mentioned in Section 4.1, such a distance measure raises questions about the unit distance relationship between \( L \) and \( BK \). Normalization of \( L \) and \( BK \) and the expansion of the scope of comparison are the topics of the following chapter.
Table 4.4: L ($M_1$) and BK ($M_2$)

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>BK</th>
<th>L/BK</th>
<th>BK/L</th>
</tr>
</thead>
<tbody>
<tr>
<td>I-ii</td>
<td>8</td>
<td>0.185</td>
<td>43.243</td>
<td>0.023</td>
</tr>
<tr>
<td>I-iii</td>
<td>7</td>
<td>0.197</td>
<td>35.533</td>
<td>0.028</td>
</tr>
<tr>
<td>ii-iii</td>
<td>8</td>
<td>0.236</td>
<td>33.898</td>
<td>0.030</td>
</tr>
<tr>
<td>iii-IV</td>
<td>8</td>
<td>0.226</td>
<td>35.398</td>
<td>0.028</td>
</tr>
<tr>
<td>ii-IV</td>
<td>7</td>
<td>0.205</td>
<td>34.146</td>
<td>0.029</td>
</tr>
<tr>
<td>iii-V</td>
<td>7</td>
<td>0.211</td>
<td>33.175</td>
<td>0.030</td>
</tr>
<tr>
<td>iii-vi</td>
<td>5</td>
<td>0.215</td>
<td>23.256</td>
<td>0.043</td>
</tr>
<tr>
<td>iii-vii</td>
<td>5</td>
<td>0.232</td>
<td>21.552</td>
<td>0.046</td>
</tr>
<tr>
<td>I-IV</td>
<td>5</td>
<td>0.194</td>
<td>25.773</td>
<td>0.039</td>
</tr>
<tr>
<td>ii-V</td>
<td>5</td>
<td>0.184</td>
<td>27.174</td>
<td>0.037</td>
</tr>
<tr>
<td>ii-vi</td>
<td>5</td>
<td>0.192</td>
<td>26.042</td>
<td>0.038</td>
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<td>ii-vii</td>
<td>7</td>
<td>0.215</td>
<td>32.558</td>
<td>0.031</td>
</tr>
<tr>
<td>I-V</td>
<td>5</td>
<td>0.165</td>
<td>30.303</td>
<td>0.033</td>
</tr>
<tr>
<td>I-vi</td>
<td>7</td>
<td>0.194</td>
<td>36.082</td>
<td>0.028</td>
</tr>
<tr>
<td>I-vii</td>
<td>8</td>
<td>0.192</td>
<td>41.667</td>
<td>0.024</td>
</tr>
<tr>
<td>IV-V</td>
<td>5</td>
<td>0.174</td>
<td>45.977</td>
<td>0.022</td>
</tr>
<tr>
<td>IV-vi</td>
<td>5</td>
<td>0.212</td>
<td>33.019</td>
<td>0.030</td>
</tr>
<tr>
<td>IV-vii</td>
<td>5</td>
<td>0.215</td>
<td>23.256</td>
<td>0.043</td>
</tr>
<tr>
<td>vi-vii</td>
<td>8</td>
<td>0.230</td>
<td>34.783</td>
<td>0.029</td>
</tr>
<tr>
<td>V-vi</td>
<td>8</td>
<td>0.186</td>
<td>43.011</td>
<td>0.023</td>
</tr>
<tr>
<td>V-vii</td>
<td>7</td>
<td>0.200</td>
<td>35.000</td>
<td>0.029</td>
</tr>
</tbody>
</table>

\[\epsilon = (5.523),(4.430)\]

\[\delta = 9.953\]
Chapter 5

Normalization and Canonical Representation of Metric Models

The distance measure defined in Chapter 4 raises questions about the equivalence of “unit distances” in metric models in general ($d^*$) and in $L$ and $BK$ in particular. Consistent unit distances are not a requirement for comparing systems. However, arguments about relationships between compared systems are made stronger by understanding equivalences and similarities in such units. While unit-distance assignments to chord pairs in models $L$ and $BK$ are, perhaps, well-founded in the context of each model (e.g., Lerdahl decides the weight of the root, fifth, and third to be particular values in order to facilitate a particular efficiency in calculation), in the context of
This chapter defines two general concepts. First (section 5.1), I define a normalization of metric models (over a key region), showing that $M_R$ is a member of the equivalence class $[M_R]$. Second (section 5.2), I define my choice of one particular model, called $M_R$, to serve as the canonical representative of all members of $[M_R]$. Once the concepts are generally defined, I apply them to the $L$ and $BK$ models (section 5.3).

In addition, I make an important connection to the theory of Chapter 3 by transforming local analytic systems $L$ and $BK$ into related global analytic systems $\mathcal{L}$ and $\mathcal{BK}$ (section 5.4).

### 5.1 Normalizing Metric Models

Having defined the distance measure $\delta$ between the two models as they are presented in Table 4.1 and Table 4.3, the question is, how meaningful is this as a comparative measure?

**Hypothesis 5.1.1 (Fundamental Hypothesis).** Each of the spaces ($L$ and $BK$) is in fact a concrete realization of an abstract system, and the particular concrete realization incorporates arbitrary choices in the representational design.

Given the specific models $L$ and $BK$, why, for example, should the com-

---

1When I refer to arbitrariness in design, I am referring to the choice of distance units. In addition, recall that in the casting of the $BK$ model into a metric model, I proportionally inverted the chord-pair weights so that (like $L$) more-related distances are smaller than less-related distances.
parison be constrained by $BK$’s choice of a scale of integers 1-7, when it may as well have been -10 to 70 or real numbers between 0 and 1. Lerdahl chose to make the root of a chord a certain weight, where he could have made it heavier or lighter. When these two spaces are compared, we must make every effort to ensure that we are comparing their essentials.

