15-453

FORMAL LANGUAGES, AUTOMATA AND COMPUTABILITY

THEOREM

For every regular language L, there exists a UNIQUE (up to re-labeling of the states) minimal DFA M_{mim} such that $L = L(M_{mim})$

Minimal means wrt number of states

PROOF

1. Let M be a DFA for L (wlog, assume no inaccessible states)
2. For pairs of states (p,q) define:

p *distinguishable* from q and
p *indistinguisable* from q (p~q).

3. Table-filling algorithm: first distinguish final from non-final states and then work backwards to distinguish more pairs.
4. What's left over are exactly the indistinguishable pairs, ie ~ related pairs. Needs proof.

PROOF

5. ~ is an equivalence relation so partitions the states into equivalence classes, $\mathbf{E_M}$
6. Define M_{min}

Define: $M_{MIN} = (Q_{MIN}, \Sigma, \delta_{MIN}, q_{0\,MIN}, F_{MIN})$

$Q_{MIN} = E_M$, $q_{0\,MIN} = [q_0]$, $F_{MIN} = \{ [q] \mid q \in F \}$

$\delta_{MIN}([q], \sigma) = [\delta(q, \sigma)]$ show well defined

Claim: $\hat{\delta}_{MIN}([q], w) = [\hat{\delta}(q, w)], w \in \Sigma^*$

So: $\hat{\delta}_{MIN}([q_0], w) = [\hat{\delta}(q_0, w)], w \in \Sigma^*$

Follows: $M_{MIN} \equiv M$

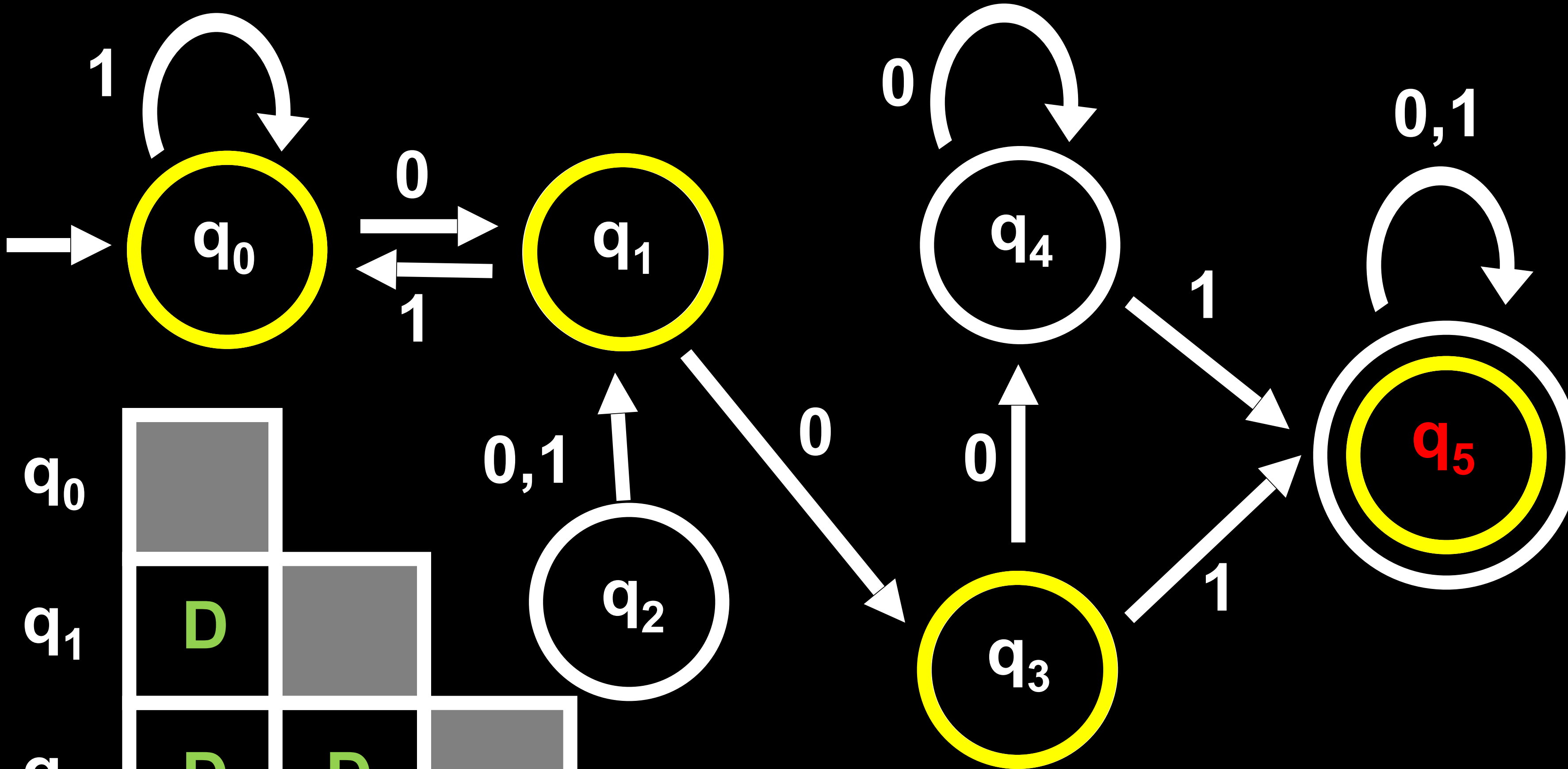
PROOF

But is M_{min} unique minimum?

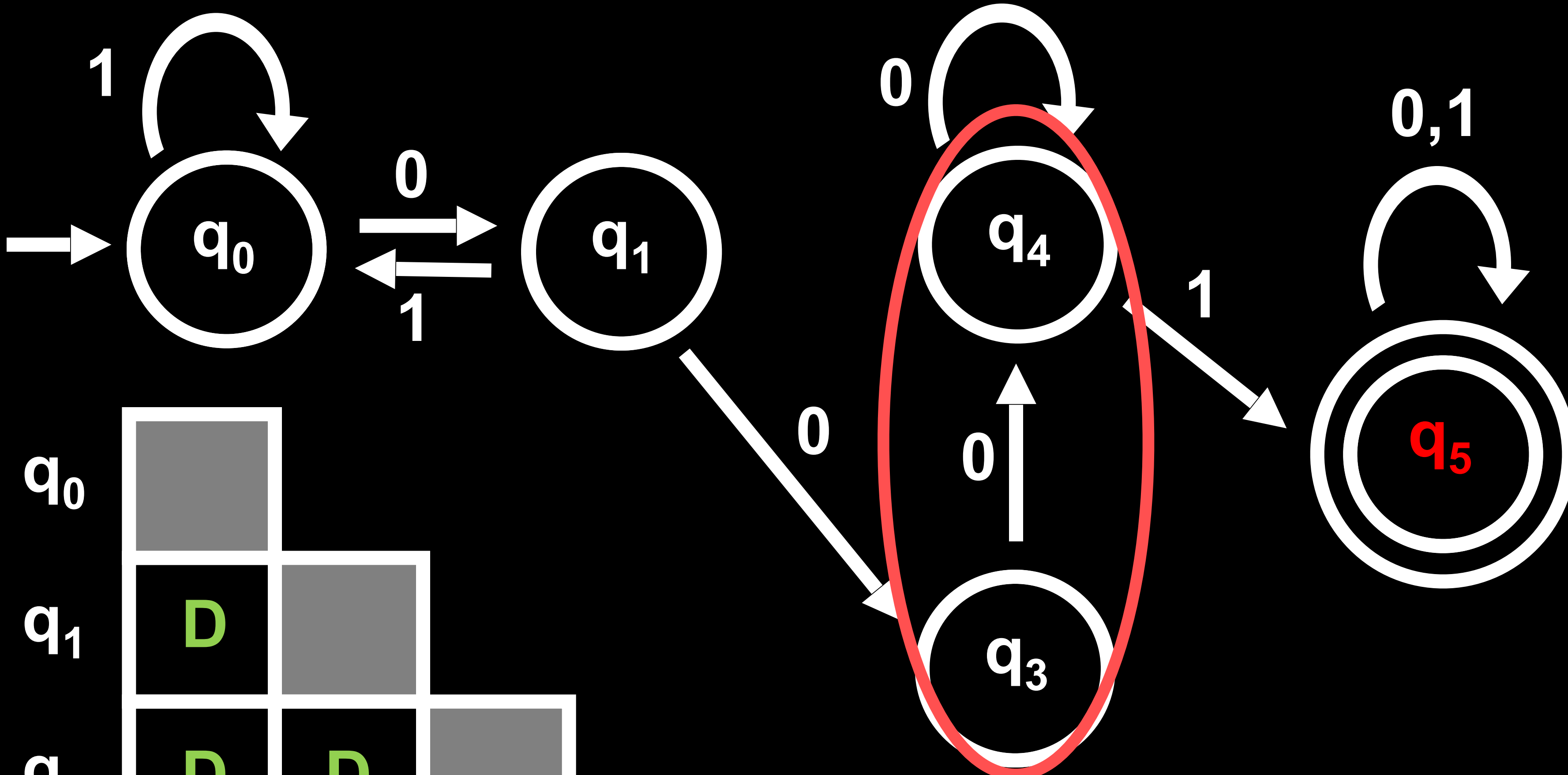
Yes, because if $M' \equiv M$ and minimum then
M' has no inaccesible states and is irreducible and
....

