NON-DETERMINISM and REGULAR OPERATIONS

UNION **THEOREM**

The union of two regular languages is also a regular language

"Regular Languages Are Closed Under Union"

INTERSECTION **THEOREM**

The intersection of two regular languages is also a regular language

Complement **THEOREM**

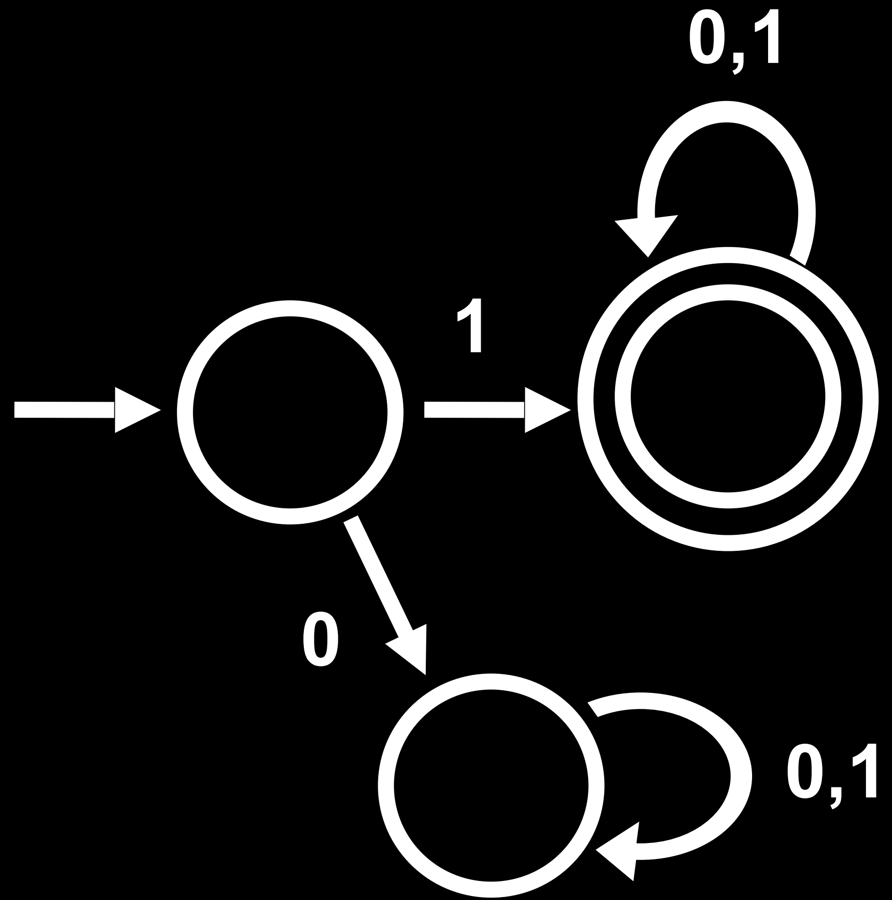
The complement of a regular language is also a regular language

