Midterm REVIEW

Midterm 1 will cover everything we have seen so far

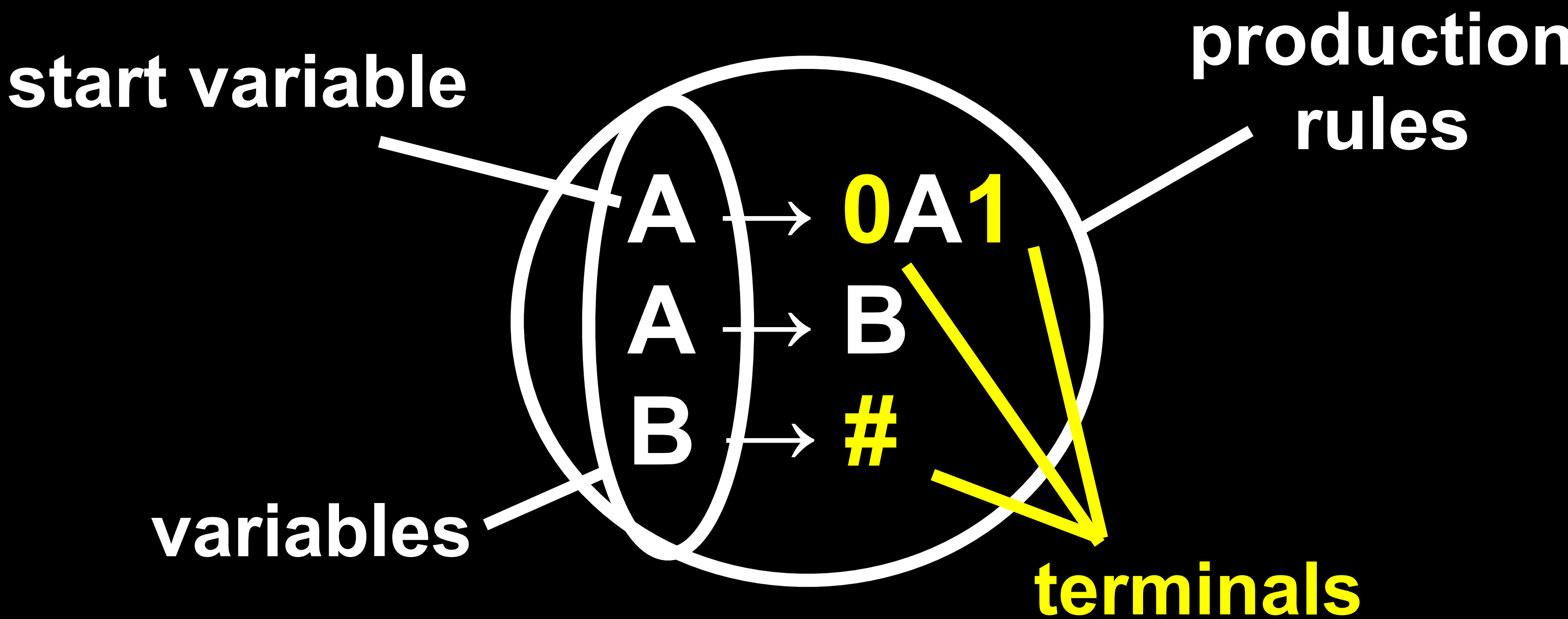
The PROBLEMS will be from Sipser,

Part I

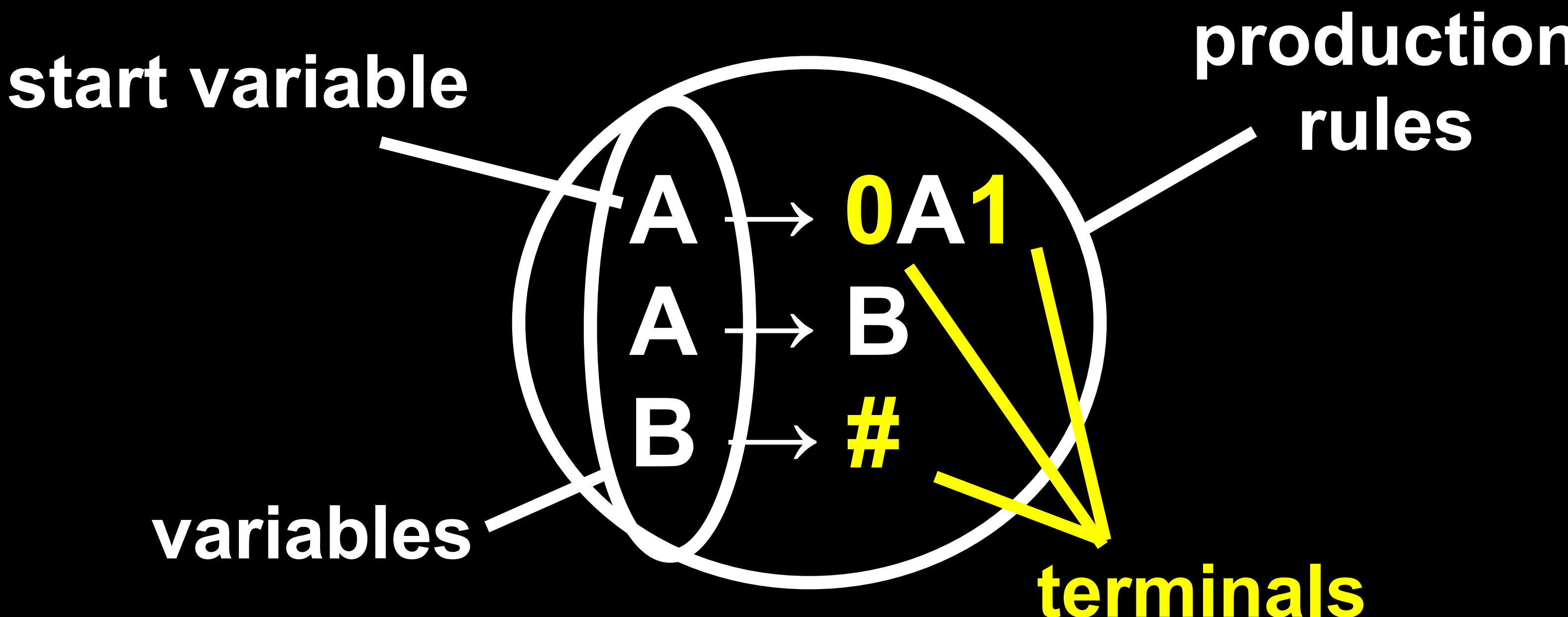
It will be Closed-Book, Closed-Everything

- 1. Deterministic Finite Automata and Regular Languages
- 2. Non-Deterministic Finite Automata
- 3. Pumping Lemma for Regular Languages; Regular Expressions
- 4. Minimizing DFAs
- 5. PDAs, CFGs;
  Pumping Lemma for CFLs
- 6. Equivalence of PDAs and CFGs
- 7. Chomsky Normal Form
- 8. Turing Machines

CONTEXT-FREE GRAMMARS



CONTEXT-FREE GRAMMARS



$$A \Rightarrow 0A1 \Rightarrow 00A11 \Rightarrow 00B11 \Rightarrow 00\#11$$

$\Rightarrow$ (yields)

Derivation

$$A \Rightarrow^{*} 00\#11$$

(derives)

CONTEXT-FREE GRAMMARS

start variable

$S \rightarrow A$

$A \rightarrow 0A1$

$A \rightarrow B$

$B \rightarrow \#$

production rules

variables

terminals

$A \Rightarrow 0A1 \Rightarrow 00A11 \Rightarrow 00B11 \Rightarrow 00\#11$

$\Rightarrow$ (yields)

Derivation

More generally, define **u** $\Rightarrow^*$ **v** (**u** derives **v**)

where **u** and **v** are strings of variables and terminals

CONTEXT-FREE GRAMMARS

A context-free grammar (CFG) is a tuple
$G = (V, \Sigma, R, S)$, where:

V is a finite set of variables

Σ is a finite set of terminals (disjoint from V)

R is set of production rules of the form $A \rightarrow W$, where $A \in V$ and $W \in (V \cup \Sigma)^*$

$S \in V$ is the start variable

CONTEXT-FREE LANGUAGES

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$L(G) = \{w \in \Sigma^* \mid S \Rightarrow^* w\}$ Strings Generated by G

A Language L is context-free if there is a CFG that generates precisely the strings in L

CHOMSKY NORMAL FORM

A context-free grammar is in Chomsky normal form if every rule is of the form:

$A \to BC$	**B and C aren't start variables**
$A \to a$	**a is a terminal**
$S \to \varepsilon$	**S is the start variable**

Any variable A that is not the start variable can only generate strings of length > 0

Theorem: If G is in CNF, $w \in L(G)$ and $|w| > 0$, then any derivation of w in G has length $2|w| - 1$

