15-453

FORMAL LANGUAGES, AUTOMATA AND COMPUTABILITY

A non-deterministic finite automaton (**NFA**) is a 5-tuple $N = (Q, \Sigma, \delta, Q_0, F)$

Q is the set of states (finite)

Σ is the alphabet (finite)

$\delta : Q \times \Sigma_\varepsilon \to 2^Q$ is the transition function *

$Q_0 \subseteq Q$ is the set of start states

$F \subseteq Q$ is the set of accept states

* 2^Q is the set of subsets of Q and $\Sigma_\varepsilon = \Sigma \cup \{\varepsilon\}$

Let $w \in \Sigma^*$ and suppose w can be written as $w_1 \ldots w_n$ where $w_i \in \Sigma_\varepsilon$ (ε is viewed as representing the empty string)

Then N accepts w if there are $r_0, r_1, \ldots, r_n \in Q$ such that

1. $r_0 \in Q_0$
2. $r_{i+1} \in \delta(r_i, w_{i+1})$ for $i = 0, \ldots, n-1$, and
3. $r_n \in F$

$L(N)$ = the language of machine N
= set of all strings machine N accepts

A language L is recognized by an NFA N if $L = L(N)$.

FROM NFA TO DFA

Input: NFA $N = (Q, \Sigma, \delta, Q_0, F)$

Output: DFA $M = (Q', \Sigma, \delta', q_0', F')$

$$Q' = 2^Q$$

$$\delta' : Q' \times \Sigma \to Q'$$

$$\delta'(R,\sigma) = \bigcup_{r \in R} \varepsilon(\delta(r,\sigma)) \quad *$$

$$q_0' = \varepsilon(Q_0)$$

$$F' = \{ R \in Q' \mid f \in R \text{ for some } f \in F \}$$

*

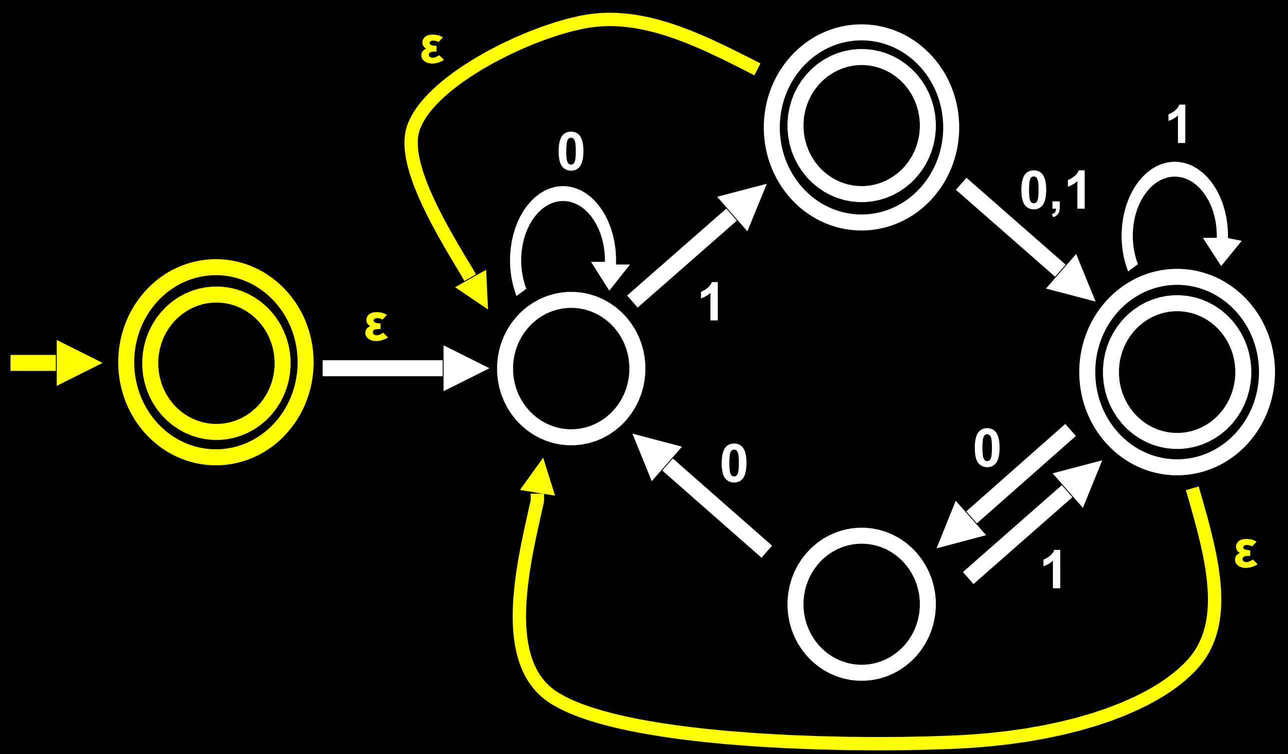
For $R \subseteq Q$, the **ε-closure** of R, $\varepsilon(R)$ = {q that can be reached from some $r \in R$ by traveling along zero or more ε arrows}

RLs ARE CLOSED UNDER **STAR**

Star: $A^* = \{ s_1 \ldots s_k \mid k \geq 0 \text{ and each } s_i \in A \}$

Let M be a DFA, and let L = L(M)