Before I address these specific models, however, I present a precise and general quantitative interpretation of hypothesis 5.1.1. First I give a formal interpretation to the notion that models “incorporate arbitrary choices in their representational design.” Given a model $M_R$ and two real numbers $\alpha, \beta$, where $\alpha \in (0, +\infty)$ and $\beta \in (-\infty, \sigma_{\text{min}}^R(M_R))$,\(^2\) I denote the $(\alpha, \beta)$-normalization of $M_R$ as

\[
\langle M_R \rangle_{\alpha, \beta} \overset{\text{def}}{=} (V, \langle d^* \rangle_{\alpha, \beta}),
\]

where $\langle d^* \rangle_{\alpha, \beta} : V \times V \to \mathbb{R}^{\geq 0}$ is defined to be

\[
\langle d^* \rangle_{\alpha, \beta}(v_1, v_2) \overset{\text{def}}{=} \alpha[d^*(v_1, v_2) - \beta]
\]

if $v_1 \neq v_2$ and 0 otherwise, for every $v_1, v_2$ in $V$.

In other words, $(\alpha, \beta)$-normalization of $M_R$ represents a linear translation of all positive distances by $-\beta$ followed by a rescaling by a factor of $\alpha$. Simply put, subtract $\beta$ and multiply by $\alpha$. Normalization parameters must be considered when assessing the similarity of two models, since the implicit

\(^2\)The ranges $(0, +\infty)$ and $(-\infty, \sigma_{\text{min}}^R(M_R))$ are open intervals, meaning that $\alpha$ can get arbitrarily close to 0 but cannot be 0 and $\beta$ can get arbitrarily close to $\sigma_{\text{min}}^R$, but cannot be the same value as $\sigma_{\text{min}}^R$. It will be clear that this is required to avoid a collapse of two points into a single point.
\[ \alpha = 1, \beta = 0 \] choices come from arbitrary choices in the formal underpinnings or experimental design. These arbitrary choices are unimportant when models are considered in isolation, but when we want to measure similarity between models, the choices exert undue influence.

In particular, given two models \( M^1_R = (V, d^*_1) \) and \( M^2_R = (V, d^*_2) \), and specific

\[ \alpha_1, \alpha_2 \in (0, +\infty) \]
\[ \beta_1 \in (-\infty, \sigma_{R}^{\min}(M^1_R)) \]
\[ \beta_2 \in (-\infty, \sigma_{R}^{\min}(M^2_R)), \]

it is difficult to make any general assertions (i.e., independent of the choices of \( \alpha_1, \alpha_2, \beta_1, \beta_2 \)) regarding the relationship between the two original models and the relationship between their normalizations. In other words, the following question does not have a uniform answer independent of the choices of \( \alpha_1, \alpha_2, \beta_1, \beta_2 \): Is \( \delta(M^1_R, M^2_R) \) less than, or equal to, or greater than \( \delta(\langle M^1_R \rangle_{\alpha_1, \beta_1}, \langle M^2_R \rangle_{\alpha_2, \beta_2}) \) ?

Two models \( M^1_R \) and \( M^2_R \) are said to be normalization-equivalent if one model is merely a linear normalization of the other, i.e.,

\[ M^1_R \simeq M^2_R \iff \langle M^1_R \rangle_{\alpha_1, \beta_1} = M^2_R, \]

for some \( \alpha_1 \in (0, +\infty), \beta_1 \in (-\infty, \sigma_{R}^{\min}(M^1_R)) \). It is easy to see that \( \simeq \) defines an equivalence relation on \( M_R \) (the set of all metric models over a region \( R \)). The equivalence class of a model \( M_R \) under the equivalence relation is

\[ [M_R] = \{ \langle M_R \rangle_{\alpha, \beta} \mid \alpha \in (0, +\infty), \beta_1 \in (-\infty, \sigma_{R}^{\min}(M^1_R)) \}. \]
Viewed as a subset of $\mathcal{M}_R$, $[M_R]$ is the set of all linear normalizations of the model $M_R$.

Putting the previous statement into a specific context, Lerdahl’s model $L$ is in fact only one member of an infinite collection $[L]$ of related models, each of which corresponds to a different normalization of $L$. The position of $L$ (the specific model) within the set $[L]$ reflects specific “arbitrary choices” in the designation of real number values to chord pairs in $L$. Indeed, both $L$ and $BK$ are laden with such arbitrary choices and it is from this arbitrariness that we must divest ourselves. We must find a way to measure the distance between models in a way that is insensitive to the arbitrary design choices affecting chord distances.

Returning to the formal scaffolding, let us define the set of all equivalence classes in $\mathcal{M}_R$ as

$$[\mathcal{M}_R] = \{[M_R] \mid M_R \in \mathcal{M}_R\}.$$  

### 5.2 Canonical Representatives

Defining a canonical representative $\overline{M}_R$ for metric models allows us to reliably choose one linear normalization of $M_R$ to stand in for all members of $[M_R]$. In the choice of $\overline{M}_R$ I indeed make some arbitrary choices. Such choices can never be completely exorcized from a comparative methodology. However, if those choices can be elevated to a meta-analytic design level, then we can identify, control, and account for these choices. I chose the particular metric model whose minimal separation ($\sigma_{\text{min}}$) is 1 and whose maximal separation ($\sigma_{\text{max}}$) is the number of vertices in the model (generally
CHAPTER 5. NORMALIZED CANONICAL REPRESENTATIVES

Figure 5.1: Canonical representation as the unique function that maps min to 1 and max to n.

called n). For every model on n vertices, there is a unique $\alpha, \beta$ realization such that $\sigma_{\min} = 1$ and $\sigma_{\max} = n$. Figure 5.1 shows the unique $\alpha, \beta$ normalization as the intercept of the minimal separation and 1 and the maximal separation and n.