Proof (by induction on $|w|$):

Base Case: If $|w| = 1$, then any derivation of w must have length 1 $(A \rightarrow a)$

Inductive Step: Assume true for any string of length at most $k \geq 1$, and let $|w| = k+1$

Since $|w| > 1$, derivation starts with $A \rightarrow BC$

So $w = xy$ where $B \Rightarrow^* x$, $|x| > 0$ and $C \Rightarrow^* y$, $|y| > 0$

By the inductive hypothesis, the length of any derivation of w must be

$$1 + (2|x| - 1) + (2|y| - 1) = 2(|x| + |y|) - 1$$

Theorem: Any context-free language can be generated by a context-free grammar in Chomsky normal form

Theorem: **Any context-free language can be generated by a context-free grammar in Chomsky normal form**

Proof Idea:

1. Add a new start variable
2. Eliminate all $A \rightarrow \varepsilon$ rules (ε rules). Repair grammar
3. Eliminate all $A \rightarrow B$ rules (unit productions). Repair

4. Convert $A \rightarrow u_1 u_2 \ldots u_k$ to $A \rightarrow u_1 A_1$, $A_1 \rightarrow u_2 A_2$, ...
If u_i is a terminal, replace u_i with U_i and add $U_i \rightarrow u_i$

Convert the following into Chomsky normal form:

$A \rightarrow BAB \mid B \mid \varepsilon$

$B \rightarrow 00 \mid \varepsilon$

$S_0 \rightarrow A$

$A \rightarrow BAB \mid B \mid \varepsilon$

$B \rightarrow 00 \mid \varepsilon$

⇒

$S_0 \rightarrow A \mid \varepsilon$

$A \rightarrow BAB \mid B \mid BB \mid AB \mid BA$

$B \rightarrow 00$

⇓

$S_0 \rightarrow BAB \mid 00 \mid BB \mid AB \mid BA \mid \varepsilon$

$A \rightarrow BAB \mid 00 \mid BB \mid AB \mid BA$

$B \rightarrow 00$

⇓

$S_0 \rightarrow BC \mid DD \mid BB \mid AB \mid BA \mid \varepsilon$, $C \rightarrow AB$,

$A \rightarrow BC \mid DD \mid BB \mid AB \mid BA$, $B \rightarrow DD$, $D \rightarrow 0$

1. Add a new start variable S_0 and add the rule $S_0 \to S$

2. Remove all $A \to \varepsilon$ rules (where A is not S_0)

For each occurrence of A on right hand side of a rule, add a new rule with the occurrence deleted

If we have the rule $B \to A$, add $B \to \varepsilon$, unless we have previously removed $B \to \varepsilon$

3. Remove unit rules $A \to B$

Whenever $B \to w$ appears, add the rule $A \to w$ unless this was a unit rule previously removed

$S_0 \to S$
$S \to 0S1$
$S \to T\#T$
$S \to T$
$T \to \varepsilon$
$S \to T\#$
$S \to \#T$
$S \to \#$
$S \to \varepsilon$
$S \to 01$
$S_0 \to 0S1$
$S_0 \to \varepsilon$

4. Convert all remaining rules into the proper form:

$S_0 \rightarrow 0S1$

$S_0 \rightarrow A_1A_2$

$A_1 \rightarrow 0$

$A_2 \rightarrow SA_3$

$A_3 \rightarrow 1$

$S_0 \rightarrow 01$

$S_0 \rightarrow A_1A_3$

$S \rightarrow 01$

$S \rightarrow A_1A_3$

$S_0 \rightarrow \varepsilon$
$S_0 \rightarrow 0S1$
$S_0 \rightarrow T\#T$
$S_0 \rightarrow T\#$
$S_0 \rightarrow \#T$
$S_0 \rightarrow \#$
$S_0 \rightarrow 01$
$S \rightarrow 0S1$
$S \rightarrow T\#T$
$S \rightarrow T\#$
$S \rightarrow \#T$
$S \rightarrow \#$
$S \rightarrow 01$

Definition: A (non-deterministic) PDA is a tuple $P = (Q, \Sigma, \Gamma, \delta, q_0, F)$, where:

- Q is a finite set of states
- Σ is the input alphabet
- Γ is the stack alphabet
- $\delta : Q \times \Sigma_\varepsilon \times \Gamma_\varepsilon \to 2^{Q \times \Gamma_\varepsilon}$
- $q_0 \in Q$ is the start state
- $F \subseteq Q$ is the set of accept states

2^Q is the set of subsets of Q and $\Sigma_\varepsilon = \Sigma \cup \{\varepsilon\}$

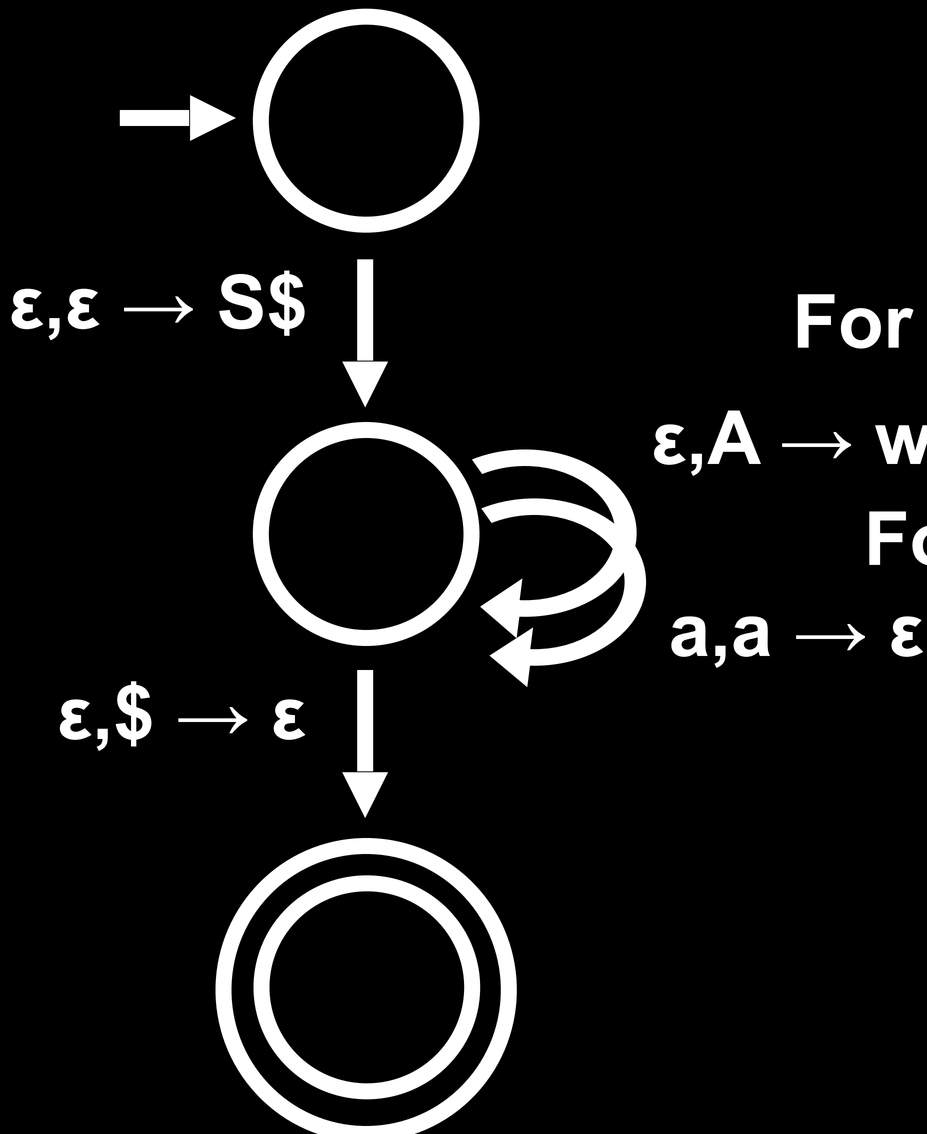
A Language L is generated by a CFG

$\Leftrightarrow$

L is recognized by a PDA

Suppose L is generated by a CFG G = (V, Σ, R, S)