Can construct an NFA N that recognizes L*



REGULAR LANGUAGES ARE CLOSED UNDER **THE REGULAR OPERATIONS**

- **Union:** $A \cup B = \{ w \mid w \in A \text{ or } w \in B \}$
- **Intersection:** $A \cap B = \{ w \mid w \in A \text{ and } w \in B \}$
- **Negation:** $\neg A = \{ w \in \Sigma^* \mid w \notin A \}$
- **Reverse:** $A^R = \{ w_1 \ldots w_k \mid w_k \ldots w_1 \in A \}$
- **Concatenation:** $A \cdot B = \{ vw \mid v \in A \text{ and } w \in B \}$
- **Star:** $A^* = \{ w_1 \ldots w_k \mid k \geq 0 \text{ and each } w_i \in A \}$

THE PUMPING LEMMA FOR REGULAR LANGUAGES and REGULAR EXPRESSIONS

WHICH OF THESE ARE REGULAR ?

$B = \{0^n 1^n \mid n \geq 0\}$

$C = \{ w \mid w$ has equal number of occurrences of 01 and 10 $\}$

$D = \{ w \mid w$ has equal number of 1s and 0s$\}$

THE PUMPING LEMMA

Let L be a regular language with $|L| = \infty$

Then there is a positive integer P s.t.

if w $\in$ L and $|w| \geq P$

then can write w = xyz, where:

1. $|y| > 0$ (y isn't ε)
2. $|xy| \leq P$
3. For *every* $i \geq 0$, $xy^iz \in L$

Why is it called the pumping lemma? The word w gets PUMPED into something longer…

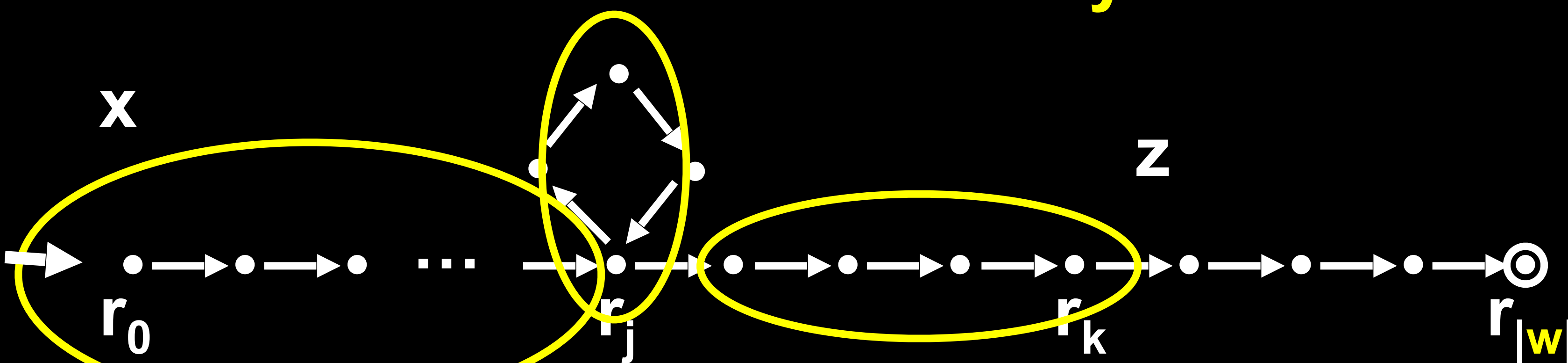
Proof: Let M be a DFA that recognizes L

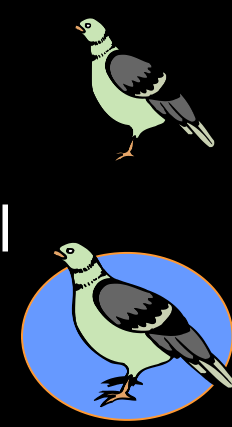
Let P be the **number of states** in M

Assume w ∈ L is such that $|w| \geq P$

We show: w = xyz

1. $|y| > 0$
2. $|xy| \leq P$
3. $xy^iz \in L$ for *all* $i \geq 0$





There must be j and k such that

$j < k \leq P$, and $r_j = r_k$ (**why?**) (Note: $k - j > 0$)

USING THE **PUMPING LEMMA**

Let's prove that
$B = \{0^n1^n \mid n \geq 0\}$ is not regular

Assume B is regular. Let $w = 0^P1^P$

If B is regular, can write $w = xyz$, $|y| > 0$, $|xy| \leq P$, and for any $i \geq 0$, xy^iz is *also* in B

y must be all 0s: Why? **$|xy| \leq P$**

$xyyz$ has more 0s than 1s

Contradiction!

USING THE **PUMPING LEMMA**

D = { w | w has equal number of 1s and 0s}

is not regular

Assume D is regular. Let $w = 0^P 1^P$ (w is in D!)

If D is regular, can write $w = xyz$, $|y| > 0$, $|xy| \leq P$, where for any $i \geq 0$, xy^iz is *also* in D

y must be all 0s: Why? **$|xy| \leq P$**

xyyz has more 0s than 1s

Contradiction!