The definition of equivalence classes and canonical representatives leads us to an important extension to the theory of metric spaces.

**Definition 5.2.1.** A function on ordered pairs of equivalence classes $\lambda : [\mathcal{M}_R] \times [\mathcal{M}_R] \rightarrow \mathbb{R}^{\geq 0}$ is defined as follows:

$$\lambda_R([M^1_R], [M^2_R]) = \delta_R(M^1_R, M^2_R)$$

**Theorem 5.2.2.** $([\mathcal{M}_R], \lambda_R)$ is a metric space.
Proof. The distance between equivalence classes is defined in terms of $\delta_R$ distance between canonical representatives, and the representatives themselves inhabit a metric space $(\mathcal{M}_R, \delta_R)$. Since a subspace of a metric space is a metric space, the theorem follows.

\[
5.3 \text{ Canonical Representatives for } [L] \text{ and } [BK]
\]

Having chosen 1 as the default minimal separation and the number of chords in a intra-regional metric model ($n$) as the value of the maximal separation, I can find the canonical representatives for $[L]$ and $[BK]$. Each metric in $L$ (resp. $BK$) is multiplied by $\frac{n-1}{\sigma^\max_R(L)-\sigma^\min_R(L)}$ where $n$ equals the number of $V$ in $L$. In other words, the distances are scaled by multiplying each $d^*$ by the difference of the desired maximal and minimal separations over the difference of the actual maximal and minimal separations of a model. Those distances are then translated by adding $1 - \sigma^\min_R(L)$ to each $d^*$ in $L$. The results, shown in Table 5.1, are the chord-pair distances of the canonical representatives $L$ and $BK$.

Normalization is important when we begin to consider the geometric relationship between $L$ and $BK$. Since the distance values comment on issues of cohesiveness, musicality, comprehensibility, we must be assured that those values are properly normalized.
Table 5.1: Canonical Representatives $\mathcal{L}$ and $\overline{BK}$

<table>
<thead>
<tr>
<th></th>
<th>$\mathcal{L}$</th>
<th>$\overline{BK}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>I-ii</td>
<td>7</td>
<td>2.690</td>
</tr>
<tr>
<td>I-iii</td>
<td>5</td>
<td>3.704</td>
</tr>
<tr>
<td>ii-iii</td>
<td>7</td>
<td>7.000</td>
</tr>
<tr>
<td>iii-IV</td>
<td>7</td>
<td>6.155</td>
</tr>
<tr>
<td>ii-IV</td>
<td>5</td>
<td>4.380</td>
</tr>
<tr>
<td>iii-V</td>
<td>5</td>
<td>4.887</td>
</tr>
<tr>
<td>iii-vi</td>
<td>1</td>
<td>5.225</td>
</tr>
<tr>
<td>iii-viio</td>
<td>1</td>
<td>6</td>
</tr>
<tr>
<td>I-IV</td>
<td>1</td>
<td>3.451</td>
</tr>
<tr>
<td>ii-V</td>
<td>1</td>
<td>2.606</td>
</tr>
<tr>
<td>ii-vi</td>
<td>1</td>
<td>3.282</td>
</tr>
<tr>
<td>ii-viio</td>
<td>5</td>
<td>5.225</td>
</tr>
<tr>
<td>I-V</td>
<td>1</td>
<td>1.000</td>
</tr>
<tr>
<td>I-vi</td>
<td>5</td>
<td>3.451</td>
</tr>
<tr>
<td>I-viio</td>
<td>7</td>
<td>3.282</td>
</tr>
<tr>
<td>IV-V</td>
<td>7</td>
<td>1.761</td>
</tr>
<tr>
<td>IV-vi</td>
<td>5</td>
<td>4.972</td>
</tr>
<tr>
<td>IV-viio</td>
<td>1</td>
<td>5.225</td>
</tr>
<tr>
<td>vi-vii</td>
<td>7</td>
<td>6.493</td>
</tr>
<tr>
<td>V-vi</td>
<td>7</td>
<td>2.775</td>
</tr>
<tr>
<td>V-vii</td>
<td>5</td>
<td>3.958</td>
</tr>
</tbody>
</table>
5.4 Considering $L$ and $BK$ as Global Systems

Following the theory established in Chapter 3, we may eventually want to consider local analytic systems (such as $L$ and $BK$) as global analytic systems. In other words, we may want to compare local systems by the degree to which each system considers $\mathbb{N}^{<\infty}$ to be typically musical. It is clear that the formal description of global analytic systems is quite different from the formal description of local analytic systems as metric models. Recall that in Chapter 1 I said that global systems and local systems are not alternatives to each other, but rather are analyses of pieces on different levels. A global system is designed to look all pieces (coded as sequences of natural numbers) and assigns a real number to each piece that represents each piece’s typical musicality.

In the specific context of $L$ and $BK$, such a reconsideration is closely connected to how these two local systems interpret pieces of music in terms of their stated goals of measuring chord relatedness. Analyses produced by the application of the $L$ model are quantifications of chord relatedness modeled by a set of real numbers. The measure of chord relatedness generated by models $L$ and $BK$ gives a strong suggestion as to how we might consider $L$ and $BK$ as global systems. In other words, in its simplest manifestation, a “pair of chords” can be considered a piece composed of such chords whose length is two. We can easily see that for $L$ and $BK$, distances between chord pairs represent a kind of musicality that is typical of each respective analytic system.