Theorem. M_{min} is isomorphic to any M' with the above properties
(need to give mapping and prove it has all the needed properties: everywhere defined , well defined, 1-1, onto, preserves transitions, and {final states} map onto {final states})

So M_{min} is isomorphic to *any* minimum $M' \equiv M$



	q_0	q_1	q_3	q_4	q_5
q_0					
q_1	D				
q_3	D	D			
q_4	D	D			
q_5	D	D	D	D	



	q0	q1	q3	q4	q5
q0					
q1	D				
q3	D	D			
q4	D	D			
q5	D	D	D	D	

How can we prove that two DFAs are equivalent?

One way: Minimize

Another way: Let $C = (\neg A \cap B) \cup (A \cap \neg B)$

Then, $A = B \Leftrightarrow C = \varnothing$

C is the "disjoint union"

CONTEXT-FREE GRAMMARS AND PUSH-DOWN AUTOMATA

TUESDAY Jan 28

NONE OF THESE ARE REGULAR

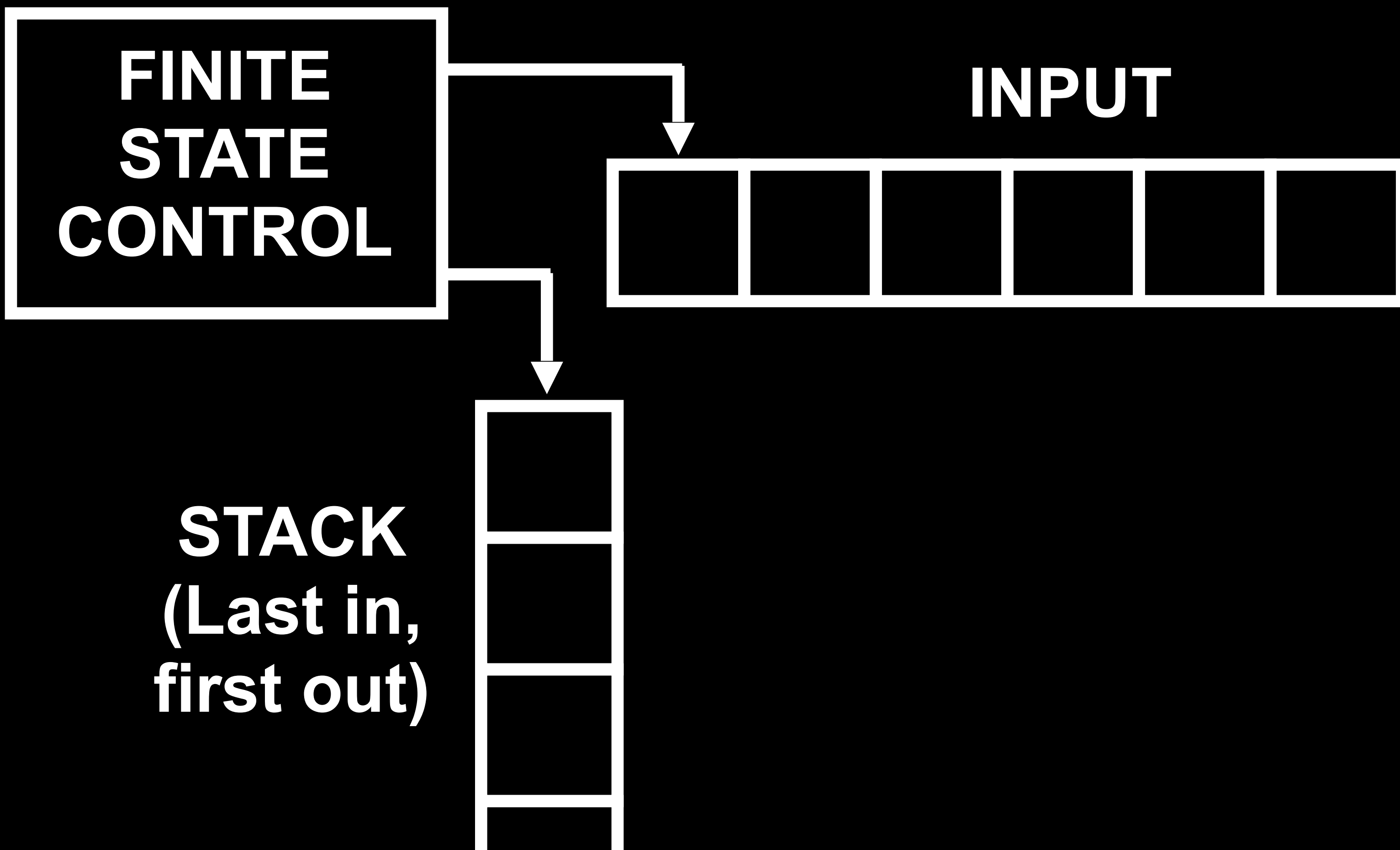
$\Sigma = \{0, 1\}$, $L = \{ 0^n 1^n \mid n \geq 0 \}$