WHAT DOES C LOOK LIKE?

C = { w | w has equal number of occurrences of 01 and 10}

= { w | w = 1, w = 0, w = ε **or**
w starts with a 0 and ends with a 0 **or**
w starts with a 1 and ends with a 1 }

$$1 \cup 0 \cup \varepsilon \cup 0(0\cup 1)^*0 \cup 1(0\cup 1)^*1$$

REGULAR **EXPRESSIONS**

(expressions representing languages)

σ is a regexp representing $\{\sigma\}$

ε is a regexp representing $\{\varepsilon\}$

$\emptyset$ is a regexp representing $\emptyset$

If R_1 and R_2 are regular expressions representing L_1 and L_2 then:

(R_1R_2) represents $L_1 \cdot L_2$

$(R_1 \cup R_2)$ represents $L_1 \cup L_2$

$(R_1)^*$ represents L_1^*

PRECEDENCE

$* \quad \cdot \quad \cup$

EXAMPLE

$$R_1{}^*R_2 \cup R_3 = ((R_1{}^*)R_2) \cup R_3$$

$\{ w \mid w \text{ has exactly a single } 1 \}$

0^*10^*

What language does $\varnothing^*$ represent?

$\{\varepsilon\}$

$\{ w \mid w \text{ has length} \geq 3 \text{ and its 3rd symbol is } 0 \}$

$$(0 \cup 1)(0 \cup 1)0(0 \cup 1)^*$$

$\{ w \mid \text{every odd position of } w \text{ is a } 1 \}$

$$(1(0 \cup 1))^*(1 \cup \varepsilon)$$

EQUIVALENCE

L can be **represented by a regexp**
$\Leftrightarrow$ L is regular

1. L can be represented by a **regexp**
$\Rightarrow$ L is regular

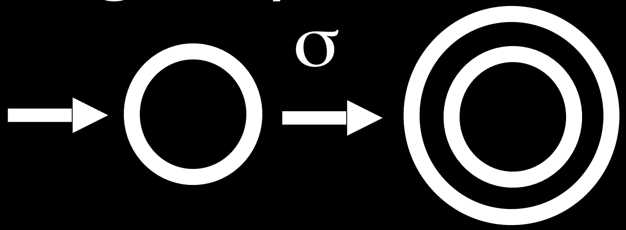
2. L can be represented by a **regexp**
$\Leftarrow$
L is a regular language

1. **Given regular expression R, we show there exists NFA N such that R represents L(N)**

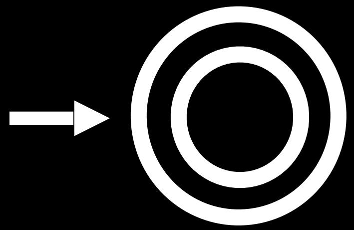
Induction on the *length* of R:

Base Cases (R has length 1):

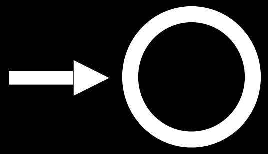
R = σ



R = ε



R = $\emptyset$



Inductive Step:

Assume R has length $k > 1$,
and that every regular expression of length $< k$ represents a regular language

Three possibilities for R:

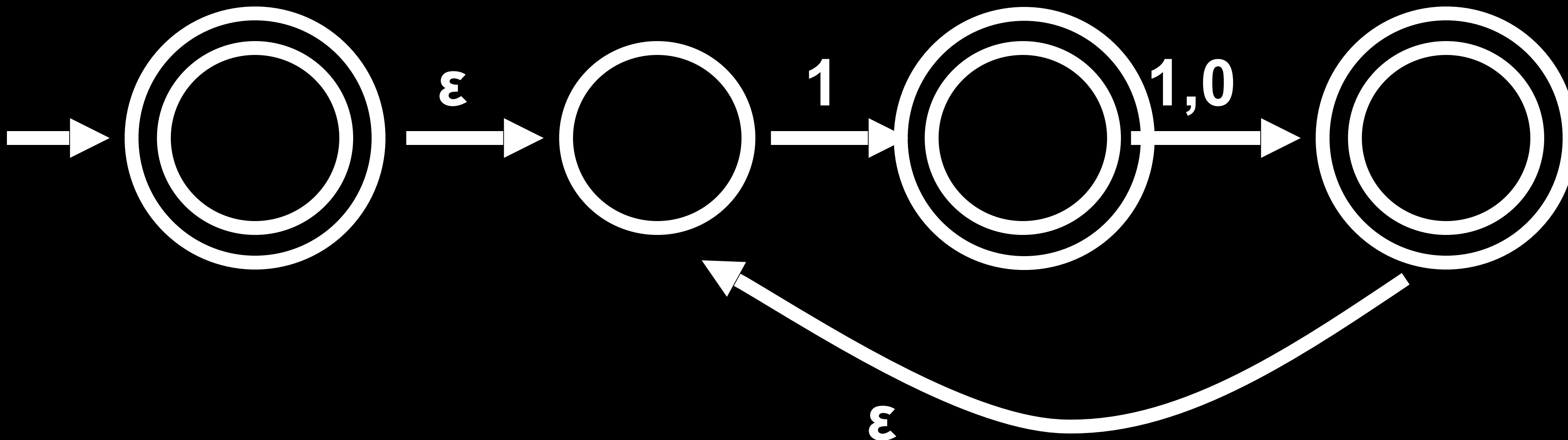
$R = R_1 \cup R_2$ **(Union Theorem!)**

$R = R_1 R_2$ **(Concatenation)**

$R = (R_1)^*$ **(Star)**

Therefore: L can be represented by a regexp
$\Rightarrow$ L is regular

Give an NFA that accepts the language represented by $(1(0 \cup 1))^*$



2. **L can be represented by a regexp**

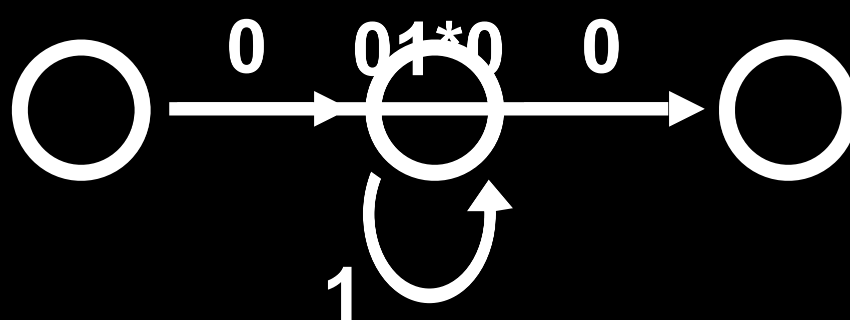
$\Leftarrow$

L is a regular language

Proof idea: Transform an NFA for L into a regular expression by removing states and re-labeling arrows with regular expressions

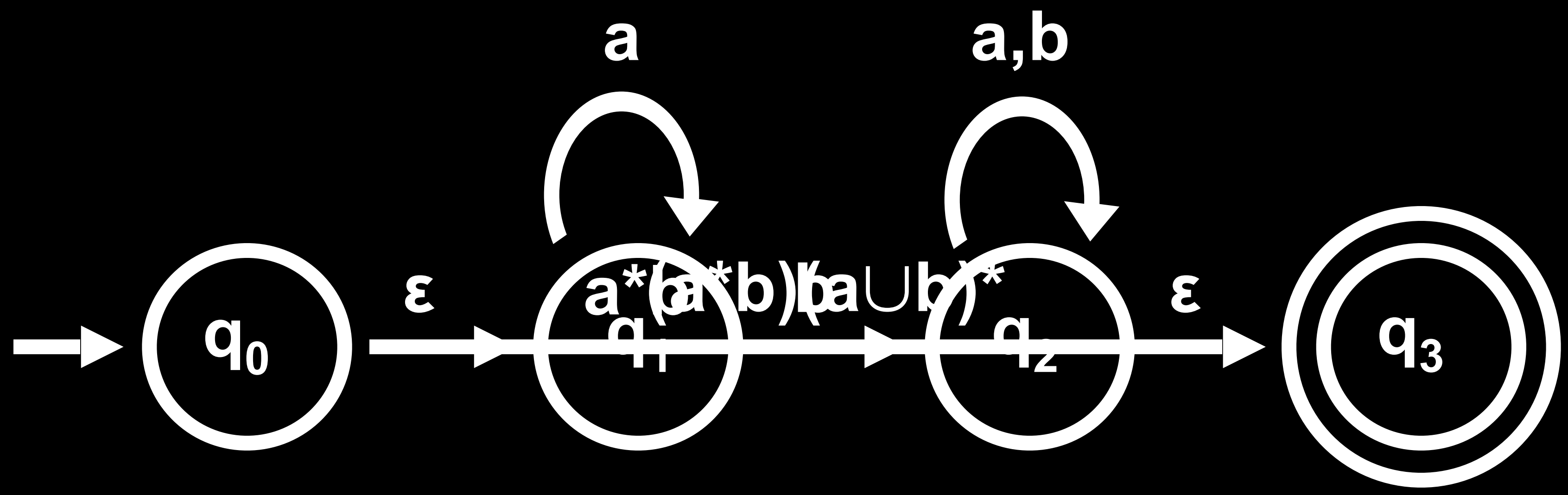
Add unique start and accept states
While machine has more than 2 states

Pick an internal state, rip it out and re-label the arrows with regexps, to account for the missing state



While machine has more than 2 states:

More generally:



$R(q_0, q_3) = (a^*b)(a \cup b)^*$

represents L(N)

Formally: Add q_{start} and q_{accept} to create G (GNFA)

Run CONVERT(G): (Outputs a regexp)

If #states = 2

return the expression on the arrow going from q_{start} to q_{accept}

If #states > 2

Formally: Add q_{start} and q_{accept} to create G (GNFA)

Run CONVERT(G): (Outputs a regexp)

If #states > 2

select $q_{rip} \in Q$ different from q_{start} and q_{accept}

$$\left.\begin{array}{l}\text{define } Q' = Q - \{q_{rip}\} \\ \text{define } R' \text{ as:}\end{array}\right\} \text{Defines: } G' \text{ (GNFA)}$$

$$R'(q_i,q_j) = R(q_i,q_{rip})R(q_{rip},q_{rip})^*R(q_{rip},q_j) \cup R(q_i,q_j)$$