I now reinterpret some of the terms used in the discussion of $L$ and $BK$ as terms used in global analysis. I define musical event types as the seven
diatonic triadic harmonies from a single region—in this case a single major key. Reserving the number 0 for the “null event,” each triad is encoded by a natural number. The alphabet, therefore, includes numbers 0 through 7.

An adequate treatment of the question of harmonic syntax, well-formed progressions, and their impact to the relevance on comparative analysis is outside the scope of my examination. I will, for the purposes of comparative analysis, limit the discussion to all sequences of length two (i.e. pairs of triads). This limit gives sequences finite support and imbues them with the identity of pieces. An ordering $\eta$ is included, but not relevant since all pieces are the same length. The set of all musical pieces $\mathbb{N}^{<\infty}$, therefore, is the enumeration of all pairs of diatonic triads shown in Table 5.2.

Having defined canonical forms for tonal models, let us consider $L$ and $BK$ as global systems $\mathcal{L}$ and $\mathcal{BK}$, and therefore as functions mapping $\mathbb{N}^{<\infty} \rightarrow \mathbb{R}$. These functions examine every piece in $\mathbb{N}^{<\infty}$ and return a real number for each piece representing the analysis of each piece’s typical musicality. As it stands, the real number values associated with each chord pair (as determined by $L$ and $BK$ and shown in Table 5.1) would seem to be inversely related to an opinion of typical musicality. In other words, the greater the value returned by the two functions on some pair of chords, the less they are related, or the farther they are apart in a cognitive-euclidean metaphor. We are able to translate the idea of “relatedness” into “typical musicality” by invoking the contextual isomorphism $\varphi$.\footnote{The function $\varphi$ is contextual because it is designed to work with $L$ and $BK$ specifically. It is not generalized to work for all $M_R$ and certainly not for all local analytic systems.}
Define
\[ \varphi(L) \overset{\text{def}}{=} \{ \varphi(d^*) \mid -1(d^*) + 8 \} . \]

In other words, for the metric model \( L \), I define a function \( \varphi \) on \( L \) that inverts and translates each chord-pair distance in \( L \). Deciding on a continuous scale of 1 through 7 with 1 being the least typically musical and 7 the most typically musical, \( \varphi \) inversely maps the analytic output of \( L \) and \( BK \) to the meta-analytic output of \( L \) and \( BK \) that measures typical musicality. This way, the most closely related chord pairs are considered the most musical pairs without affecting the normalized unit distance. Invoking \( \varphi \) allows us to preserve the geometric qualities already inherent in the regional spaces of \( L \) and \( BK \)–a valuable metaphor. Examining Table 5.2 we see that no chord pair is considered “typically unmusical.” This clearly supports the beliefs of Lerdahl and Bharucha, et. al. that all chord pairs are discernible as being “related,” with some being more “related” than others. If I were to consider pieces longer than length two, then I would need to employ an additional function called an \textit{aggregator} that would return a single real number for each analysis regardless of the piece’s length.

5.4.1 Geometry of \( L \) and \( BK \)

The lengths of \( L \) and \( BK \) from \( I \) are 20.32 and 18.88 respectively. \( L \) interprets the set of all musical pieces as more typically musical than \( BK \). The relative meaning of norms in \( l_2(F) \) make it difficult to say how much more typically musical \( L \) “believes” the set to be over what \( BK \) believes the set to be. They are disposed at a 38.12° angle indicating that the two systems share a good deal of similar positive opinions. This could have
Table 5.2: Analysis of $N$ under $L$ and $BK$

<table>
<thead>
<tr>
<th>$N$</th>
<th>$\eta$</th>
<th>$L$</th>
<th>$BK$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>12</td>
<td>1.0</td>
<td>5.310</td>
</tr>
<tr>
<td>2</td>
<td>13</td>
<td>3.0</td>
<td>4.296</td>
</tr>
<tr>
<td>3</td>
<td>23</td>
<td>1.0</td>
<td>1.000</td>
</tr>
<tr>
<td>4</td>
<td>34</td>
<td>1.0</td>
<td>1.845</td>
</tr>
<tr>
<td>5</td>
<td>24</td>
<td>3.0</td>
<td>3.620</td>
</tr>
<tr>
<td>6</td>
<td>35</td>
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<td>3.113</td>
</tr>
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<td>7</td>
<td>36</td>
<td>7.0</td>
<td>2.775</td>
</tr>
<tr>
<td>8</td>
<td>37</td>
<td>7.0</td>
<td>1.338</td>
</tr>
<tr>
<td>9</td>
<td>14</td>
<td>7.0</td>
<td>4.549</td>
</tr>
<tr>
<td>10</td>
<td>25</td>
<td>7.0</td>
<td>5.394</td>
</tr>
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<td>11</td>
<td>26</td>
<td>7.0</td>
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<tr>
<td>14</td>
<td>16</td>
<td>3.0</td>
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</tr>
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<td>15</td>
<td>17</td>
<td>1.0</td>
<td>4.718</td>
</tr>
<tr>
<td>16</td>
<td>45</td>
<td>1.0</td>
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</tr>
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<td>3.028</td>
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<tr>
<td>18</td>
<td>47</td>
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</tr>
<tr>
<td>19</td>
<td>67</td>
<td>1.0</td>
<td>1.507</td>
</tr>
<tr>
<td>20</td>
<td>56</td>
<td>1.0</td>
<td>5.225</td>
</tr>
<tr>
<td>21</td>
<td>57</td>
<td>3.0</td>
<td>4.042</td>
</tr>
</tbody>
</table>
been partially predicted because, as mentioned above, \( L \) and \( B \) have no negative (i.e., typically unmusical) opinions.