Construct P = (Q, Σ, Γ, δ, q, F) that recognizes L

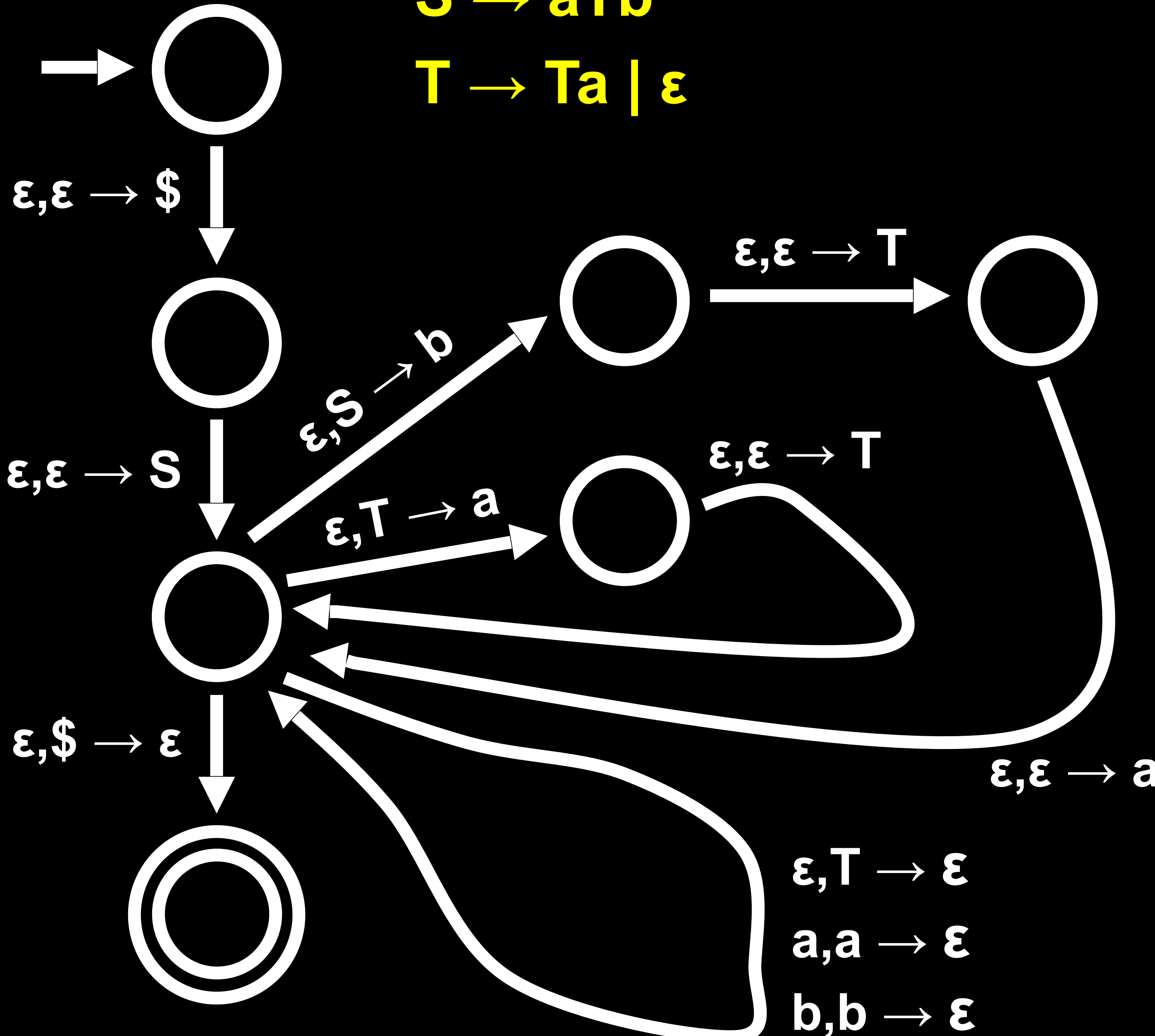


For each rule 'A → w' ∈ R:

For each terminal a ∈ Σ:

S → aTb

T → Ta | ε



A Language L is generated by a CFG

$\Leftarrow$

L is recognized by a PDA

Given PDA P = (Q, Σ, Γ, δ, q, F)

Construct a CFG G = (V, Σ, R, S) such that L(G) = L(P)

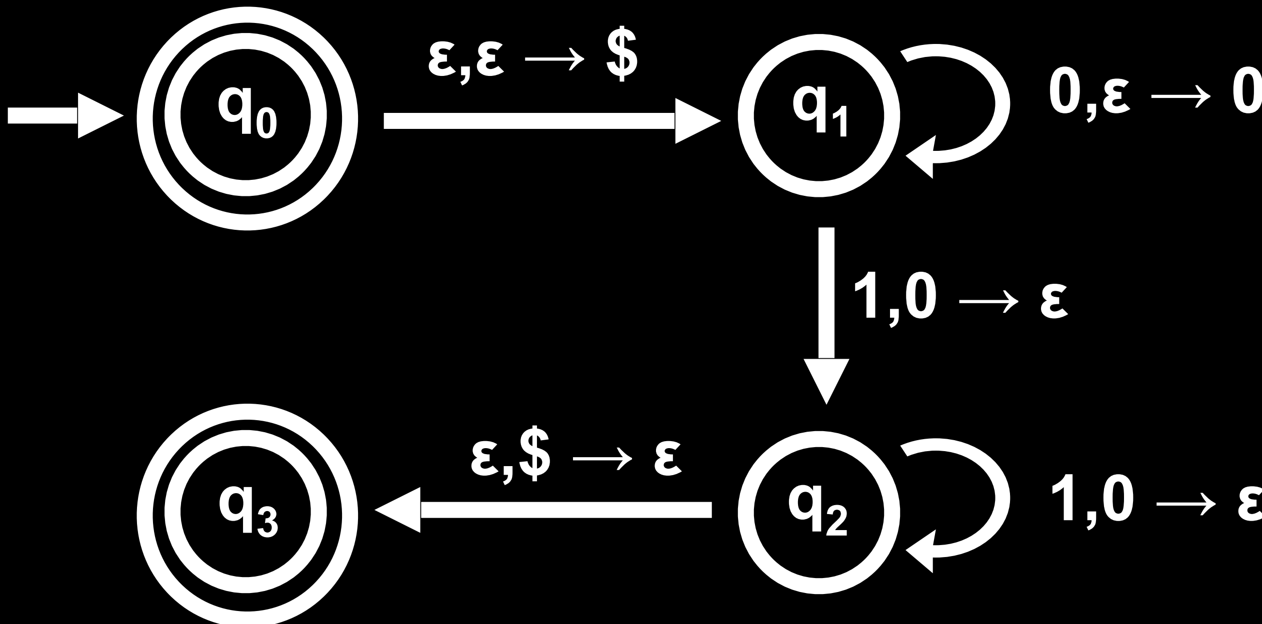
First, simplify P to have the following form:

(1) It has a single accept state, q_{accept}

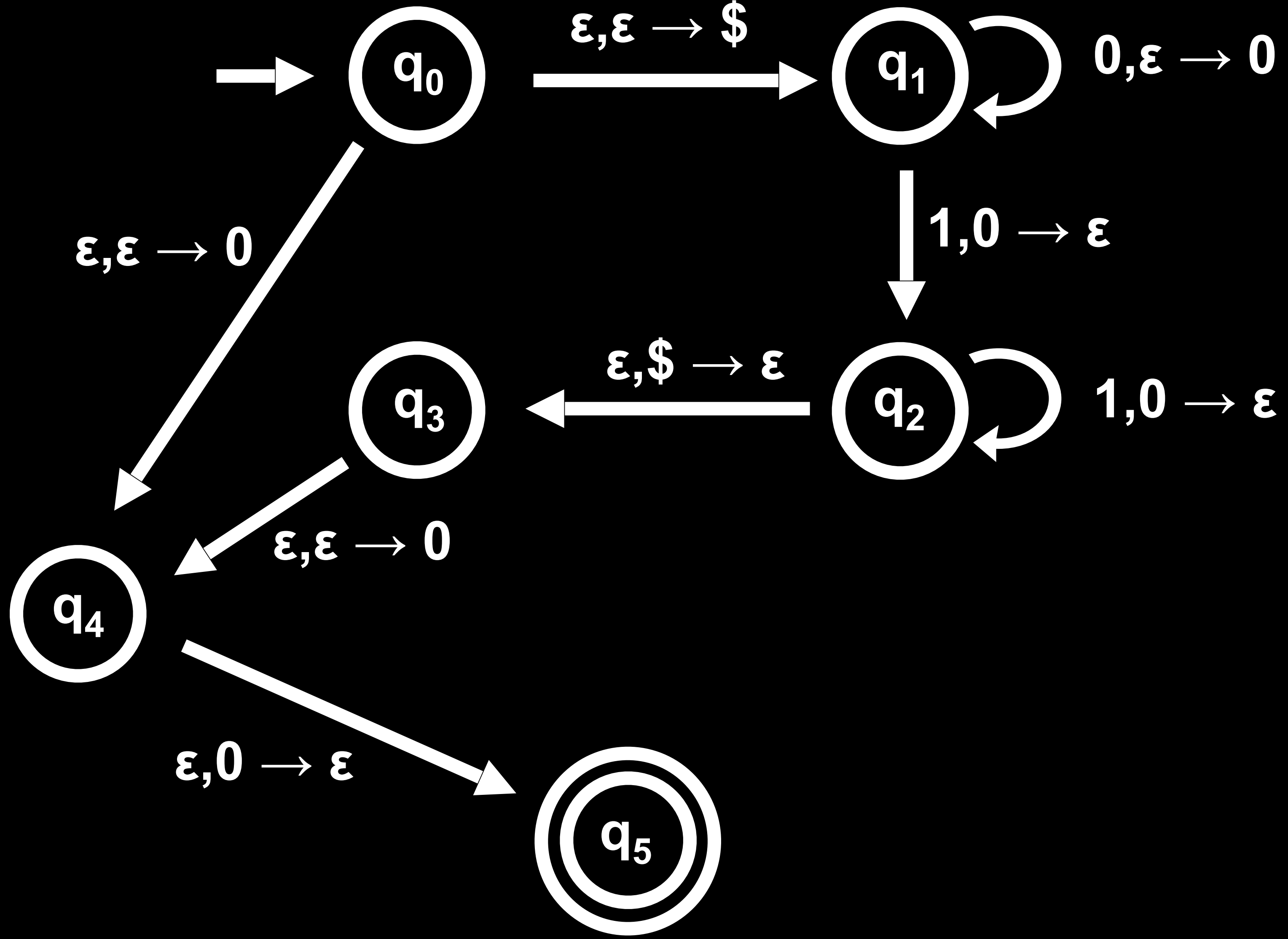
(2) It empties the stack before accepting