$\Sigma = \{a, b, c, \ldots, z\}$, $L = \{ w \mid w = w^R \}$

$\Sigma = \{ (,) \}$, L = { balanced strings of parens }

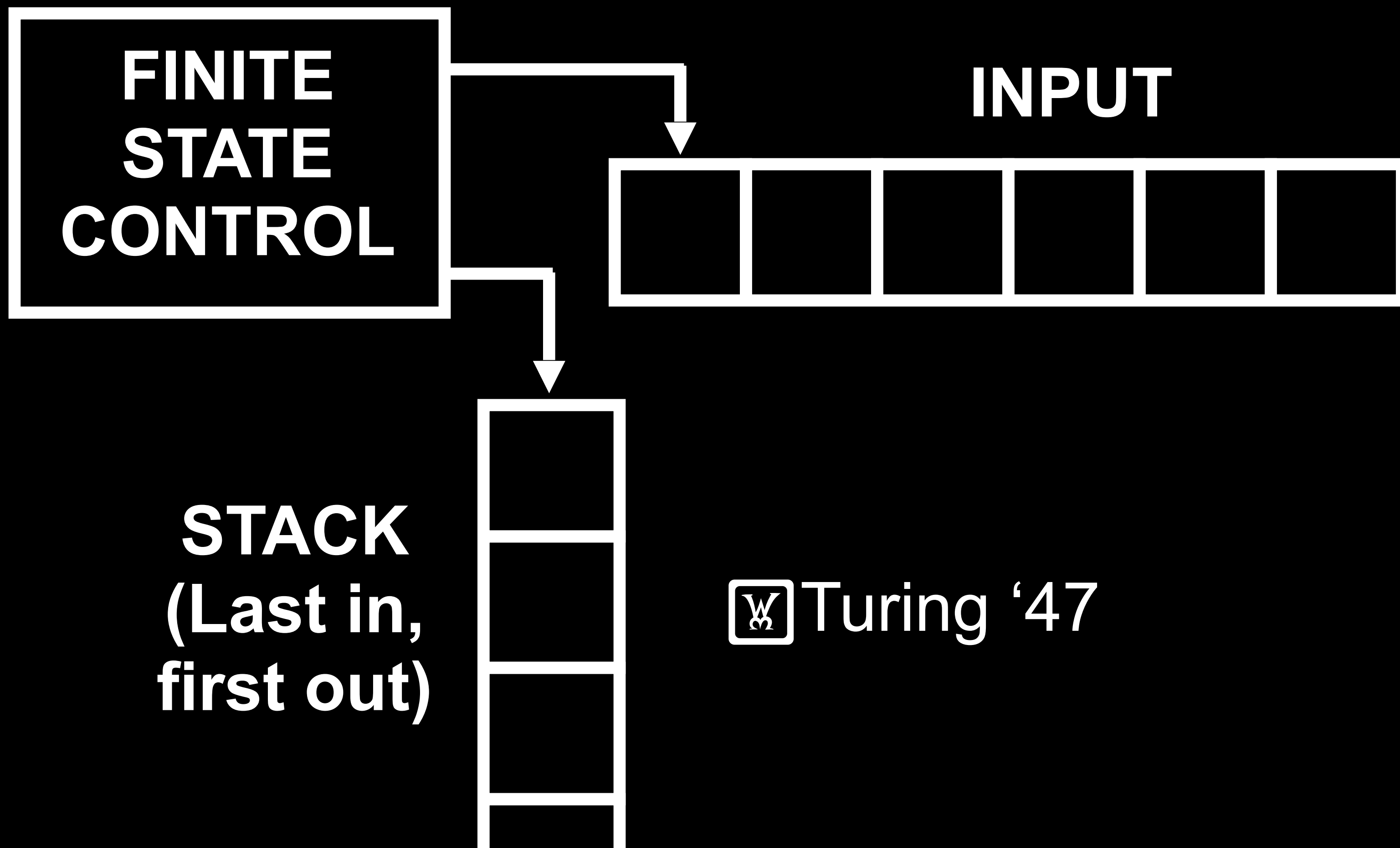
(), ()(), (()()) are in L, (, ()), ())(() are **not** in L

PUSHDOWN AUTOMATA (PDA)



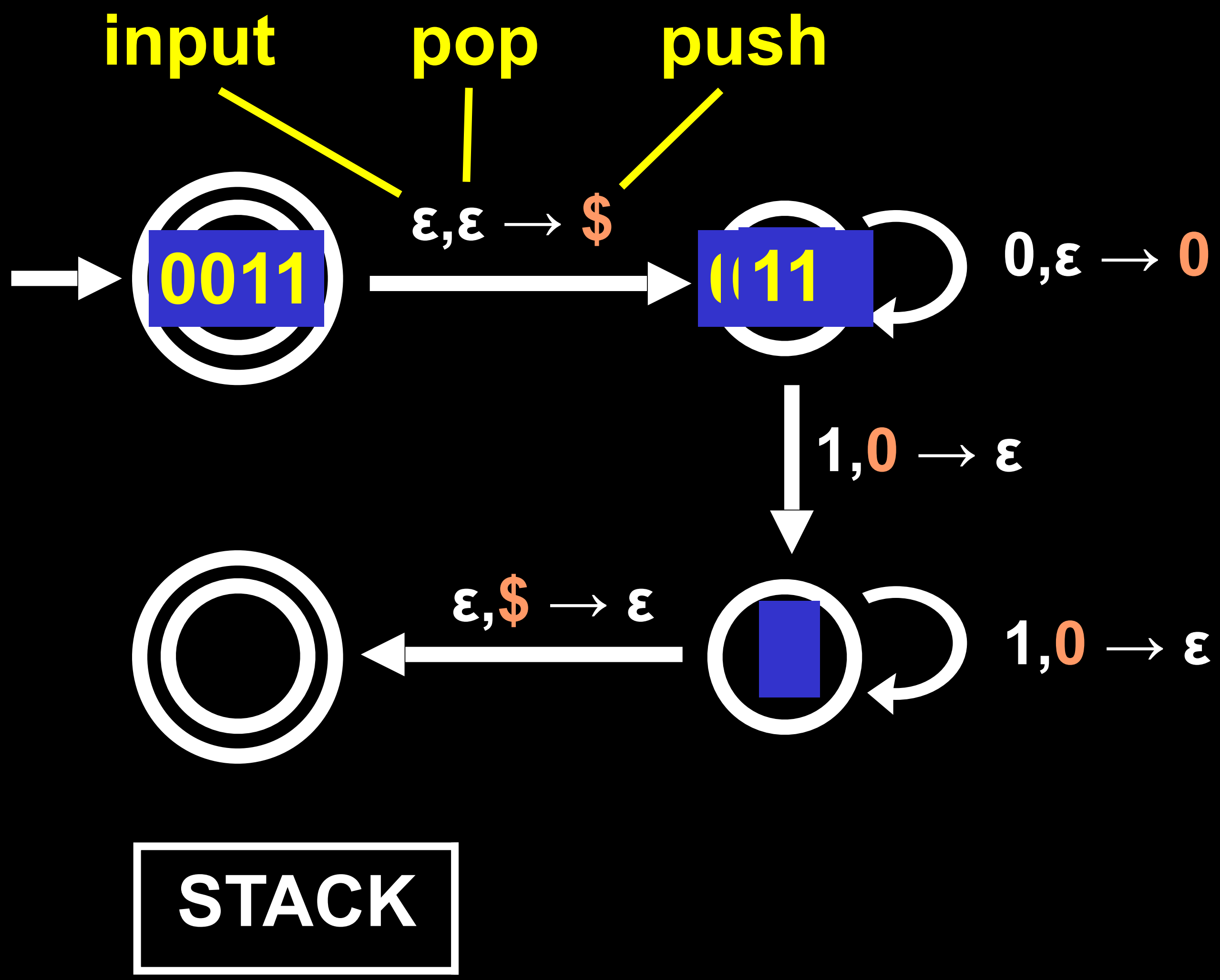
Newell, A., Shaw, J.C., & Simon, H.A. ”Report on a general problem-solving program in Information Processing”, Proc. International Conference, UNESCO Paris 1959

PUSHDOWN AUTOMATA (PDA)