(R′ = the regexps for edges in G′)

We note that G and G′ are *equivalent*

return CONVERT(G′)

Claim: CONVERT(G) is *equivalent* to G

Proof by **induction on k** (number of states in G)

Base Case:

✓ k = 2

Inductive Step:

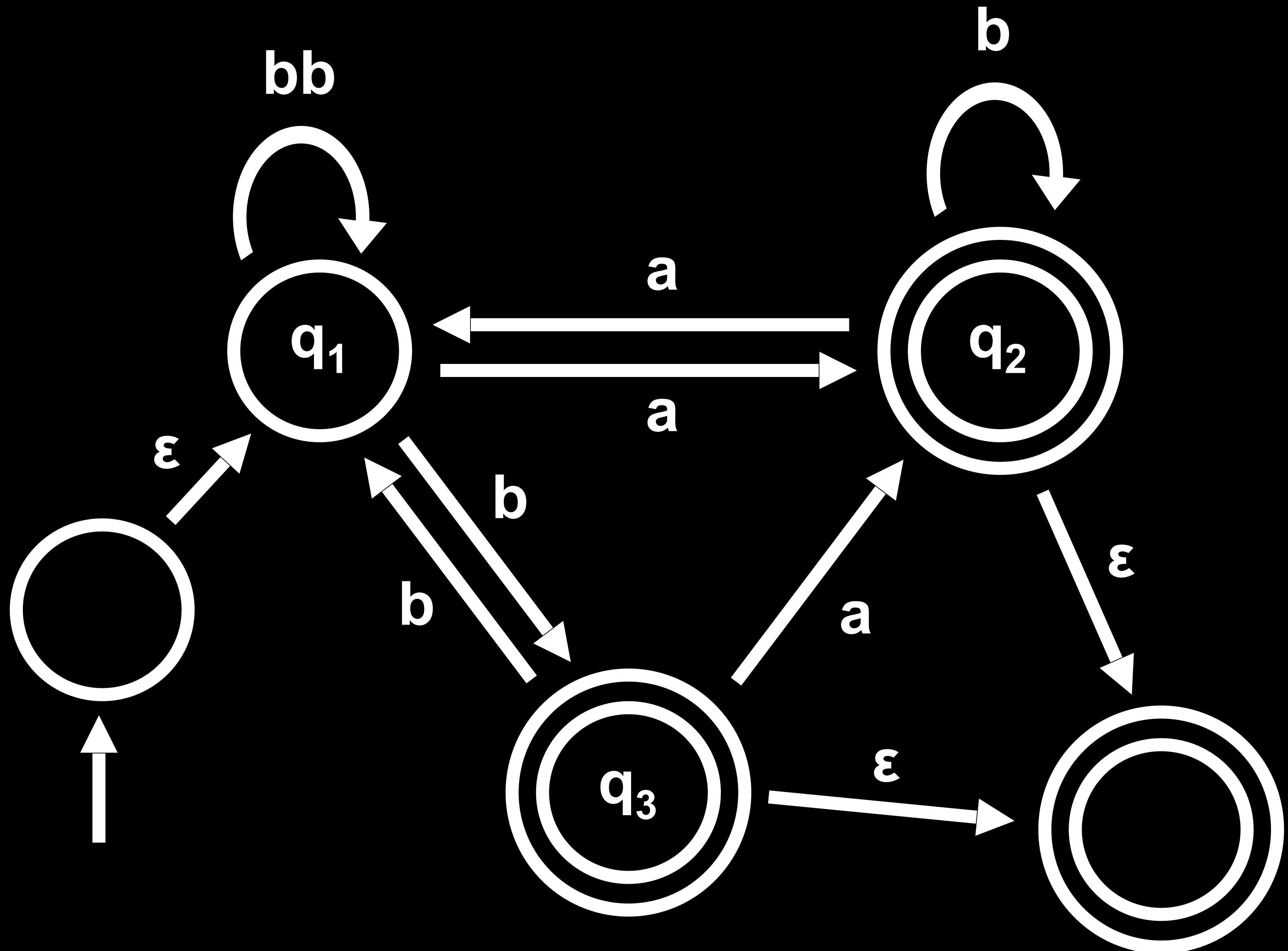
Assume claim is true for k-1 state GNFAs