(3) Each transition either pushes a symbol or pops a symbol, but not both at the same time

SIMPLIFY



SIMPLIFY



Idea For Our Grammar G:

For every pair of states p and q in PDA P,

G will have a variable A_{pq} which generates all strings x that can take:

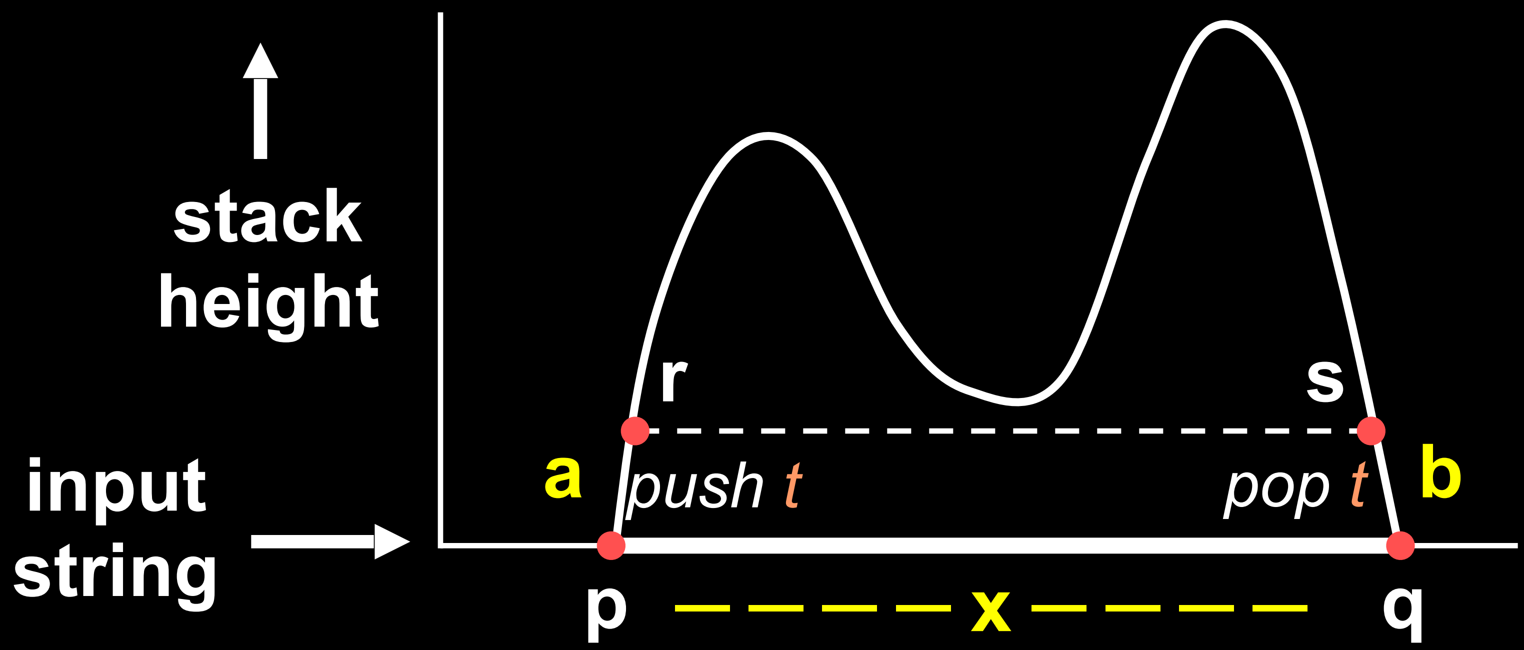
P from p with an empty stack
to q with an empty stack

$$V = \{A_{pq} \mid p,q \in Q\}$$

$$S = A_{q_0 q_{acc}}$$

x = ayb takes **p** with empty stack to **q** with empty stack

1. The symbol t popped at the end is exactly the one pushed at the beginning

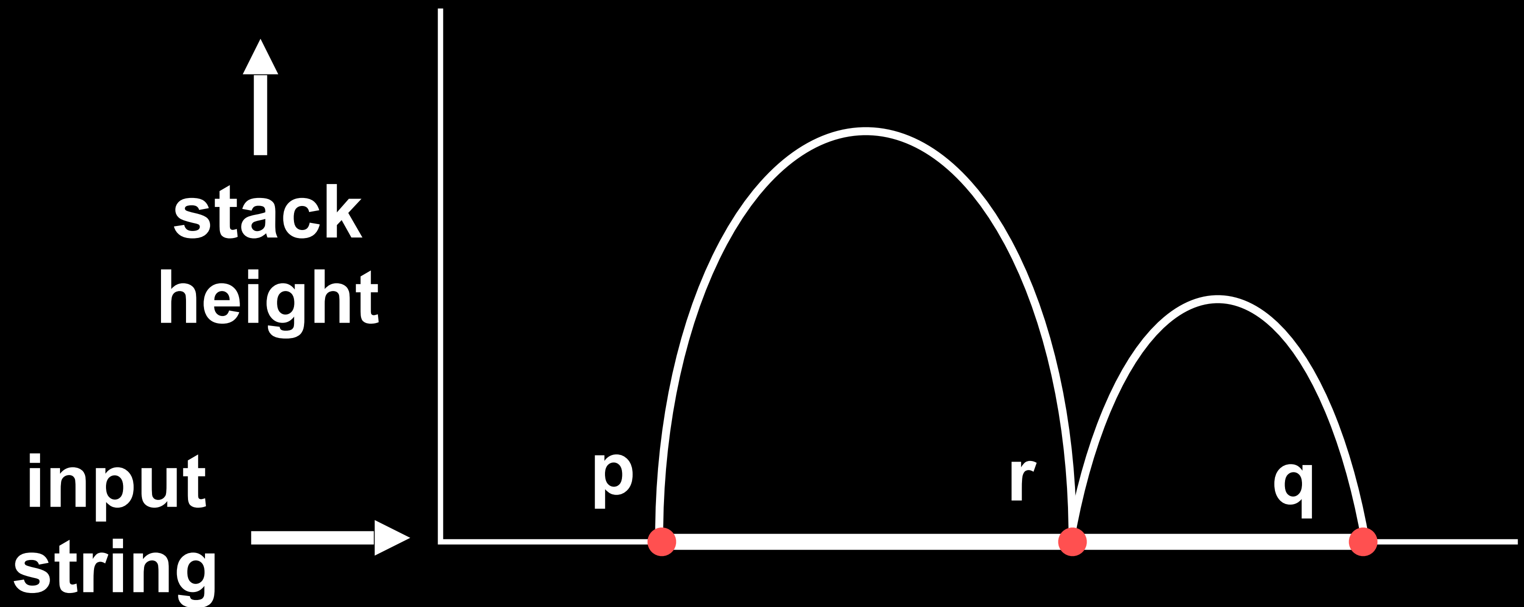


$\delta(p, a, \varepsilon) \rightarrow (r, t)$

$\delta(s, b, t) \rightarrow (q, \varepsilon)$

$$A_{pq} \rightarrow aA_{rs}b$$

2. The symbol popped at the end is **not** the one pushed at the beginning



$$A_{pq} \rightarrow A_{pr}A_{rq}$$

Formally:

$V = \{A_{pq} \mid p, q \in Q\}$

$S = A_{q_0 q_{acc}}$

For every p, q, r, s $\in$ Q, t $\in \Gamma$ and a, b $\in \Sigma_\varepsilon$

If $(r, t) \in \delta(p, a, \varepsilon)$ and $(q, \varepsilon) \in \delta(s, b, t)$