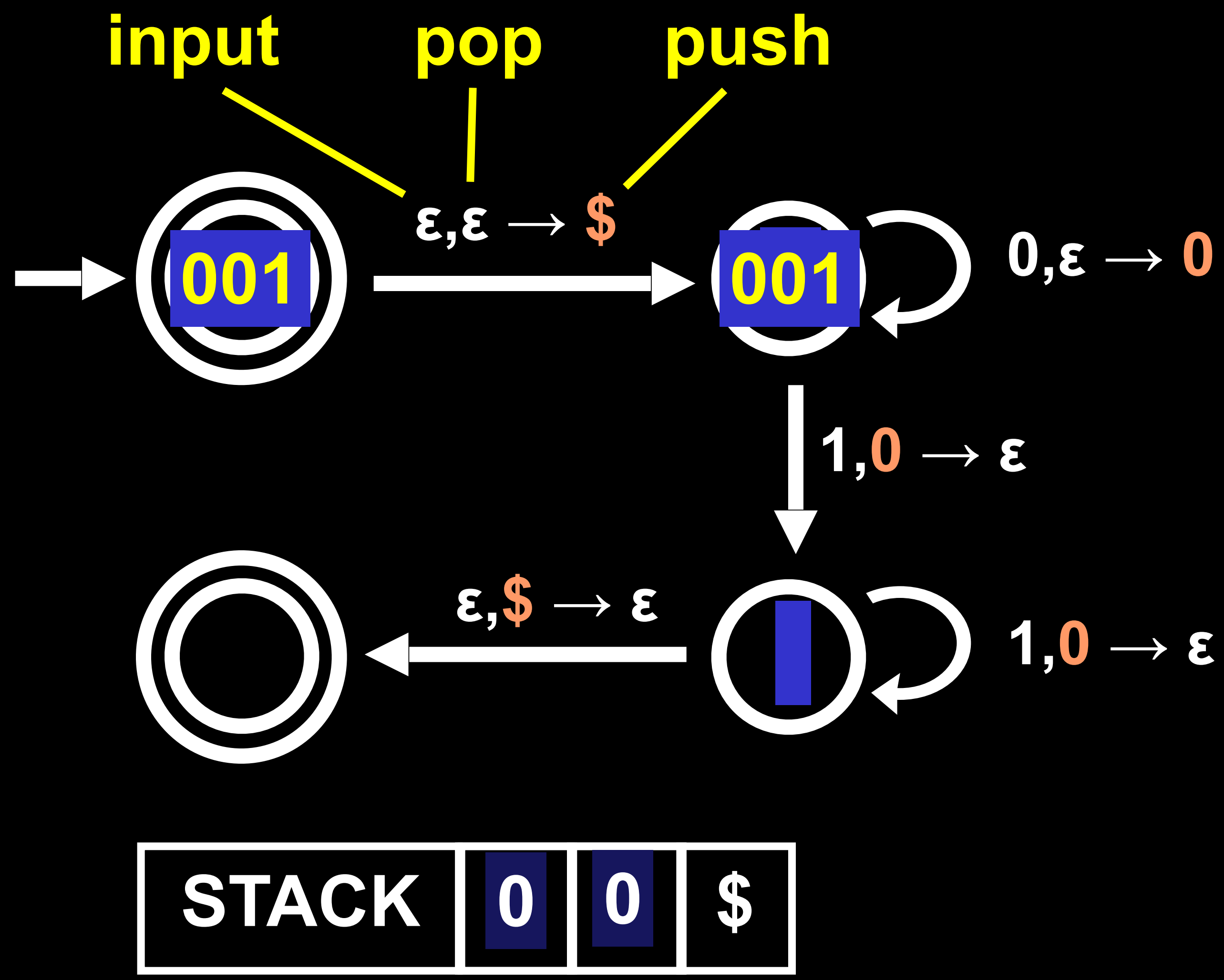


Turing '47

A brief history of the stack, Sten Henriksson,
Computer Science Department, Lund University, Lund, Sweden.



Non-deterministic



PDA that recognizes $L = \{ 0^n1^n \mid n \geq 0 \}$

Definition: A (*non-deterministic*) PDA is a 6-tuple $P = (Q, \Sigma, \Gamma, \delta, q_0, F)$, where:

Q is a finite set of states

Σ is the input alphabet

Γ is the stack alphabet

pop

push

$\delta : Q \times \Sigma_\varepsilon \times \Gamma_\varepsilon \rightarrow 2^{Q \times \Gamma_\varepsilon}$

$q_0 \in Q$ is the start state

$F \subseteq Q$ is the set of accept states

$2^{Q \times \Gamma_\varepsilon}$ is the set of subsets of $Q \times \Gamma_\varepsilon$

$\Sigma_\varepsilon = \Sigma \cup \{\varepsilon\}$, $\Gamma_\varepsilon = \Gamma \cup \{\varepsilon\}$

Let $w \in \Sigma^*$ and suppose w can be written as $w_1 \ldots w_n$ where $w_i \in \Sigma_\varepsilon$ (recall $\Sigma_\varepsilon = \Sigma \cup \{\varepsilon\}$)

Then P accepts w if there are

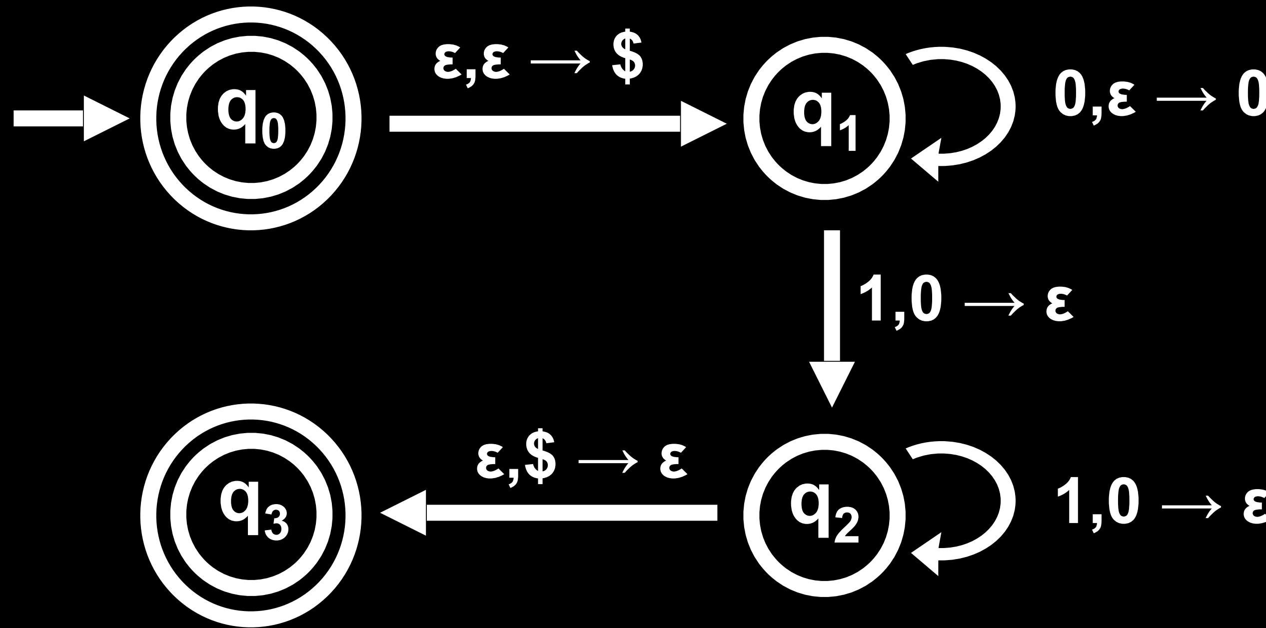
$r_0, r_1, \ldots, r_n \in Q$ and

$s_0, s_1, \ldots, s_n \in \Gamma^*$ (sequence of stacks) such that

1. $r_0 = q_0$ and $s_0 = \varepsilon$ (P starts in q_0 with empty stack)

2. For $i = 0, \ldots, n-1$:
$(r_{i+1}, b) \in \delta(r_i, w_{i+1}, a)$, where $s_i = at$ and $s_{i+1} = bt$ for some $a, b \in \Gamma_\varepsilon$ and $t \in \Gamma^*$
(P moves correctly according to state, stack and symbol read)

3. $r_n \in F$ (P is in an accept state at the end of its input)