### 5.4.2 Commentary

How a music analytic system interprets something to be or not to be music is a difficult feature to quantify. However, within each music analytic model is a set of assumptions that can be expounded upon and used as a guideline for determining how a coordinated global system “views” music. In the case presented above, I chose to make a rather explicit translation from the product of a tonal model to the output of a global music analytic system.
Chapter 6

A Meta-Analytical
Ramification of the General
Theory of Comparative
Music Analysis

We can now see the geometry of the global systems $L$ and $B\Omega$ and compare $L$ and $B\Omega$ as local systems. The perspective this methodology affords us can be used to diagnose what may be called systematic deficiencies or design problems. Looking at Table 6.1, we see that the largest points of divergence occur at chord pairs IV-V and iii-vii. In terms of $B\Omega$, $L$ greatly amplifies the distance between iii and vii, and minimizes the distance between IV and V. These discrepancies summarize the difference between the two models. Lerdahl’s algorithmic approach privileges all fifth-related harmonies by giving
them the lowest value (i.e., representing the “closest” cognitive relations). By doing so, he distorts some important relations, the most important being the IV-V progression.

Let us return to considering the analytic application of local systems $\mathcal{L}$ and $\mathcal{BK}$. This is reflected clearly in the interpretation of the “Madamina” progression (Figure 6.1) using the canonical representatives $\mathcal{L}$ and $\mathcal{BK}$ shown in Figure 6.2. The difference is clear. The length of $\mathcal{L}(\text{Madamina}) = 9$ and $\mathcal{BK}(\text{Madamina}) = 6.212$.

The progression of the subdominant (IV) to the dominant (V) needs no real introduction. For Hugo Riemann, the relation between these harmonies was paramount. The theory of tonal functions of chords outlined by Riemann in Simplified Harmony (1896) and preempted by Systematische Modulationslehre (1887) focuses on three primary triads, described in the former as $T$ (for tonic; I), $S$ (subdominant; IV), and $D$ (dominant; V). The
Table 6.1: Canonical Representatives $L$ and $BK$

<table>
<thead>
<tr>
<th></th>
<th>$L$</th>
<th>$BK$</th>
<th>$L/BK$</th>
<th>$BK/L$</th>
</tr>
</thead>
<tbody>
<tr>
<td>I-ii</td>
<td>7.0</td>
<td>2.690</td>
<td>2.602</td>
<td>0.384</td>
</tr>
<tr>
<td>I-iii</td>
<td>5.0</td>
<td>3.704</td>
<td>1.350</td>
<td>0.741</td>
</tr>
<tr>
<td>ii-iii</td>
<td>7.0</td>
<td>7.000</td>
<td>1.000</td>
<td>1.000</td>
</tr>
<tr>
<td>iii-IV</td>
<td>7.0</td>
<td>6.155</td>
<td>1.137</td>
<td>0.879</td>
</tr>
<tr>
<td>ii-IV</td>
<td>5.0</td>
<td>4.380</td>
<td>1.141</td>
<td>0.876</td>
</tr>
<tr>
<td>iii-V</td>
<td>5.0</td>
<td>4.887</td>
<td>1.023</td>
<td>0.977</td>
</tr>
<tr>
<td>iii-vi</td>
<td>1.0</td>
<td>5.225</td>
<td>0.191</td>
<td>5.225</td>
</tr>
<tr>
<td>iii-viio</td>
<td>1.0</td>
<td>6.662</td>
<td>0.150</td>
<td>$\textbf{6.662}$</td>
</tr>
<tr>
<td>I-IV</td>
<td>1.0</td>
<td>3.451</td>
<td>0.290</td>
<td>3.451</td>
</tr>
<tr>
<td>ii-V</td>
<td>1.0</td>
<td>2.606</td>
<td>0.384</td>
<td>2.606</td>
</tr>
<tr>
<td>ii-vi</td>
<td>1.0</td>
<td>3.282</td>
<td>0.305</td>
<td>3.282</td>
</tr>
<tr>
<td>ii-viio</td>
<td>5.0</td>
<td>5.225</td>
<td>0.191</td>
<td>5.225</td>
</tr>
<tr>
<td>I-V</td>
<td>1.0</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
</tr>
<tr>
<td>I-vi</td>
<td>5.0</td>
<td>3.451</td>
<td>1.449</td>
<td>0.690</td>
</tr>
<tr>
<td>I-viio</td>
<td>7.0</td>
<td>3.282</td>
<td>2.133</td>
<td>0.469</td>
</tr>
<tr>
<td>IV-V</td>
<td>7.0</td>
<td>1.761</td>
<td>$\textbf{3.976}$</td>
<td>0.252</td>
</tr>
<tr>
<td>IV-vi</td>
<td>5.0</td>
<td>4.972</td>
<td>1.006</td>
<td>0.994</td>
</tr>
<tr>
<td>IV-viio</td>
<td>1.0</td>
<td>5.225</td>
<td>0.191</td>
<td>5.225</td>
</tr>
<tr>
<td>vi-vii</td>
<td>7.0</td>
<td>6.493</td>
<td>1.078</td>
<td>0.928</td>
</tr>
<tr>
<td>V-vi</td>
<td>7.0</td>
<td>2.775</td>
<td>2.523</td>
<td>0.396</td>
</tr>
<tr>
<td>V-vii</td>
<td>5.0</td>
<td>3.958</td>
<td>1.263</td>
<td>0.792</td>
</tr>
</tbody>
</table>
(I) TONIC: opening assertion.¹

(II) SUBDOMINANT: conflict.

(III) DOMINANT: resolution of the conflict.

(IV) TONIC: confirmation, conclusion.