Then add the rule $A_{pq} \to aA_{rs}b$

For every p, q, r $\in$ Q,

add the rule $A_{pq} \to A_{pr}A_{rq}$

For every p $\in$ Q,

add the rule $A_{pp} \to \varepsilon$

THE PUMPING LEMMA

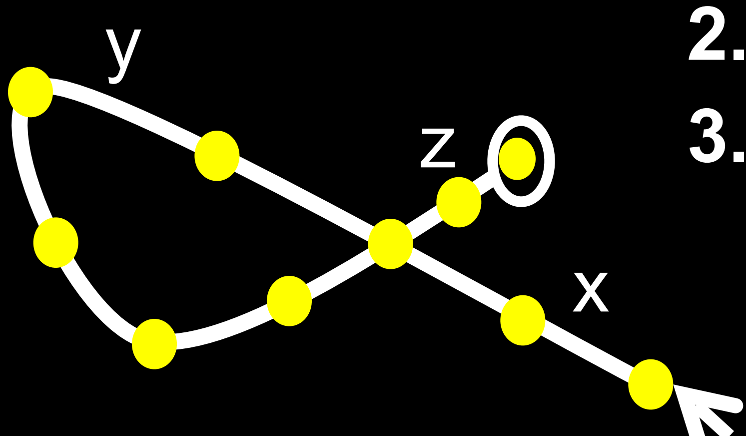
(for Regular Languages)

Let L be a regular language with |L| = ∞

Then there is an integer P such that

if w ∈ L and |w| ≥ P

then can write w = xyz, where:



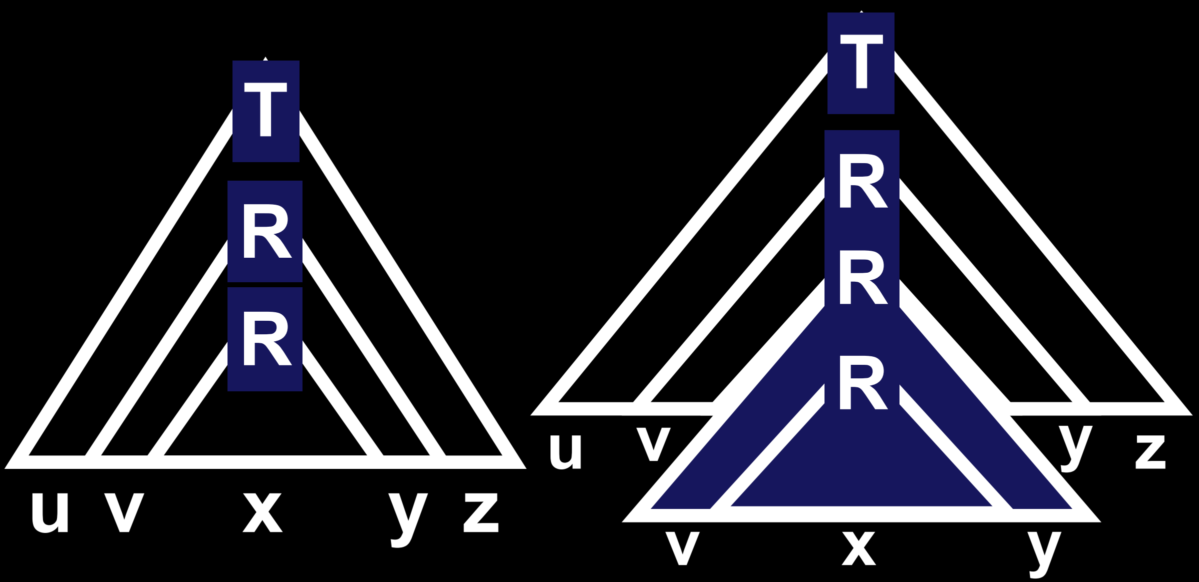
1. $|y| > 0$
2. $|xy| \le P$
3. $xy^iz \in L$ for any $i \ge 0$

THE PUMPING LEMMA
(for Context Free Grammars)

Let L be a context-free language with $|L| = \infty$

Then there is an integer P such that
if $w \in L$ and $|w| \geq P$

then can write $w = uvxyz$, where:



1. $|vy| > 0$
2. $|vxy| \leq P$
3. $uv^ixy^iz \in L$, for any $i \geq 0$

Machines	Syntactic Rules
DFAs	
↕	
NFAs ⟷	Regular Expressions
PDAs ⟷	Context-Free Grammars

deterministic DFA

A ^ finite automaton ^ is a 5-tuple $M = (Q, \Sigma, \delta, q_0, F)$

Q is the set of states (finite)

Σ is the alphabet (finite)

$\delta : Q \times \Sigma \to Q$ is the transition function

$q_0 \in Q$ is the start state

$F \subseteq Q$ is the set of accept states

Let $w_1, \ldots, w_n \in \Sigma$ and **w** $= w_1 \ldots w_n \in \Sigma^*$

Then **M accepts w** if there are $r_0, r_1, \ldots, r_n \in Q$, s.t.

1. $r_0 = q_0$
2. $\delta(r_i, w_{i+1}) = r_{i+1}$, for $i = 0, \ldots, n-1$, and
3. $r_n \in F$

Let $w \in \Sigma^*$ and suppose w can be written as $w_1 \ldots w_n$ where $w_i \in \Sigma_\varepsilon$ (ε = empty string)

Then N accepts w if there are $r_0, r_1, \ldots, r_n \in Q$ such that

1. $r_0 \in Q_0$
2. $r_{i+1} \in \delta(r_i, w_{i+1})$ for $i = 0, \ldots, n-1$, and
3. $r_n \in F$

L(N) = the language recognized by N
= set of all strings machine N accepts

A language L is recognized by an NFA N if L = L(N).

Let $w \in \Sigma^*$ and suppose w can be written as $w_1 \ldots w_n$ where $w_i \in \Sigma_\varepsilon$ (recall $\Sigma_\varepsilon = \Sigma \cup \{\varepsilon\}$)

Then P **accepts** w if there are

$r_0, r_1, \ldots, r_n \in Q$ and

$s_0, s_1, \ldots, s_n \in \Gamma^*$ (sequence of stacks) such that

1. $r_0 = q_0$ and $s_0 = \varepsilon$ (P starts in q_0 with empty stack)

2. For $i = 0, \ldots, n-1$:
$(r_{i+1}, b) \in \delta(r_i, w_{i+1}, a)$, where $s_i = at$ and $s_{i+1} = bt$ for some $a, b \in \Gamma_\varepsilon$ and $t \in \Gamma^*$
(P moves correctly according to state, stack and symbol read)

3. $r_n \in F$ (P is in an accept state at the end of its input)

THE REGULAR OPERATIONS

Union: $A \cup B = \{ w \mid w \in A \text{ or } w \in B \}$

Intersection: $A \cap B = \{ w \mid w \in A \text{ and } w \in B \}$

Negation: $\neg A = \{ w \in \Sigma^* \mid w \notin A \}$

Reverse: $A^R = \{ w_1 \ldots w_k \mid w_k \ldots w_1 \in A \}$

Concatenation: $A \cdot B = \{ vw \mid v \in A \text{ and } w \in B \}$

Star: $A^* = \{ s_1 \ldots s_k \mid k \geq 0 \text{ and each } s_i \in A \}$

REGULAR **EXPRESSIONS**

σ is a regexp representing $\{\sigma\}$

ε is a regexp representing $\{\varepsilon\}$

$\emptyset$ is a regexp representing $\emptyset$

If R_1 and R_2 are regular expressions representing L_1 and L_2 then:

(R_1R_2) represents $L_1 \cdot L_2$

$(R_1 \cup R_2)$ represents $L_1 \cup L_2$

$(R_1)^*$ represents L_1^*

How can we test if two regular expressions are the same?