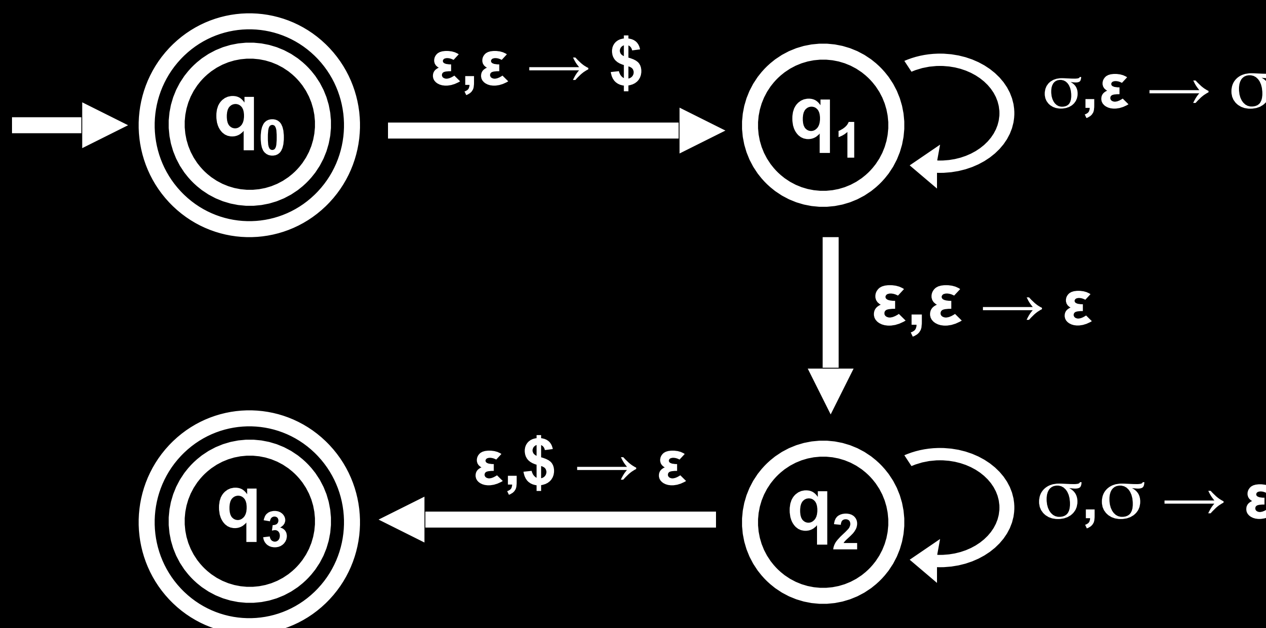
$Q = \{q_0, q_1, q_2, q_3\}$ $\Sigma = \{0,1\}$ $\Gamma = \{\$,0,1\}$

$\delta : Q \times \Sigma_\varepsilon \times \Gamma_\varepsilon \rightarrow 2^{Q \times \Gamma_\varepsilon}$

$\delta(q_1,1,0) = \{ (q_2,\varepsilon) \}$ $\delta(q_2,1,1) = \varnothing$

EVEN-LENGTH **PALINDROMES**

$\Sigma = \{a, b, c, \ldots, z\}$

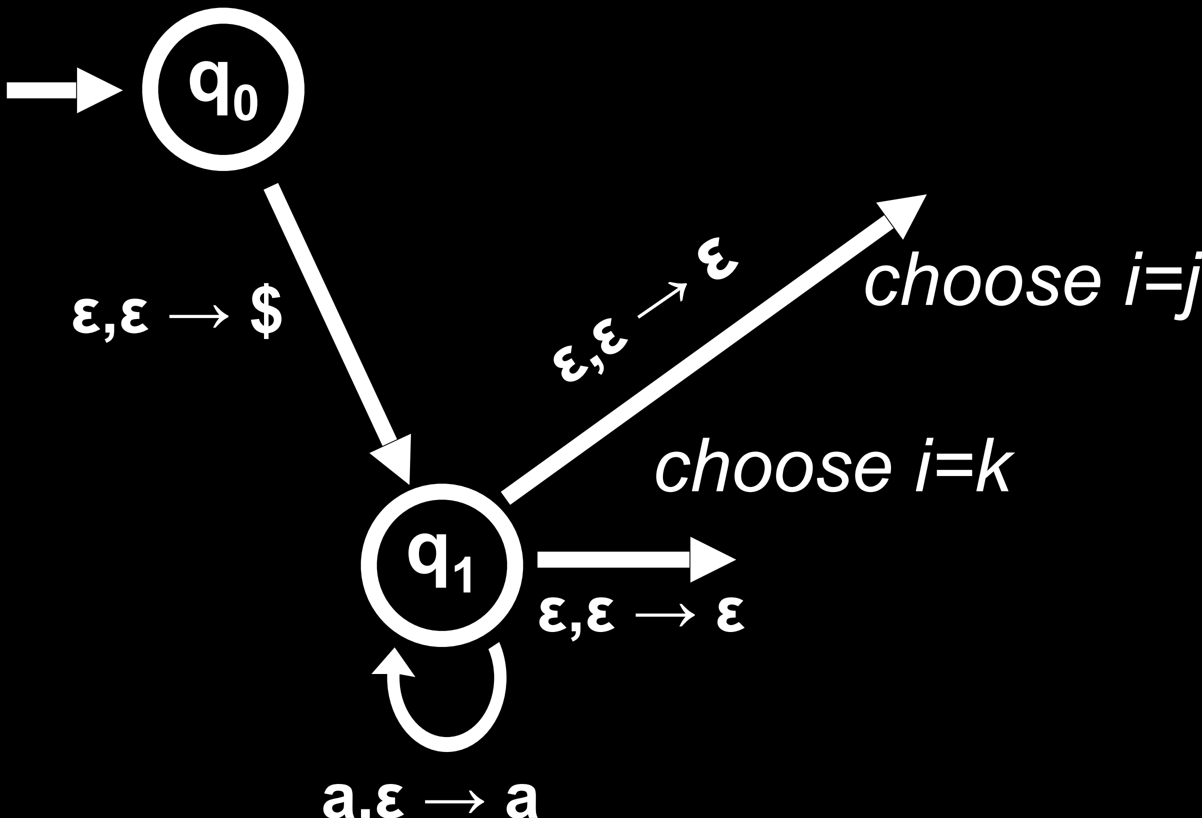


zeus sees suez

Madamimadam

(How to recognize odd-length palindromes?)

Build a PDA to recognize $L = \{ a^i b^j c^k \mid i, j, k \geq 0$ **and** $(i = j \text{ or } i = k) \}$



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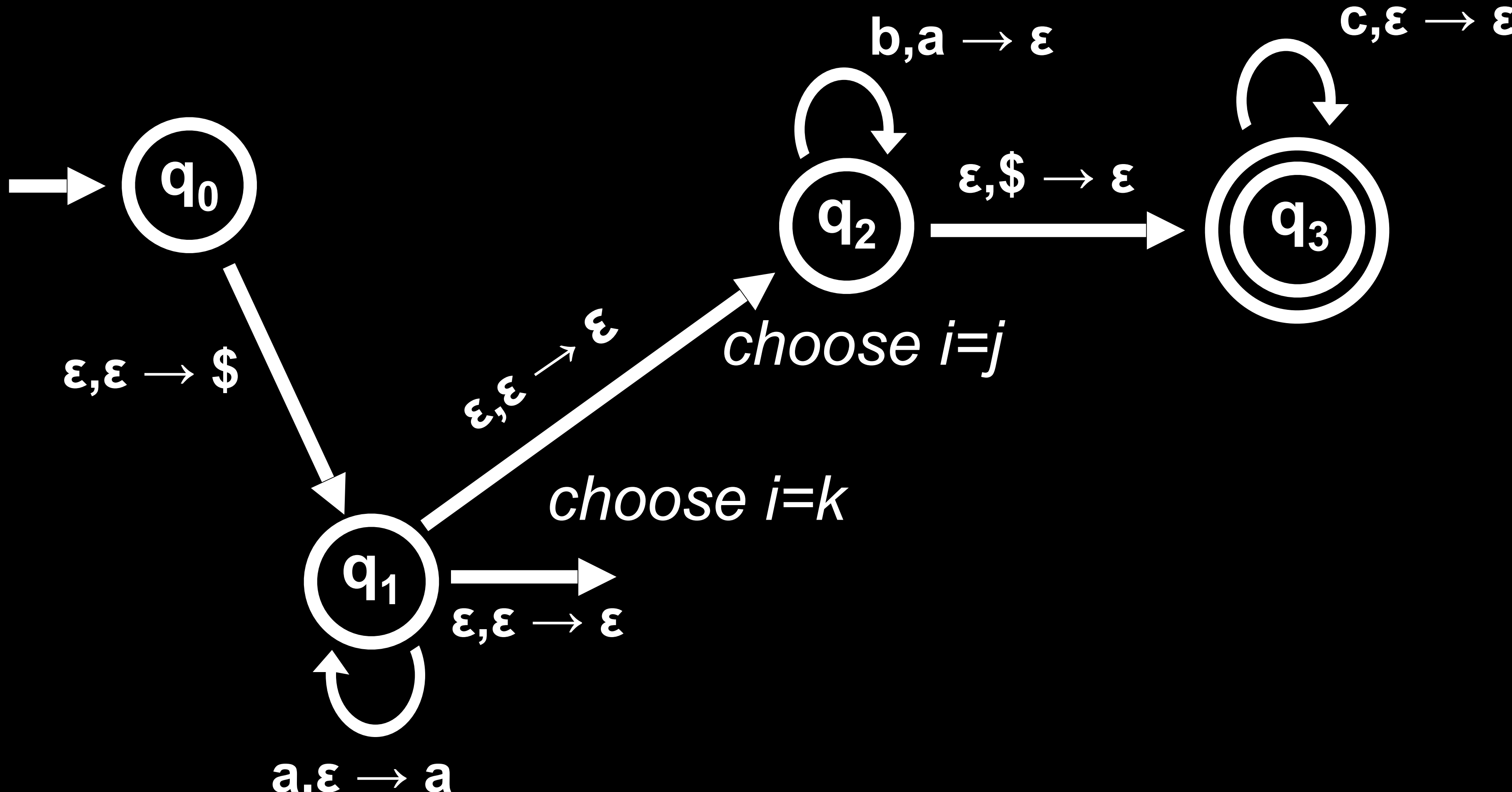