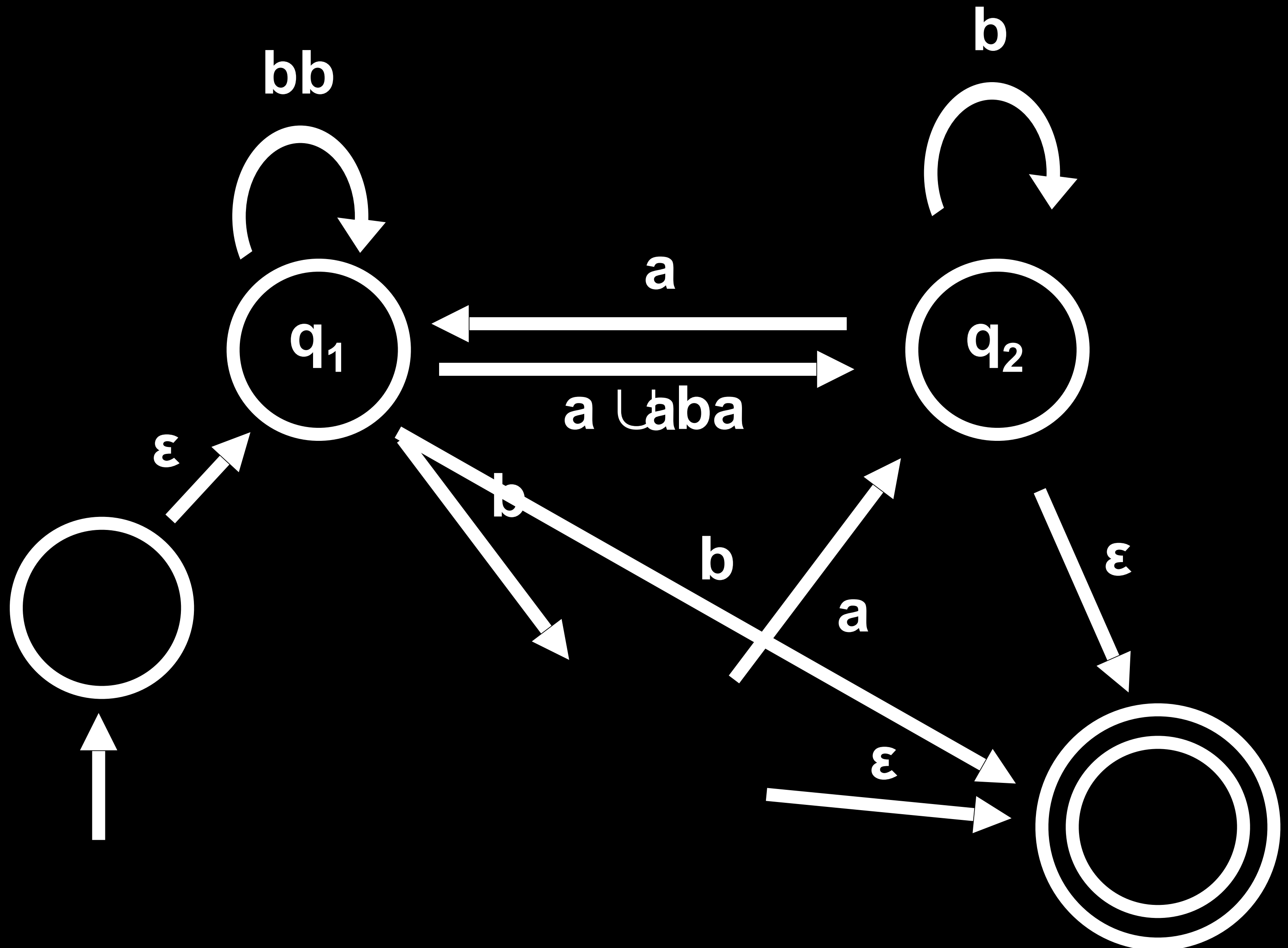
Recall that G and G′ are *equivalent*

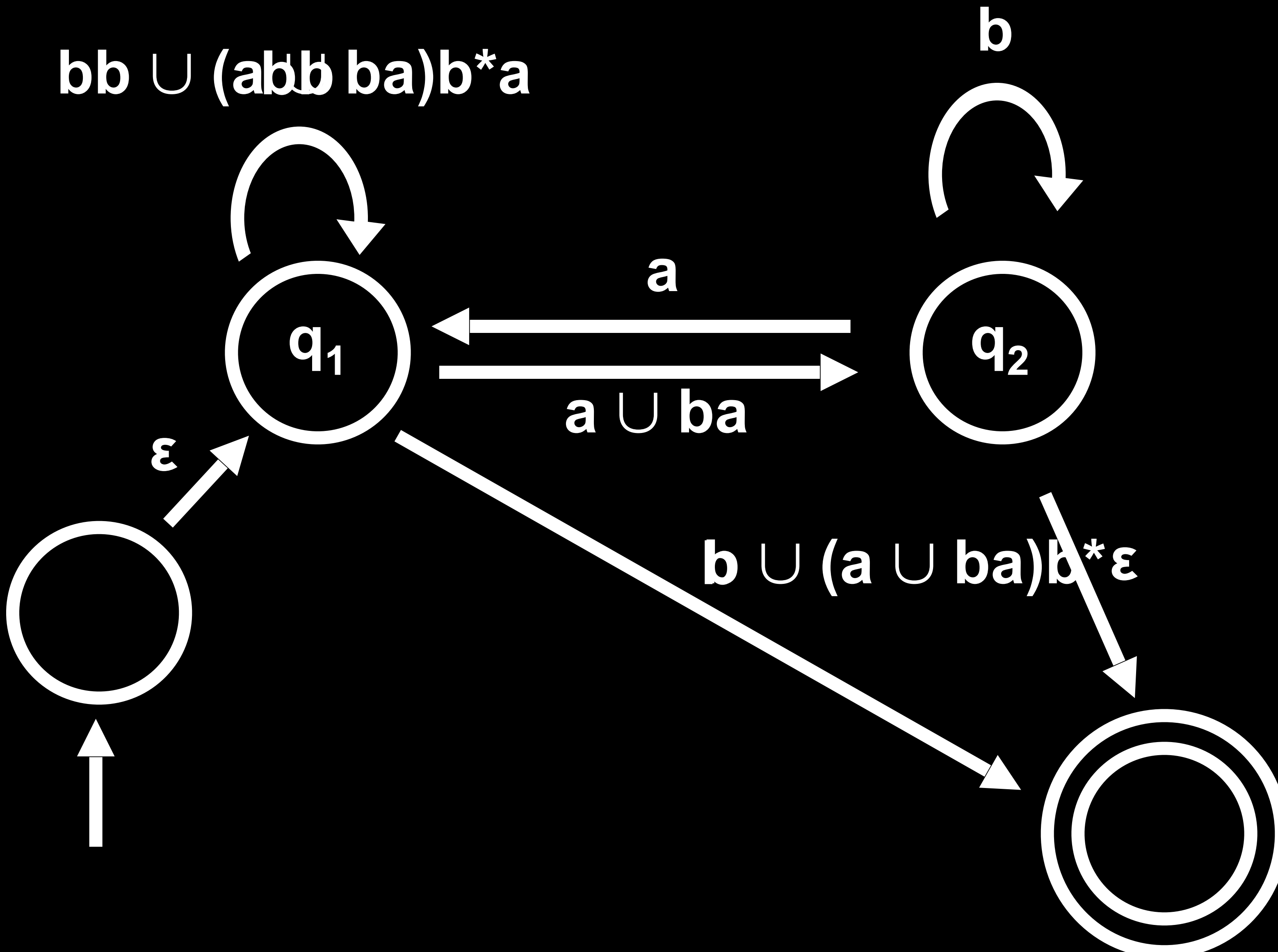
But, by the induction hypothesis, G′ is *equivalent* to CONVERT(G′)