Riemann emphasizes the meaning of individual chords (the subdominant is “conflict,” the dominant is “resolution”) and thus moves away from earlier dialectic conceptions of chord pairs while still highlighting the importance of the IV-V progression. A metric space, of course, comprises pairwise distances, bundling chords together and resisting individual identities. Nevertheless, there is a correlation between Riemann’s cadence model and BK’s model as BK assigns the shortest (e.g., perceptually closest) distances to IV-V and V-I.

¹The italicized terms represent the connection Riemann hoped to make with Hegel.
6.0.3 Reducing $\lambda_R$ using a new model $F$

The disagreement between pairwise distances in $L$ and their partners in $\overline{BK}$ invites us to question in specific ways the algorithm used to generate $L$. With a clearer view of the differences between these two models that the distance measure affords, we face the question of whether a new a chord distance algorithm can minimize the distortion encountered switching between $L$ and $\overline{BK}$. The answer of course is “yes, it can.” One simple solution is based loosely on Riemann’s harmonic functions. The four-stage cadence model cited above suggests the shortest path, which agrees more with $\overline{BK}$ than does $L$.

Using the three harmonic functions all intra-regional chords are categorized as either strongly or weakly belonging to either $T$, $S$, or $D$. I assign strong function chords the value 1 and weak function chords the value 2. Pairwise distance between any two chords is assigned the metric which is the sum of the values of the two chords. Chords I, IV, and V strongly belong to $T$, $S$, and $D$, respectively. Chords iii and vi weakly belong to $T$, chord ii weakly belongs to $S$, and vii weakly belongs to $D$. Table 6.2 shows the pairwise chord distances of the new intra-regional model I call $F$. Table 6.3 shows the maximum pairwise distortions experienced switching back and forth from canonical representatives $\overline{F}$ and $\overline{BK}$.

6.0.4 Recursive Metric Models

Because we define a single, symmetric distance between metric models ($d^*$), their canonical representatives ($\delta_R$), and their equivalence classes ($\lambda_R$), following Theorem 5.2.2, I extend the notion of a metric space to $[M_R]$. Ta-
### Table 6.2: Functional Harmonic Distances.

<table>
<thead>
<tr>
<th>First Chord</th>
<th>I</th>
<th>II</th>
<th>III</th>
<th>IV</th>
<th>V</th>
<th>VI</th>
<th>VII</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>II</td>
<td>3</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>III</td>
<td>3</td>
<td>4</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>IV</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>V</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>2</td>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>VI</td>
<td>3</td>
<td>4</td>
<td>3</td>
<td>3</td>
<td>2</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>VII</td>
<td>3</td>
<td>4</td>
<td>4</td>
<td>3</td>
<td>3</td>
<td>4</td>
<td>0</td>
</tr>
</tbody>
</table>

### Figure 6.3: Analysis of the Madamina Progression Under $L$, $BK$, and $F$.

$L$ distance: $I/I \leftrightarrow IV/I \leftrightarrow V/I \leftrightarrow I/I$

$BK$ distance: $I/I \leftrightarrow IV/I \leftrightarrow V/I \leftrightarrow I/I$

$F$ distance: $I/I \leftrightarrow IV/I \leftrightarrow V/I \leftrightarrow I/I$

$L$ distance: 1 \ 7 \ 1 = 9

$BK$ distance: 3.451 \ 1.761 \ 1 = 6.212

$F$ distance: 2 \ 2 \ 2 = 6

Figure 6.3: Analysis of the Madamina Progression Under $L$, $BK$, and $F$. 
Table 6.3: Canonical Representatives $\mathcal{F}$ and $\mathcal{B}K$

<table>
<thead>
<tr>
<th></th>
<th>$\mathcal{F}$</th>
<th>$\mathcal{B}K$</th>
<th>$\mathcal{F}/\mathcal{B}K$</th>
<th>$\mathcal{B}K/\mathcal{F}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>I-ii</td>
<td>4</td>
<td>2.690</td>
<td>1.487</td>
<td>0.673</td>
</tr>
<tr>
<td>I-iii</td>
<td>4</td>
<td>3.704</td>
<td>1.080</td>
<td>0.926</td>
</tr>
<tr>
<td>ii-iii</td>
<td>7</td>
<td>7.000</td>
<td>1.000</td>
<td>1.000</td>
</tr>
<tr>
<td>iii-IV</td>
<td>4</td>
<td>6.155</td>
<td>0.650</td>
<td>1.539</td>
</tr>
<tr>
<td>ii-IV</td>
<td>4</td>
<td>4.380</td>
<td>0.913</td>
<td>1.095</td>
</tr>
<tr>
<td>iii-V</td>
<td>4</td>
<td>4.887</td>
<td>0.818</td>
<td>1.222</td>
</tr>
<tr>
<td>iii-vi</td>
<td>4</td>
<td>5.225</td>
<td>0.765</td>
<td>1.306</td>
</tr>
<tr>
<td>iii-viio</td>
<td>7</td>
<td>6.662</td>
<td>1.051</td>
<td>0.952</td>
</tr>
<tr>
<td>I-IV</td>
<td>1</td>
<td>3.451</td>
<td>0.290</td>
<td>3.451</td>
</tr>
<tr>
<td>ii-V</td>
<td>4</td>
<td>2.606</td>
<td>1.535</td>
<td>0.651</td>
</tr>
<tr>
<td>ii-vi</td>
<td>7</td>
<td>3.282</td>
<td>2.133</td>
<td>0.469</td>
</tr>
<tr>
<td>ii-viio</td>
<td>7</td>
<td>5.225</td>
<td>1.340</td>
<td>0.746</td>
</tr>
<tr>
<td>I-V</td>
<td>1</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
</tr>
<tr>
<td>I-vi</td>
<td>4</td>
<td>3.451</td>
<td>1.159</td>
<td>0.863</td>
</tr>
<tr>
<td>I-viio</td>
<td>4</td>
<td>3.282</td>
<td>1.219</td>
<td>0.820</td>
</tr>
<tr>
<td>IV-V</td>
<td>1</td>
<td>1.761</td>
<td>0.568</td>
<td>1.761</td>
</tr>
<tr>
<td>IV-vi</td>
<td>4</td>
<td>4.972</td>
<td>0.805</td>
<td>1.243</td>
</tr>
<tr>
<td>IV-viio</td>
<td>4</td>
<td>5.225</td>
<td>0.765</td>
<td>1.306</td>
</tr>
<tr>
<td>vi-vii</td>
<td>7</td>
<td>6.493</td>
<td>1.078</td>
<td>0.928</td>
</tr>
<tr>
<td>V-vi</td>
<td>4</td>
<td>2.775</td>
<td>1.442</td>
<td>0.694</td>
</tr>
<tr>
<td>V-vii</td>
<td>4</td>
<td>3.958</td>
<td>1.011</td>
<td>0.989</td>
</tr>
</tbody>
</table>
Table 6.4 models the metric space \([\mathcal{M}_R], \lambda_R\) using the three equivalence classes \([L]\), \([BK]\), and \([F]\).