Build a PDA to recognize
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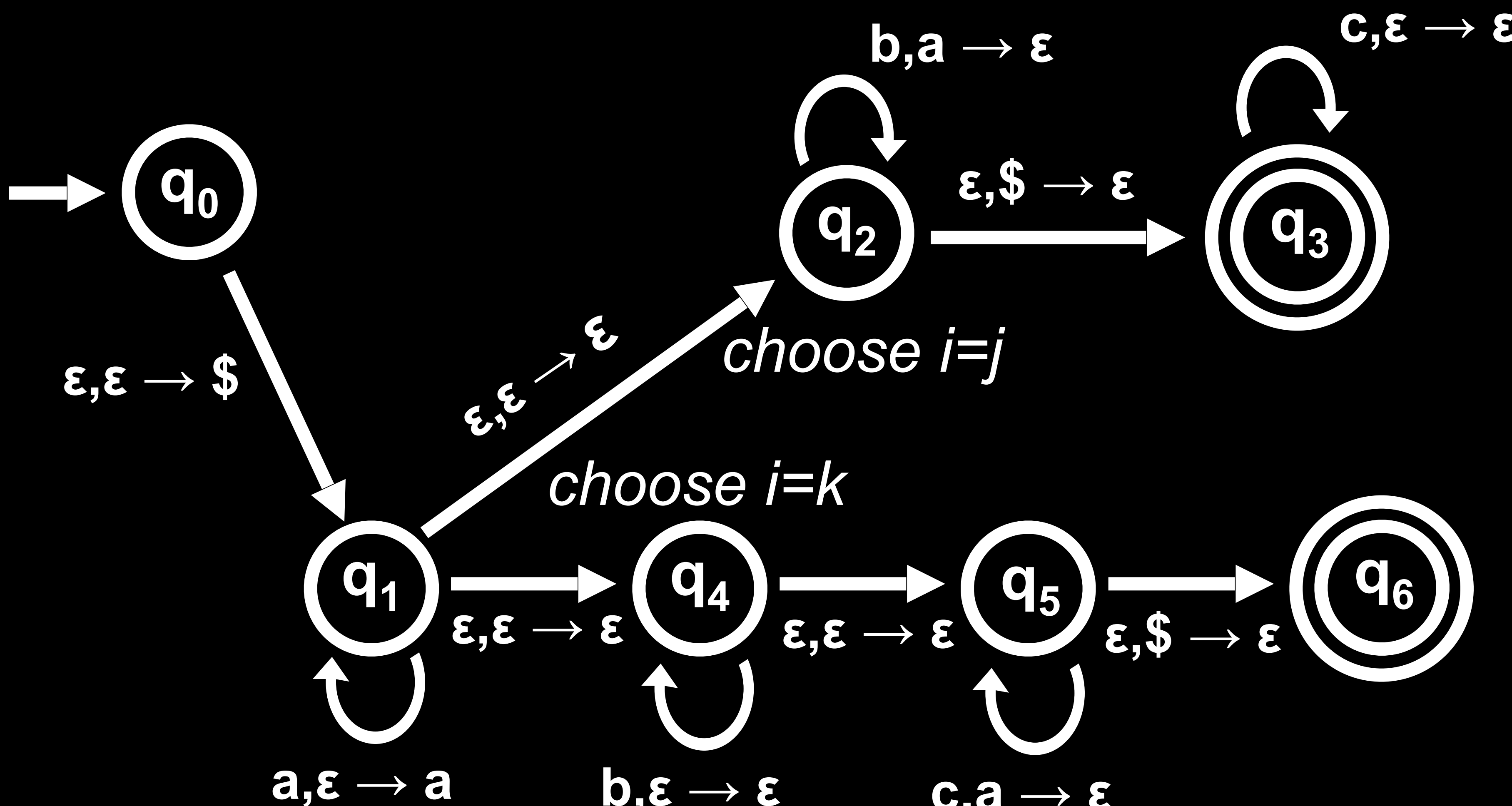

CONTEXT-FREE GRAMMARS

“Colorless green ideas sleep furiously.”

Noam Chomsky (1957)

CONTEXT-FREE GRAMMARS

start variable

production rules

$$A \rightarrow 0A1$$
$$A \rightarrow B$$
$$B \rightarrow \#$$

variables

terminals

$$A \Rightarrow 0A1 \Rightarrow 00A11 \Rightarrow 00B11 \Rightarrow 00\#11$$

$\Rightarrow$ (yields)

Derivation

$$A \Rightarrow^* 00\#11$$

(derives)

Non-deterministic

We say: **00#11** is *generated* by the Grammar

CONTEXT-FREE GRAMMARS

$A \rightarrow 0A1$
$A \rightarrow B$
$B \rightarrow \#$

$A \Rightarrow 0A1 \Rightarrow 00A11 \Rightarrow 00B11 \Rightarrow 00\#11$

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Deterministic CFGs??

CONTEXT-FREE GRAMMARS

$A \rightarrow 0A1$

$A \rightarrow B$

$B \rightarrow \#$

$A \rightarrow 0A1 \mid B$

$B \rightarrow \#$

SNOOP'S GRAMMAR

(courtesy of Luis von Ahn)

<PHRASE> → <FILLER><PHRASE>

<PHRASE> → <START WORD><END WORD>DUDE

<FILLER> → LIKE

<FILLER> → UMM

<START WORD> → FO

<START WORD> → FA

<END WORD> → SHO

<END WORD> → SHAZZY

<END WORD> → SHEEZY

<END WORD> → SHIZZLE

SNOOP'S GRAMMAR

(courtesy of Luis von Ahn)

Generate:

Umm Like Umm Umm Fa Shizzle Dude

Fa Sho Dude

CONTEXT-FREE GRAMMARS

A context-free grammar (CFG) is a tuple $G = (V, \Sigma, R, S)$, where:

V is a finite set of variables

Σ is a finite set of terminals (disjoint from V)

R is set of production rules of the form $A \to W$, where $A \in V$ and $W \in (V \cup \Sigma)^*$

$S \in V$ is the start variable

CONTEXT-FREE LANGUAGES

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$L(G) = \{w \in \Sigma^* \mid S \Rightarrow^* w\}$ Strings Generated by G

A Language L is context-free if there is a CFG that generates precisely the strings in L

CONTEXT-FREE LANGUAGES

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$G = \{ \{S\}, \{0,1\}, R, S \}$ $\quad$ $R = \{ S \to 0S1, S \to \varepsilon \}$

$L(G) =$

CONTEXT-FREE LANGUAGES

A context-free grammar (CFG) is a tuple
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$G = \{ \{S\}, \{0,1\}, R, S \}$ $\quad$ $R = \{ S \to 0S1, S \to \varepsilon \}$