Thus: CONVERT(G′) *equivalent* to CONVERT(G)

QED

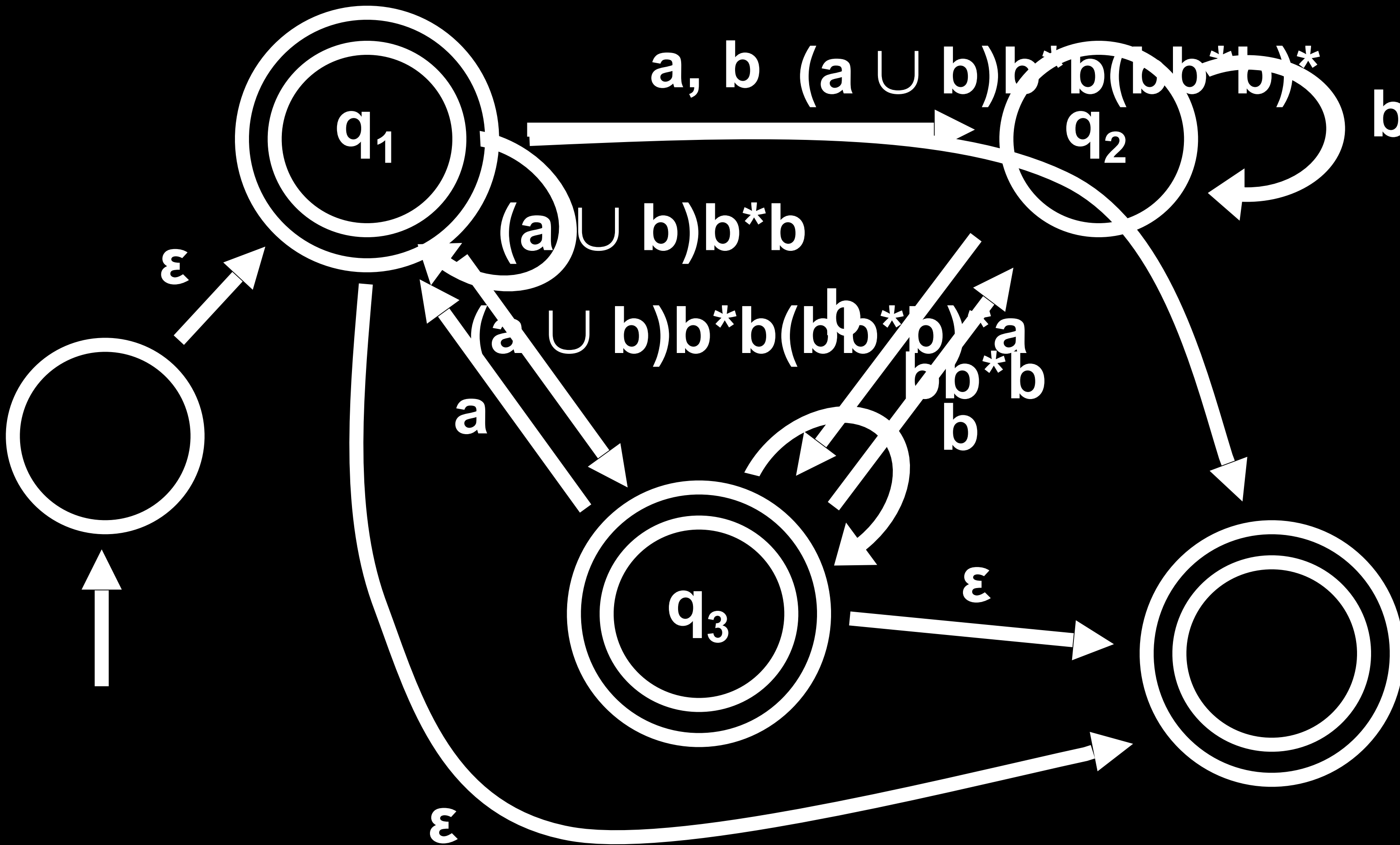

bb
b
a
a
q_1
q_2
q_3
ε
b
b
a
ε
ε


bb
b
a
q_1
q_2
$a \cup aba$
ε
b
b
a
ε
ε

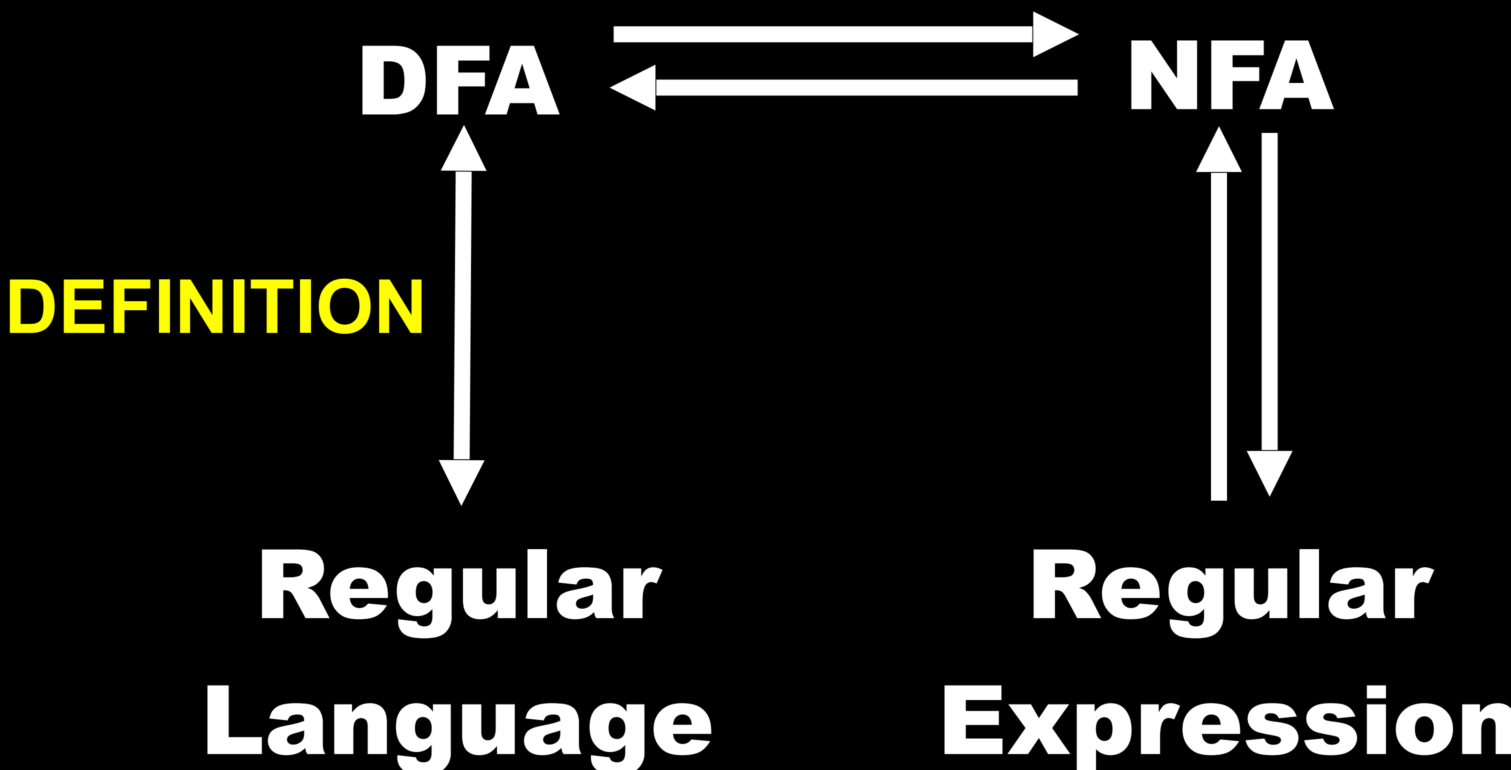


(bb ∪ (a ∪ ba)b*a)* (b ∪ (a ∪ ba)b*)

Convert the NFA to a regular expression



$$((a \cup b)b^*b(bb^*b)^*a)^* \cup$$
$$((a \cup b)b^*b(bb^*b)^*a)^*(a \cup b)b^*b(bb^*b)^*$$


DFA
NFA
DEFINITION
Regular Language
Regular Expression

Finish Chapter 1 for next time.