Table 6.4: The Metric Space \([\mathcal{M}_R], \lambda_R\).

<table>
<thead>
<tr>
<th></th>
<th>([L])</th>
<th>([BK])</th>
<th>([F])</th>
</tr>
</thead>
<tbody>
<tr>
<td>([L])</td>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>([BK])</td>
<td>4.727</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>([F])</td>
<td>5.615</td>
<td>2.880</td>
<td>0</td>
</tr>
</tbody>
</table>

The distance measure \(\lambda_R([F], [BK]) = 2.880\). It confirms that there is less distortion in switching between \(\overline{F}\) and \(\overline{BK}\) than was found in switching between \(\overline{L}\) and \(\overline{BK}\). This supports the claim that \(\overline{F}\) better correlates with \(\overline{BK}\) than does \(\overline{L}\). The Madamina progression is reinterpreted in terms of all three models in Figure 6.3 and \(\overline{F}\) does exactly what was asked of it: interpret the progression more like \(\overline{BK}\) than \(\overline{L}\) was able to do. By comparative extension, note that \(\lambda_R([L], [F]) = 5.615\).

What is significant about this comparative analysis is how it informs the choice and design of local analytic systems. This chapter has shown how the distance between canonical representatives provided a point of reference used to inform the design of a third model, \(F\). Finally, these three tonal models are members of three different equivalence classes, whose representatives come from a single metric space of tonal models, and hence, since every subset of a metric space is a metric space, the three models form a metric space.
CHAPTER 6. A META-ANALYTICAL RAMIFICATION

6.1 Multidimensional Global Geometry

Invoking the function $\varphi$, $\mathcal{F}$ can be incorporated into a geometric comparison by transforming it into $\mathcal{F}$. Including a third global system shifts the comparative reference from a two-dimensional geometry to three-dimensional geometry. $l_2(F)$ is in fact an infinite dimensional space. Adding a fourth system would push us into four-dimensional space. The three global analytic systems in relation to $I$ form a tetragon as represented in Figure 6.4. Angles between line segments $\mathcal{L}$ and $\mathcal{B}K$ and $\mathcal{F}$ are $\theta_{\mathcal{L},\mathcal{F}} = 39.85$ and $\theta_{\mathcal{F},\mathcal{B}K} = 19.11$. Each system is related to $I$ and therefore related to each other by angles.

Figure 6.4: Three-dimensional geometry of $\mathcal{F}$, $\mathcal{B}K$, and $\mathcal{L}$. 
Chapter 7

Conclusion

7.1 Summary

This dissertation began by asking the question: “how do we compare pieces of music if we consider the identity of pieces to be inseparable from some associated analytic methodology?” The first step toward an answer was to explain what a piece of music is and how it depends on analysis. I began with Husserl’s idea of the intentional object, arguing that music ought to be thought of as a consciously interpreted entity.

I defined a musical piece as an intentional object created when a set of putative musical events is filled out according to some set of values or relations. As I stated both the musical piece and the sequence of putative musical events reside in the same set. The putative events are musical when they are shown to be associated with an analytic function. My decision to assert a set of all pieces was partly motivated by the question, “where does
the process of analysis begin and end?” This is an incredibly difficult (if not impossible) question to answer. And it is exactly this difficulty that motivated my proposed definition of local analysis. If we can imagine the process of analysis as a sequence of analytic stages, where each stage is a set of musical events, then our definition only requires that you take one of those stages and connect it to another. Questions about how each stage is formed and where, in respect to each other, each stage exists are unrelated (which is not to say that they are unimportant or uninteresting) to questions about intentional connection. For example given an analytic system that maps a “triad” to a roman numeral, we are not necessarily concerned with how the triad came to be, what happened to registral ordering, pitch-class multiplicities, or coutrapuntal elaborations, etc. Accepting the existence of “triads” (or any other “pre-analytic musical event”) as a priori frees us to focus different kinds questions.

I defined two types of analytic systems as functions with the principal formal difference between the two being their range. Global analytic systems acted on the entire set of musical pieces and took the real numbers as their range. The set of all local analytic systems mapped the set of all musical pieces to itself.