Length n	R_1		R_2
	↓		↓
O(n) states	N_1		N_2
	↓		↓
$O(2^n)$ states	M_1		M_2
	↓		↓
	$M_{1\ MIN}$	?=	$M_{2\ MIN}$

THEOREMS and CONSTRUCTIONS

CONVERTING NFAs TO DFAs

Input: NFA **N** $= (Q, \Sigma, \delta, Q_0, F)$

Output: DFA **M** $= (Q', \Sigma, \delta', q_0', F')$

$$Q' = 2^Q$$

$$\delta' : Q' \times \Sigma \rightarrow Q'$$

$$\delta'(R,\sigma) = \bigcup_{r \in R} \varepsilon(\ \delta(r,\sigma)\ ) \quad *$$

$$q_0' = \varepsilon(Q_0)$$

$$F' = \{\ R \in Q' \mid f \in R \text{ for some } f \in F\ \}$$

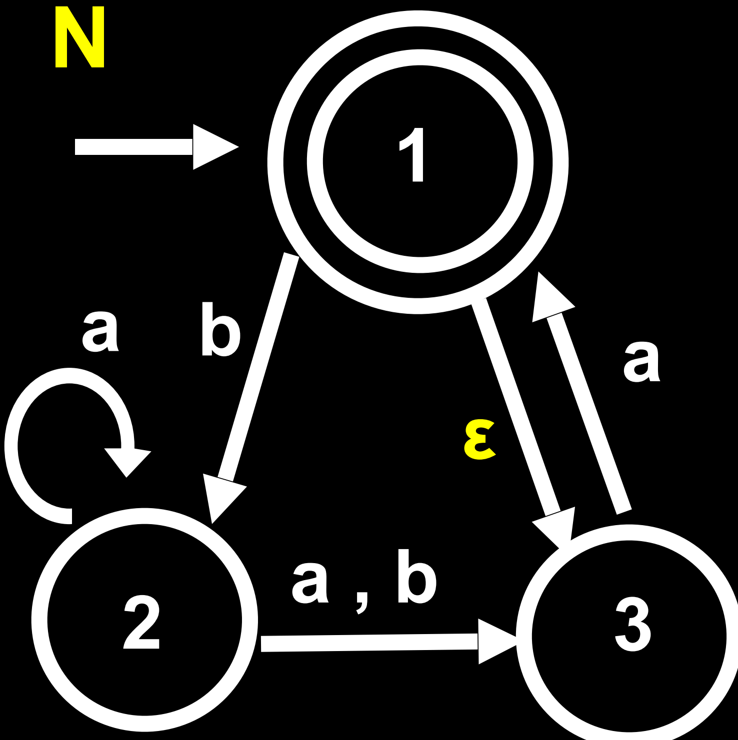
* For $R \subseteq Q$, the **ε-closure** of R, $\varepsilon(R)$ = {q that can be reached from some $r \in R$ by traveling along zero or more ε arrows}

Given: NFA N = ({1,2,3}, {a,b}, δ , {1}, {1})

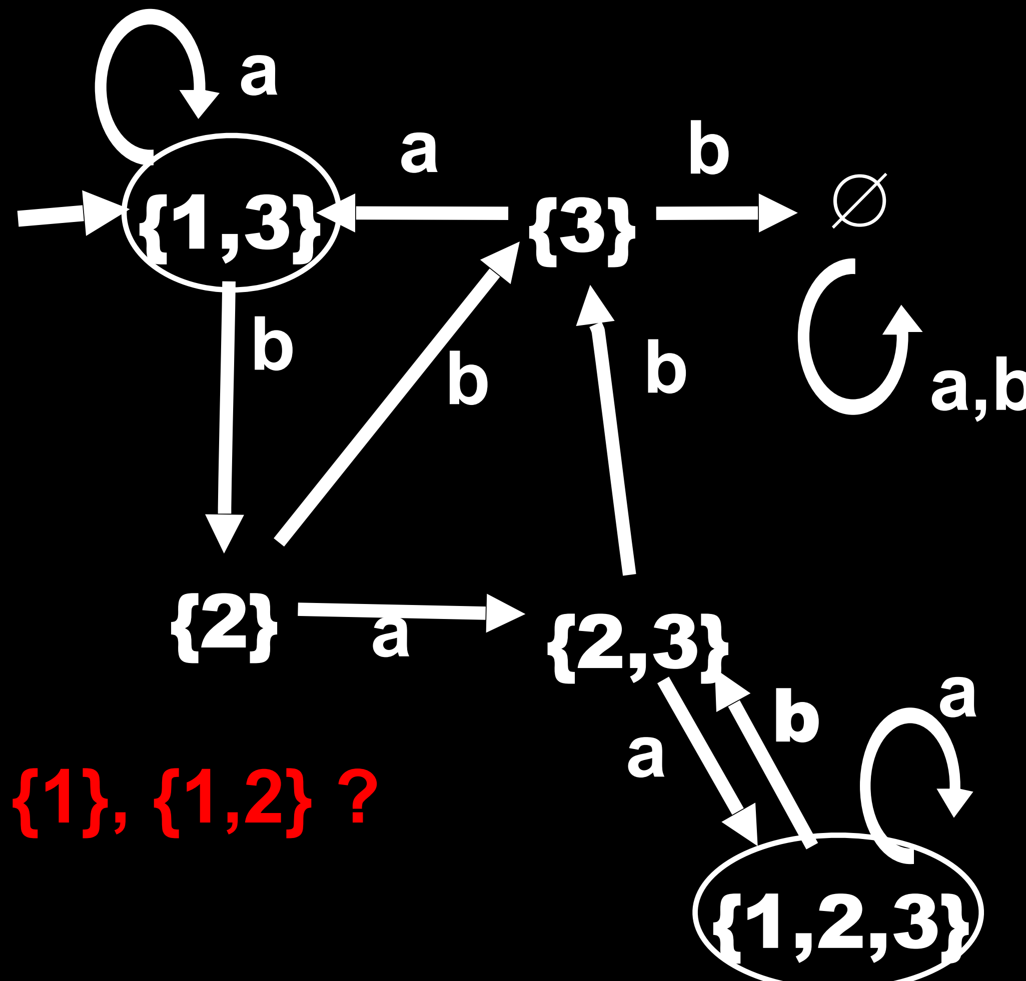
Construct: Equivalent DFA M

$M = (2^{\{1,2,3\}}, \{a,b\}, \delta', \{1,3\}, \ldots)$

N



$\varepsilon(\{1\}) = \{1,3\}$



{1}, {1,2} ?

EQUIVALENCE

L can be represented by a regexp

$\Leftrightarrow$

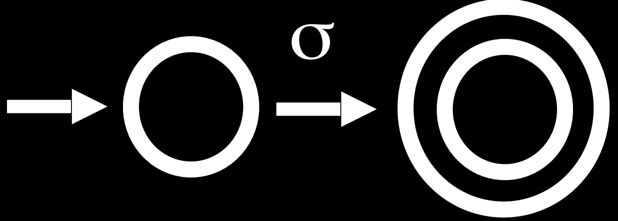
L is a regular language

✓ L can be represented by a regexp $\Rightarrow$ L is a regular language

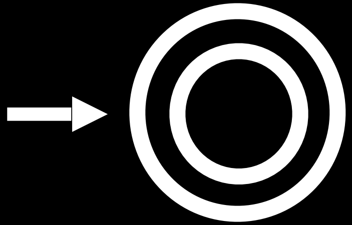
Induction on the length of R:

Base Cases (R has length 1):

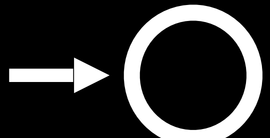
$R = \sigma$



$R = \varepsilon$



$R = \emptyset$



Inductive Step:

Assume R has length $k > 1$,
and that every regexp of length $< k$
represents a regular language

Three possibilities for what R can be:

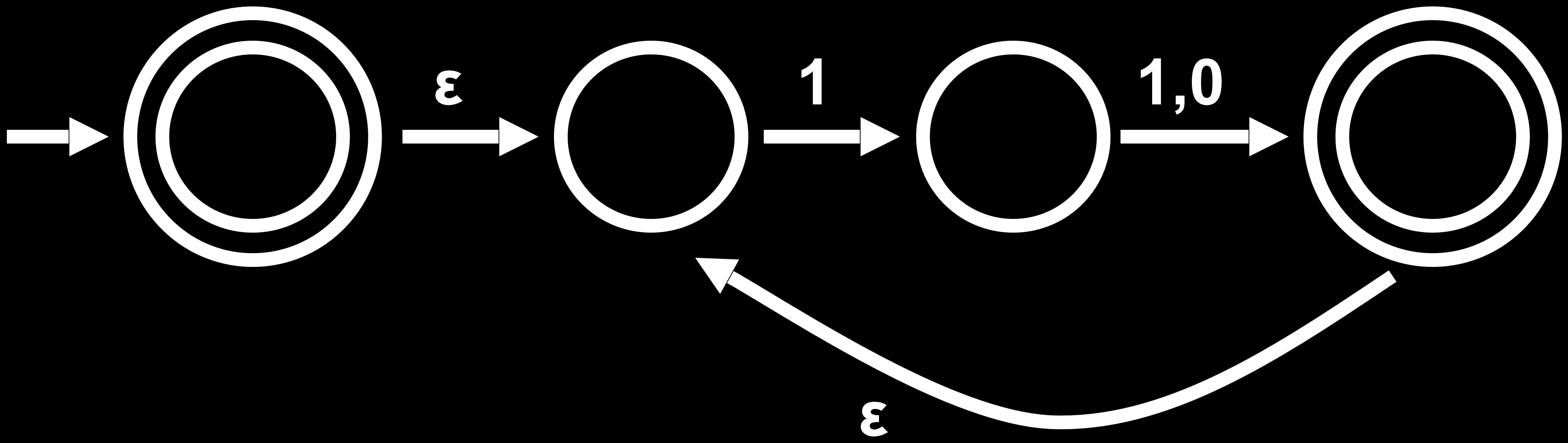
$R = R_1 \cup R_2$ (Closure under Union)

$R = R_1 R_2$ (Closure under Concat.)