$L(G) = \{ 0^n 1^n \mid n \geq 0 \}$ Strings Generated by G

WRITE A CFG FOR EVEN-LENGTH PALINDROMES

S $\rightarrow$ σ**S**σ **for all** $\sigma \in$ **Σ**

S $\rightarrow$ **ε**

WRITE A CFG FOR THE EMPTY SET

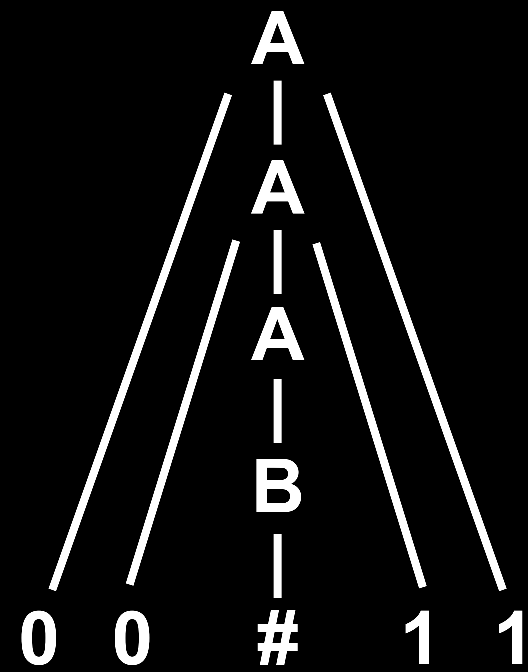
$$G = \{ \{S\}, \Sigma, \varnothing, S \}$$

PARSE TREES

$A \rightarrow 0A1$

$A \rightarrow B$

$B \rightarrow \#$



$$A \Rightarrow 0A1 \Rightarrow 00A11 \Rightarrow 00B11 \Rightarrow 00\#11$$

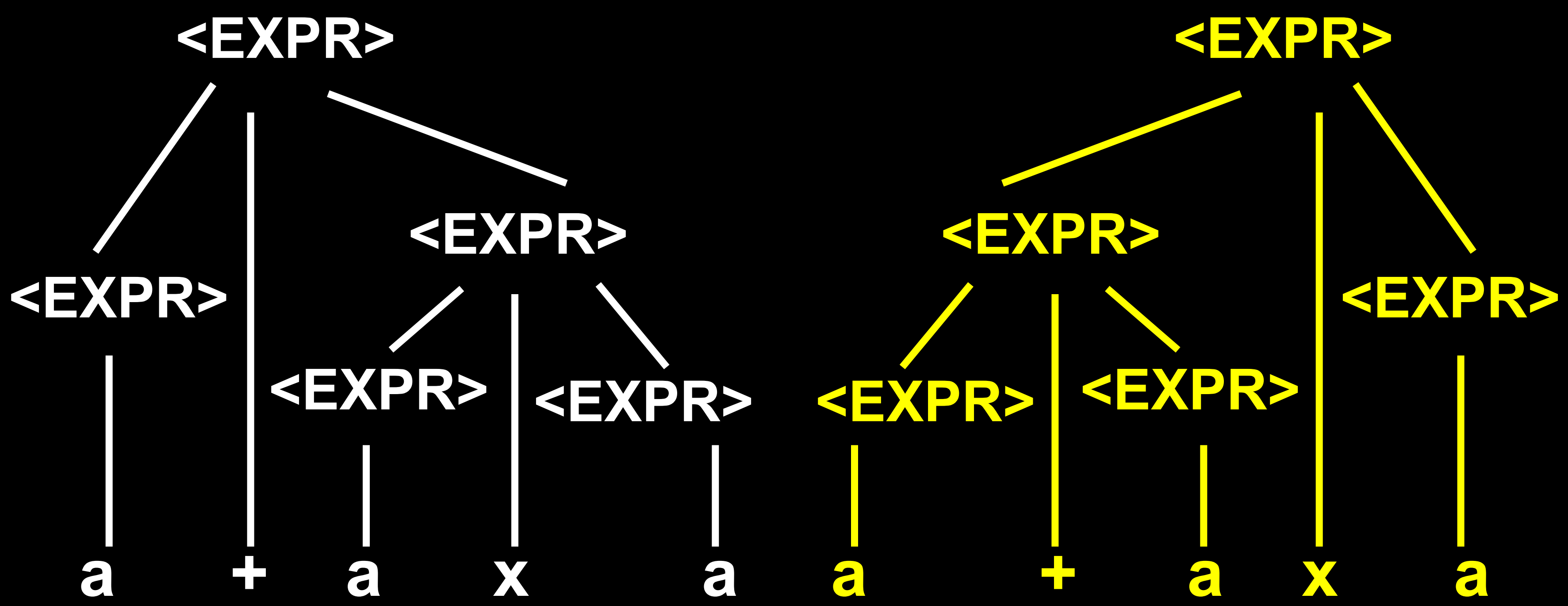
<EXPR> → <EXPR> + <EXPR>

<EXPR> → <EXPR> x <EXPR>

<EXPR> → (<EXPR>)

<EXPR> → a

Build a parse tree for a + a x a



Definition: a string is derived **ambiguously** in a context-free grammar if it has more than one parse tree

Definition: a grammar is **ambiguous** if it generates some string ambiguously

See G_4 for **unambiguous standard arithmetic precedence** [adds parens (,)]

$L = \{ a^i b^j c^k \mid i, j, k \geq 0$ **and** $(i = j \text{ or } j = k) \}$
is *inherently ambiguous* (**xtra credit**)

Undecidable to tell if a language has unambiguous parse trees (**Post's problem**)

NOT REGULAR

$\Sigma = \{0, 1\}$, $L = \{ 0^n 1^n \mid n \geq 0 \}$

But L is **CONTEXT FREE**

$A \rightarrow 0A1$
$A \rightarrow \varepsilon$

WHAT ABOUT?

$\Sigma = \{0, 1\}$, $L_1 = \{ 0^n 1^n 0^m \mid m,n \geq 0 \}$

$\Sigma = \{0, 1\}$, $L_2 = \{ 0^n 1^m 0^n \mid m,n \geq 0 \}$

$\Sigma = \{0, 1\}$, $L_3 = \{ 0^m 1^n 0^n \mid m=n \geq 0 \}$

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WHAT ABOUT?

$\Sigma = \{0, 1\}$, $L_1 = \{ 0^n 1^n 0^m \mid m, n \geq 0 \}$

S -> AB

A -> 0A1 | **ε**

B -> 0B | **ε**

$\Sigma = \{0, 1\}$, $L_2 = \{ 0^n 1^m 0^n \mid m, n \geq 0 \}$

S -> 0S0 | A

A -> 1A | **ε**

$\Sigma = \{0, 1\}$, $L_3 = \{ 0^m 1^n 0^n \mid m=n \geq 0 \}$

THE PUMPING LEMMA FOR CFGs

Let L be a context-free language

Then there is a P such that
if $w \in L$ and $|w| \geq P$

then can write $w = uvxyz$, where:

1. $|vy| > 0$
2. $|vxy| \leq P$
3. For *every* $i \geq 0$, $uv^ixy^iz \in L$

WHAT ABOUT?

$\Sigma = \{0, 1\}$, $L_3 = \{ 0^m 1^n 0^n \mid m = n \geq 0 \}$

Choose $w = 0^P 1^P 0^P$.

By the Pumping Lemma, we can write

$w = uvxyz$ with $|vy| > 0$, $|vxy| \leq P$ such that pumping v together with y will produce another word in L_3

Since $|vxy| \leq P$, $vxy = 0^a 1^b$, or $vxy = 1^a 0^b$.

WHAT ABOUT?

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Since $|vxy| \leq P$, $vxy = 0^a 1^b$, or $vxy = 1^a 0^b$.

Pumping in the first case will unbalance with the 0's at the end; in the second case, will unbalance with the 0's at the beginning. Contradiction.