The idea of typical musicality is specific to this research. Every local analytic system has the capacity to act on any piece of music. I hypothesized that every local system suggests a musicality. This musicality is akin to a set of conditions a piece in order for it to be considered “music” in terms of the local system. The metric for how well a piece meet these conditions was termed the “typical musicality,” because it models the degree of musicality of a piece that is typical of the analytic system. The idea of typical musicality
owes a great debt to Boretz, who writes:

\[
\ldots\text{the compulsions to reify a literature, to find some general structural paradigm as some particular structural level that makes every composition a member of some group of a certain kind, to force everything into some existing model of musical structure, or to accept a greatly reduced standard of musical coherence, are considerably relieved when musical coherence is regarded as a direction on a relativistic scale rather than an absolute attribute, and when \ldots everything likely to be regarded as a potential piece can be shown to be coherent to at least a certain degree if it is admissible at all—and all it has to be to be admissible is a finite succession of discriminable (and discriminated) \ldots phenomena that someone wants to regard as music (1995, 247-48).}
\]

Although Boretz is referring to the compositions stemming from a supersyntactical system, his comment can be considered from an analytic point of view. In lieu of such a super-syntactical analytic system, I chose to elevate any and all analytic systems to the same level, asking only that we define how these systems value particular musical events. The “common denominator,” so to speak, is the set of all pieces, which is identical for each analytic system.

Analysis of global systems was carried out under the auspices of functional analysis. The analysis produced by a global system is a real number that represents the typical musicality of the set of all pieces. I showed that each real number is a position in a vector space called $l_2(F)$ and described the relationship between two or more points geometrically. The length of
a line segment from the point of no-opinion and a given analytic position represents how typically musical the set of all pieces is in terms of the analytic system. The relationship between two global systems is represented by the angle between two line segments formed at the unique point of no-opinion. Acute angles represent greater agreement and obtuse angles represent greater disagreement. This type of analysis is possible because both systems are looking at the same set—the set of pieces of music. The vector space is infinitely dimensional and the number of analytic systems compared determines the number of dimensions. I initially compared two systems and later compared three systems.

As mentioned above, a local system maps one piece to another and the set of all local systems maps the set of all pieces to itself. In the example presented in this dissertation, the local system \( L \) (derived from Lerdahl’s Tonal Pitch Space Model) took the seven diatonic triads (chords) and connected each pair to a real number that represented their hypothesized perceptual closeness. I constructed a second local system, \( BK \), from experimental data describing relationships between pairs from the same set of chords in the same numerical way.

I asserted that local systems could only be reasonably compared if they could be represented formally at a higher level. To wit, I redefined each space as a metric model and defined a distance measure between them. The distance measure represented the amount of distortion required to turn one model into the other. The distance measure was symmetric and, therefore, order-insensitive. Because the distance between models is symmetric, two or more models also form a metric space.
The distance measure between models raised questions about the equivalence of unit distances in each model. I defined a normalization algorithm and chose the normalized model to be the canonical representative of all normalization equivalent models. Lastly, I defined a contextual transformation that converted these local analytic systems into global analytic systems.

Comparing canonical representatives of the models and to converting them into global analytic systems gave me the perspective necessary to propose a third analytic system that reduced the amount of distortion found in switching between models. In addition, the analysis of three global analytic systems produced a three-dimensional comparative geometry.

7.2 Suggestions for Future Research

Some trajectories for future research are clear. For example, defining the typical musicality of specific analytic gestures in specific analytic systems (e.g., Schenker theory or neo-Riemannian theory) would be a fruitful (albeit quite challenging) endeavor. It is apparent that the interpretation of magnitudes of typical musicality is contextual and I have not tried to portray them otherwise. However, through critical analysis of published analyses it may be possible to achieve some intersubjective agreement on numerical assignments for typical musicality.

Following the examples given in Chapters 4, 5, and 6, there are many local analytic systems that can be compared by developing contextual comparative methodologies. For example, the formal similitude I required for comparisons is readily available in systems based on commutative groups.
In addition, category theory could be incorporated into local comparisons. Different families of local analytic systems (a family being a group of systems modeled by the same formal structure) could be defined as categories connected by functors. One of my goals has been to show that comparisons of analyses and analytic systems can be as creative as “analysis” in the traditional sense. This proposal gets to the heart of my problem with existing comparative studies. None of them are capable of being reinterpreted so that they become a specific instance of a general theory.

The idea of canonical representatives of analytic systems raises questions about equivalence classes of local analytic systems in general. If this idea were extended to other formal structures, then different normalization algorithms would be required. It would be interesting to know how different normalization processes affect canonical representatives and whether or not effective comparative measures between different formal structures could be defined.

The idea that an analytic system can “emerge” from information gained by comparing two or more systems is new and rich in potential. For example, an “emergent” system could be used to inform experimental design. Also, an emerged system could be used to conceptually bridge parent local systems that might otherwise be thought of as alternatives to each other?

I have taken a formal tone in my discussion of global analysis and global analytic systems in order to promote clarity. However, there is nothing wrong with a less formal notion about how an analytic system contributes to the notion of musicality. We make choices everyday about which analytic system to use to interpret a particular music. It has been my intuition that
these choices reflect a musicality inherent in each piece/system interaction. It has been my goal to show that this musicality can serve as the basis for a general theory of comparative music analysis.
Bibliography

Adler, Guido (1885). “Umfang, Methode und Ziel der Musikwissenschaft”.

Armstrong, Paul B. (1997). “Phenomenology”. In M. Groden and
M. Kreiswirth (Eds.), Johns Hopkins Guide to Literary Theory and Crit-
icism (Online ed.). Johns Hopkins University Press.

College Music Symposium 5: 49–60.

Bharucha, Jamshed and Carol Krumhansl (1983). “The Representation of
Harmonic Structures in Music: Hierarchies of Stability as a Function of


Brooks, Barry S. (1969). The Analysis of Communication Content: Devel-
lopments in Scientific Theories and Computer Techniques, Chapter Style
and Content Analysis in Music: The Simplified “Plaine and Easie Code”,


Heinichen, Johann David (1728). *Der Generalbass in Der Composition*. Dresden.