$R = (R_1)^*$ (Closure under Star)

Therefore: L can be represented by a regexp
$\Rightarrow$ L is regular

Transform (1(0 ∪ 1))* to an NFA



✓

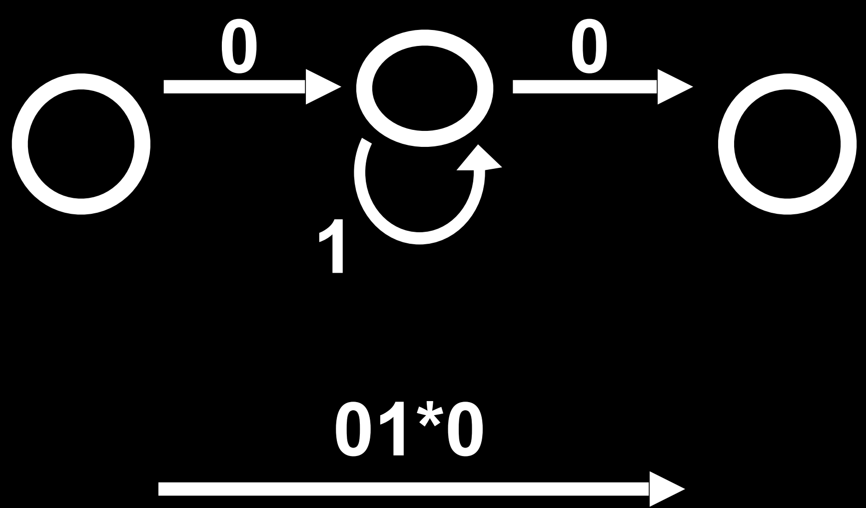
L is a regular language ⇒
L can be represented by a regexp

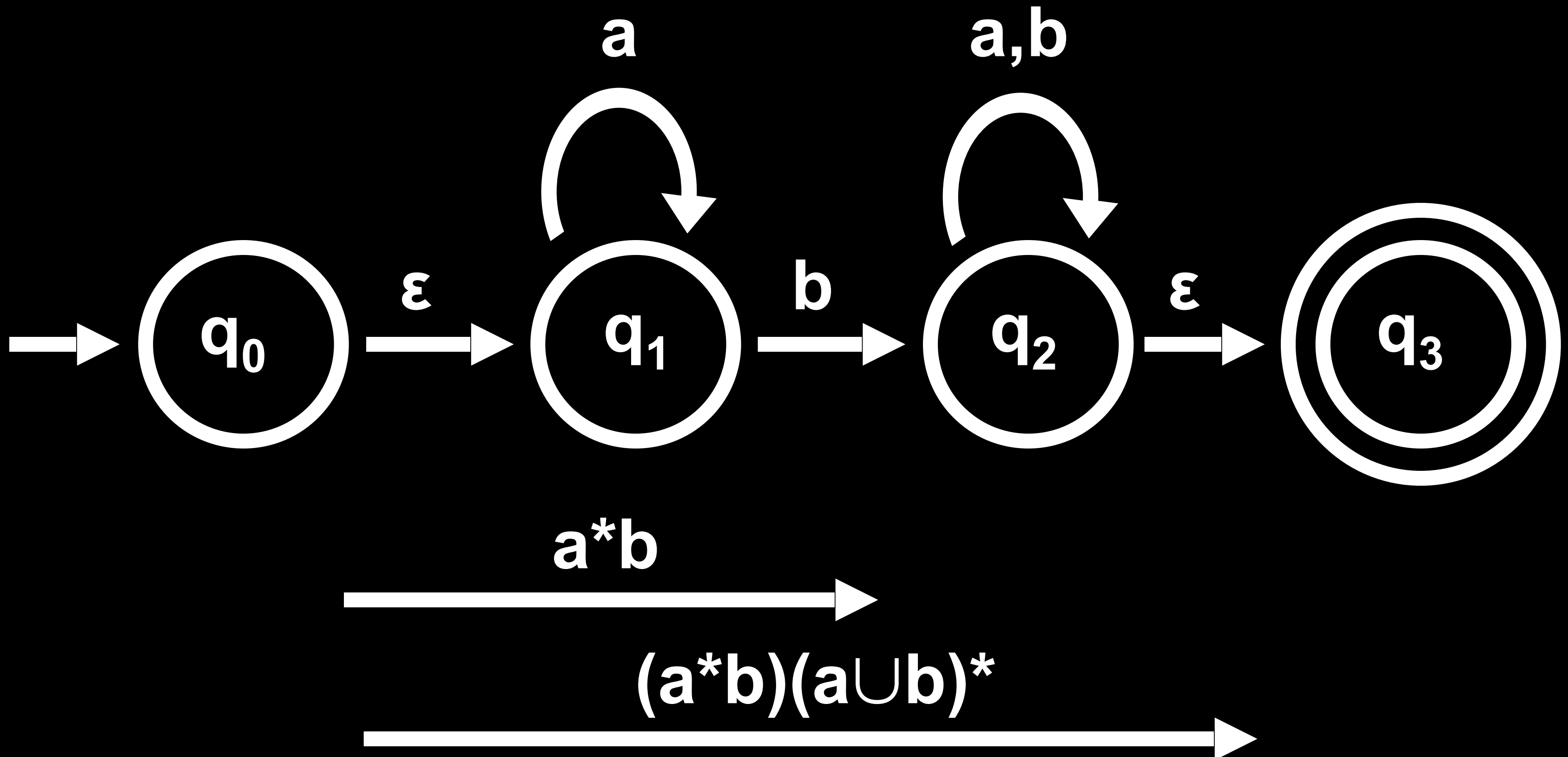
Proof idea: Transform an NFA for L into a regular expression by **removing states** and re-labeling the arrows with regular expressions

Add unique and distinct start and accept states

While machine has more than 2 states:

Pick an internal state, rip it out and re-label the arrows with regexps, to account for the missing state





$R(q_0,q_3) = (a^*b)(a \cup b)^*$

THEOREM

For every regular language L, there exists a UNIQUE (up to re-labeling of the states) minimal DFA M such that L = L(M)

EXTENDING δ

Given DFA **M** = $(Q, \Sigma, \delta, q_0, F)$, extend δ

to $\hat{\delta} : Q \times \Sigma^* \rightarrow Q$ as follows:

$$\hat{\delta}(q, \varepsilon) = q$$

$$\hat{\delta}(q, \sigma) = \delta(q, \sigma)$$

$$\hat{\delta}(q, w_1 \ldots w_{k+1}) = \delta(\hat{\delta}(q, w_1 \ldots w_k), w_{k+1})$$

Note: $\hat{\delta}(q_0, w) \in F \Leftrightarrow$ M accepts w

String $w \in \Sigma^*$ **distinguishes** states q_1 and q_2 iff exactly ONE of $\hat{\delta}(q_1, w)$, $\hat{\delta}(q_2, w)$ is a final state

Fix $M = (Q, \Sigma, \delta, q_0, F)$ and let $p, q, r \in Q$

Definition:

$p \sim q$ iff p is indistinguishable from q

$p \not\sim q$ iff p is distinguishable from q

Proposition: $\sim$ is an equivalence relation

$p \sim p$ (reflexive)

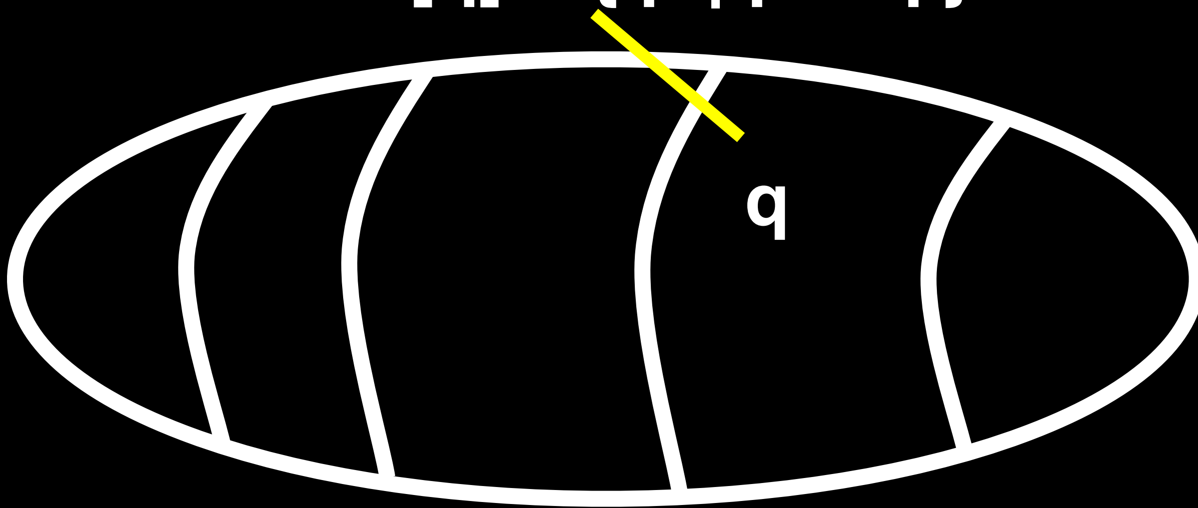
$p \sim q \Rightarrow q \sim p$ (symmetric)