THE PUMPING LEMMA FOR CFGs

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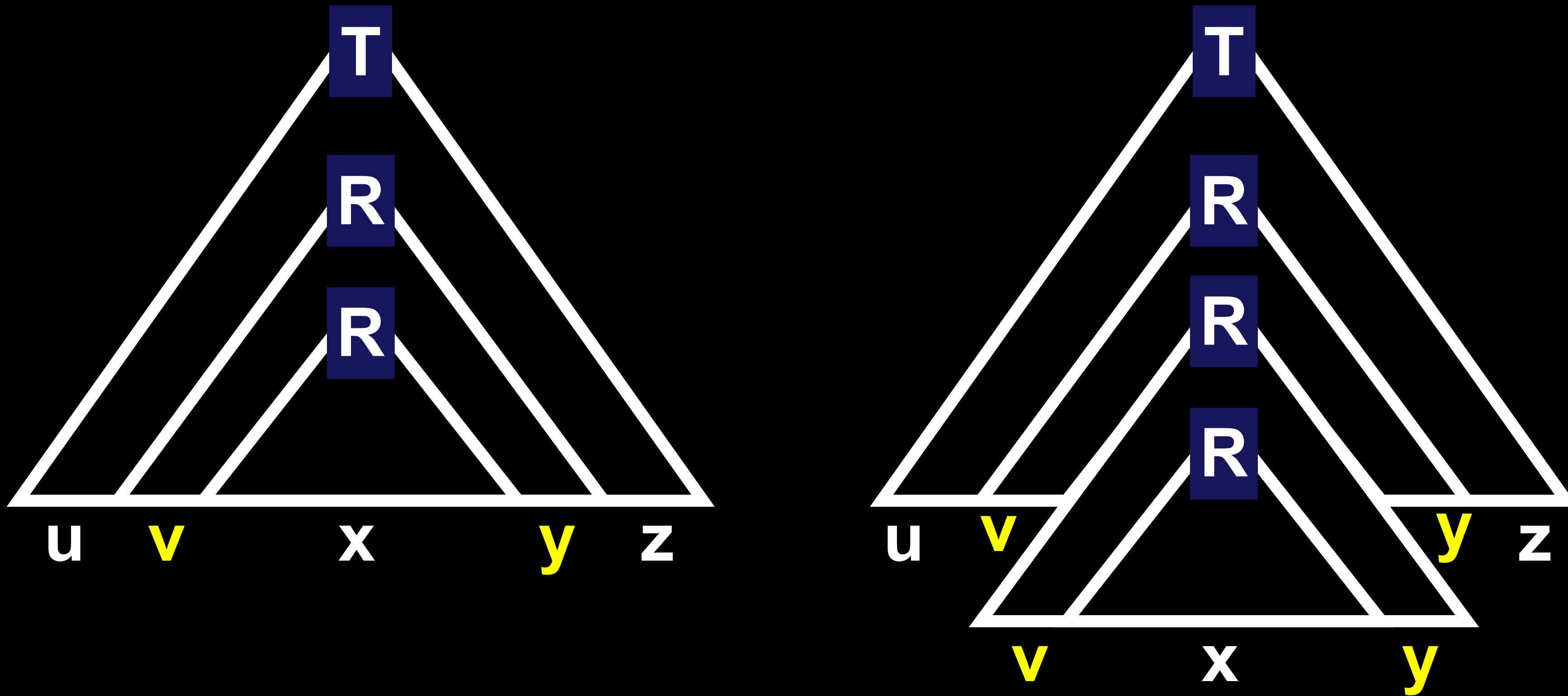
Idea of Proof: If w is long enough, then any parse tree for w must have a path that contains a variable more than once



Formal Proof:

Let b be the maximum number of symbols (length) on the right-hand side of any rule

If the height of a parse tree is h, the length of the string generated by that tree is at most: b^h

---

Let $|V|$ be the number of variables in G

Define $P = b^{|V|+1}$

Let w be a string of length at least P

Let T be a parse tree for w with a *minimum* number of nodes.

$$b^{|V|+1} = P \leq |w| \leq b^h$$

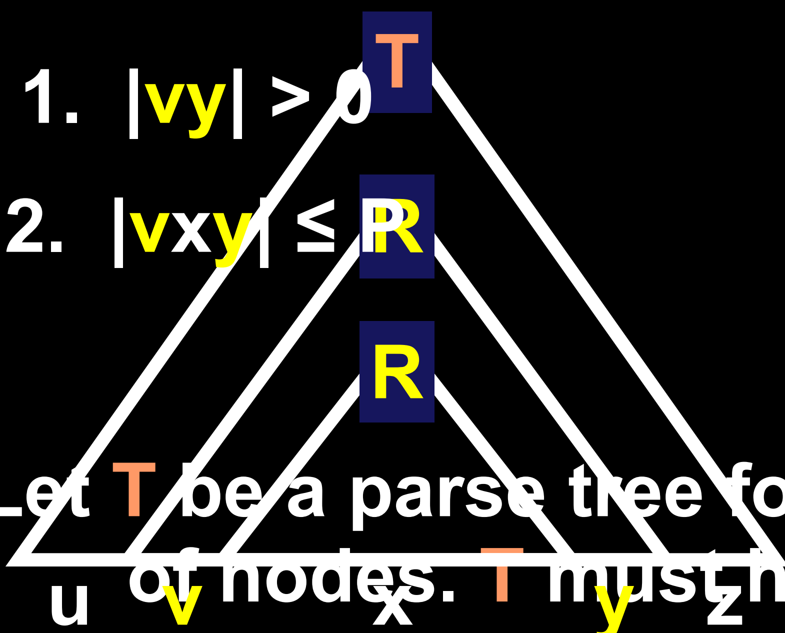
T must have height h at least $|V|+1$

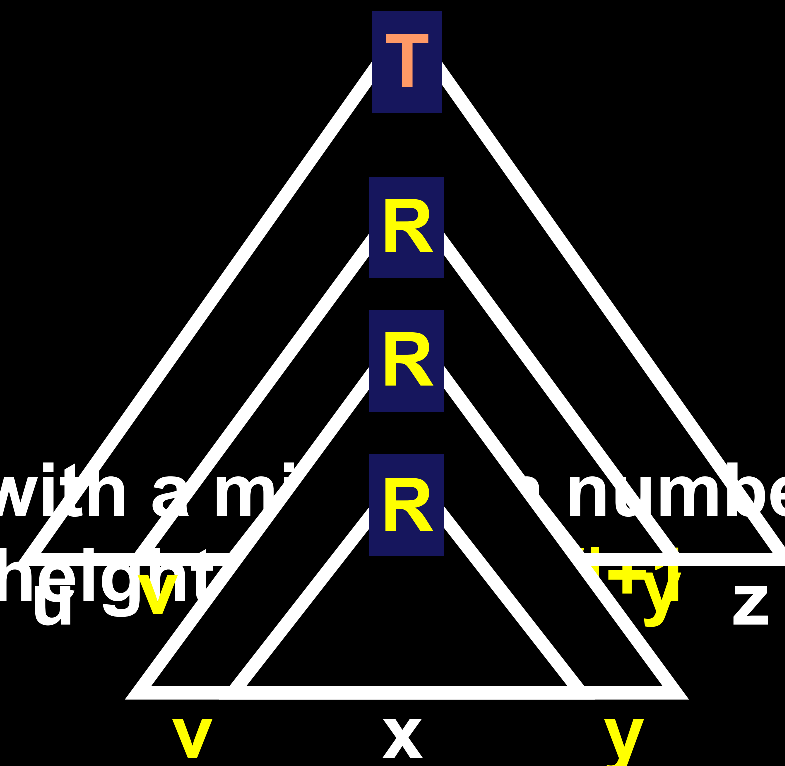
The longest path in T must have ≥ |V|+1 variables

Select R to be a variable that repeats among the lowest |V|+1 variables (in the path)

1. |vy| > 0

2. |vxy| ≤ p





Let T be a parse tree for w with a mi[...] number of nodes. T must have height [...]

The longest path in T must have $\geq |V|+1$ variables

Select R to be a variable in T that repeats, among the lowest $|V|+1$ variables in the tree

1. $|vy| > 0$ since T has minimun # nodes

2. $|vxy| \leq P$ since $|vxy| \leq b^{|V|+1} = P$