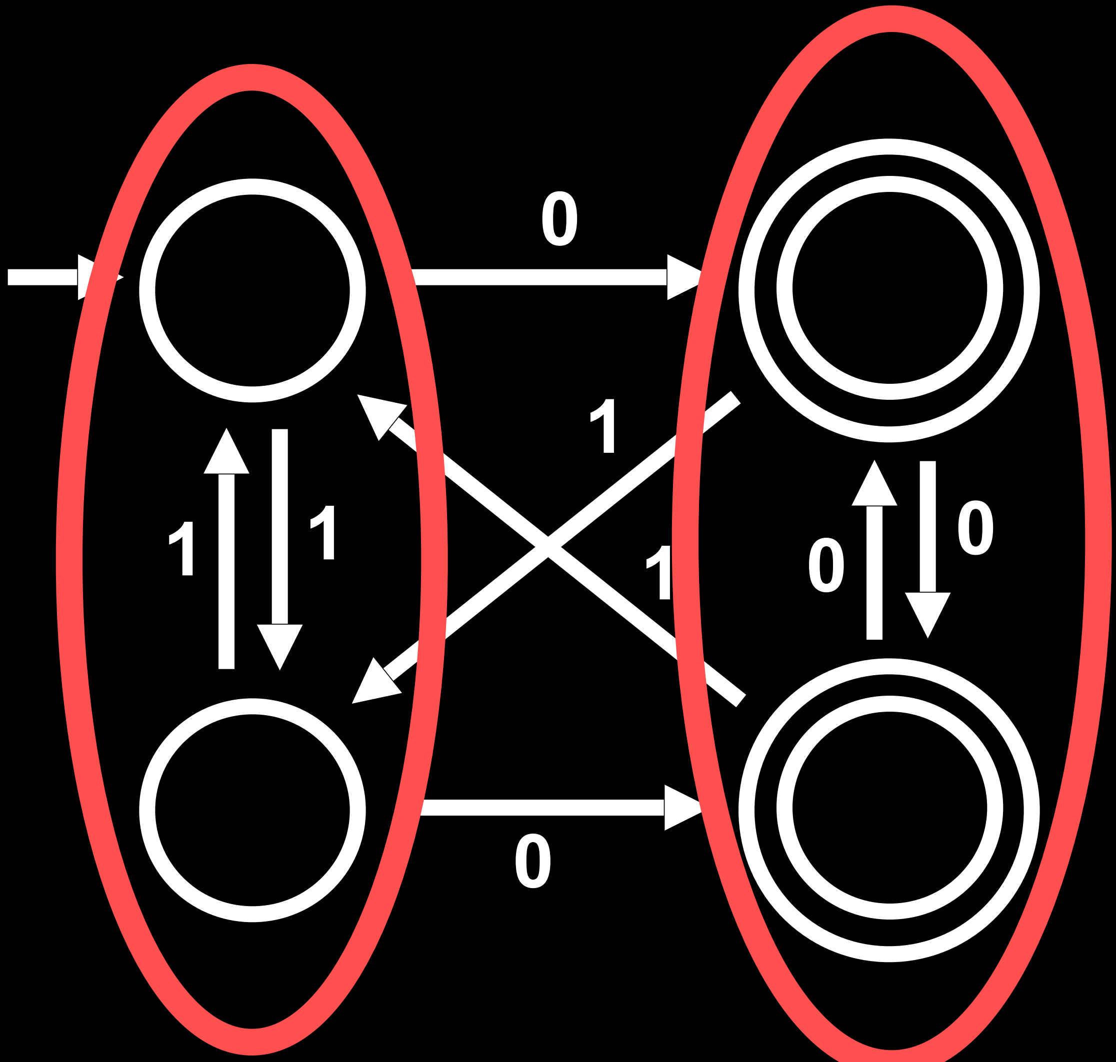
$p \sim q$ and $q \sim r \Rightarrow p \sim r$ (transitive)

Proposition: ~ is an **equivalence relation**

so ~ partitions the set of states of M into disjoint equivalence classes

$[q] = \{ p \mid p \sim q \}$





0
1
1
1
1
0
0
0

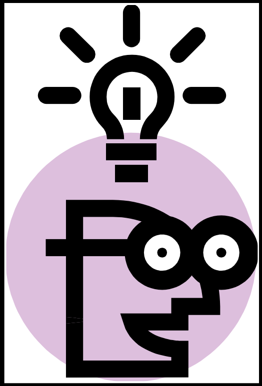
TABLE-FILLING ALGORITHM

Input: DFA $M = (Q, \Sigma, \delta, q_0, F)$

Output: (1) $D_M = \{ (p,q) \mid p,q \in Q \text{ and } p \not\sim q \}$

(2) $E_M = \{ [q] \mid q \in Q \}$

IDEA:



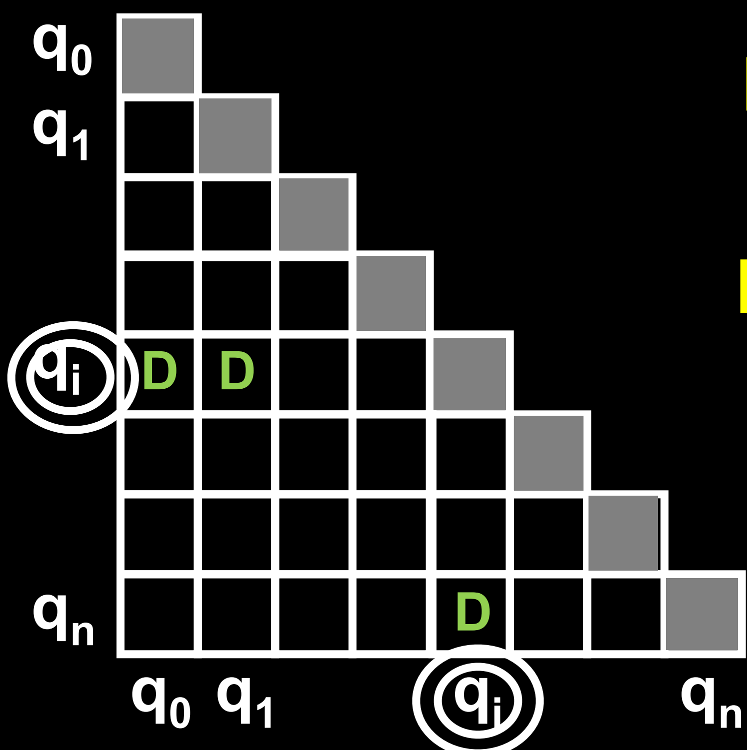
- **We know how to find those pairs of states that ε distinguishes…**
- **Use this and recursion to find those pairs distinguishable with *longer* strings**
- **Pairs left over will be indistinguishable**

TABLE-FILLING ALGORITHM

Input: DFA $M = (Q, \Sigma, \delta, q_0, F)$

Output: (1) $D_M = \{ (p,q) \mid p,q \in Q \text{ and } p \not\sim q \}$

(2) $E_M = \{ [q] \mid q \in Q \}$



Base Case: p accepts and q rejects $\Rightarrow p \not\sim q$

Recursion: if there is $\sigma \in \Sigma$ and states p', q' satisfying

$$\begin{array}{l} \delta(p, \sigma) = p' \\ \qquad\qquad \not\sim \qquad \Rightarrow \quad p \not\sim q \\ \delta(q, \sigma) = q' \end{array}$$

Repeat until no more new **D**'s

Algorithm MINIMIZE

Input: DFA M

Output: DFA M_{MIN}

(1) Remove all inaccessible states from M

(2) Apply Table-Filling algorithm to get
$E_M = \{ [q] \mid q \text{ is an accessible state of } M \}$

$$M_{MIN} = (Q_{MIN}, \Sigma, \delta_{MIN}, q_{0\,MIN}, F_{MIN})$$

$$Q_{MIN} = E_M, \quad q_{0\,MIN} = [q_0], \quad F_{MIN} = \{ [q] \mid q \in F \}$$

$$\delta_{MIN}([q],\sigma) = [\delta(q,\sigma)]$$

Claim: $M_{MIN} \equiv M$