In other words,

if L is regular than so is ¬L,

where $\neg L = \{ w \in \Sigma^* \mid w \notin L \}$

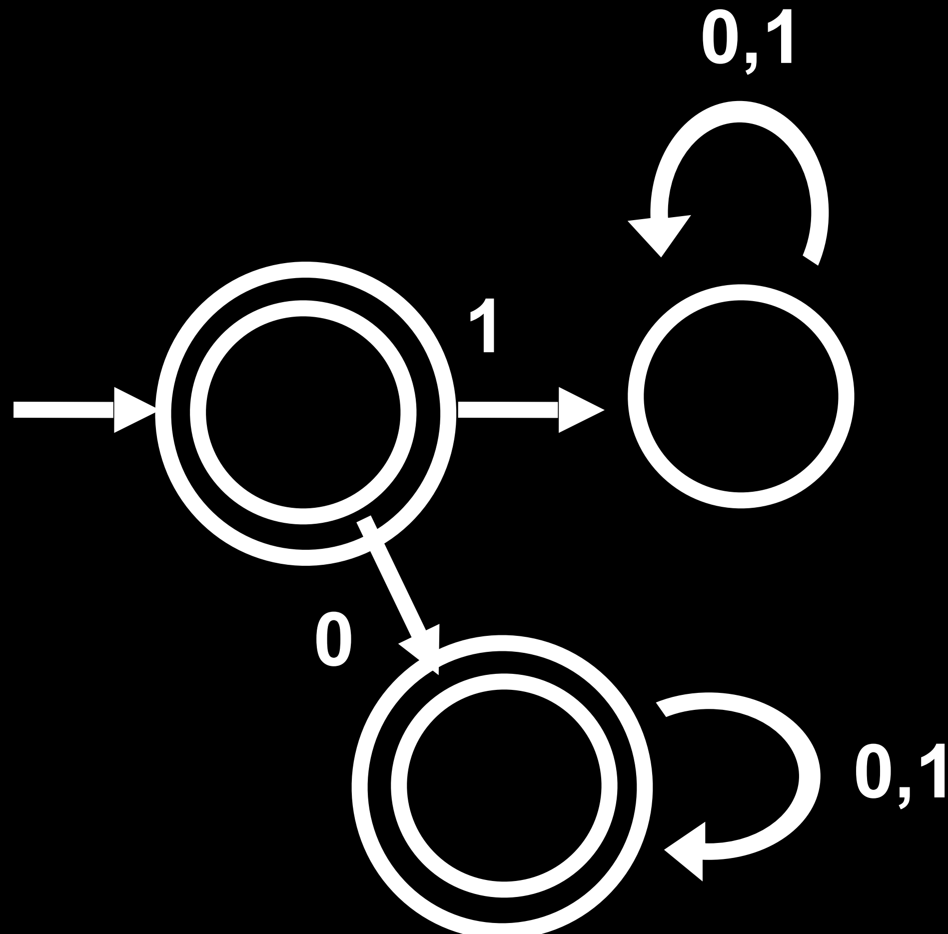
Proof ?



L(M) = { w | w begins with 1}

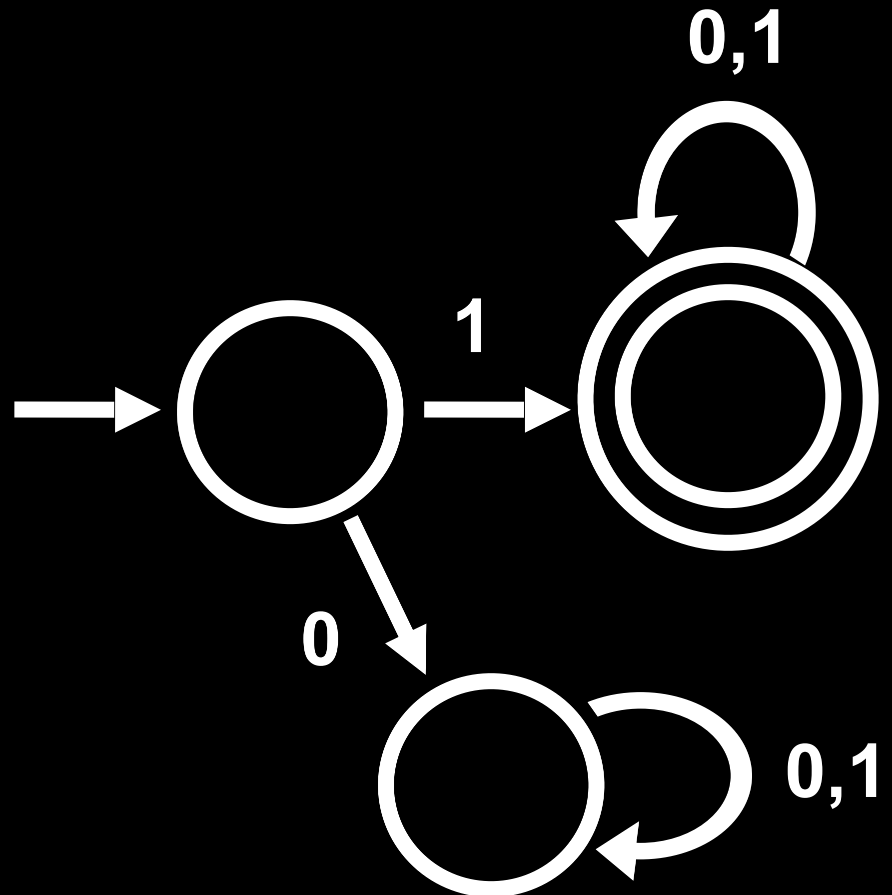
¬L(M) = { w | w does not begin with 1}

Is ¬L(M) regular?



$\neg L(M) = \{ w \mid w \text{ does not begin with } 1\}$

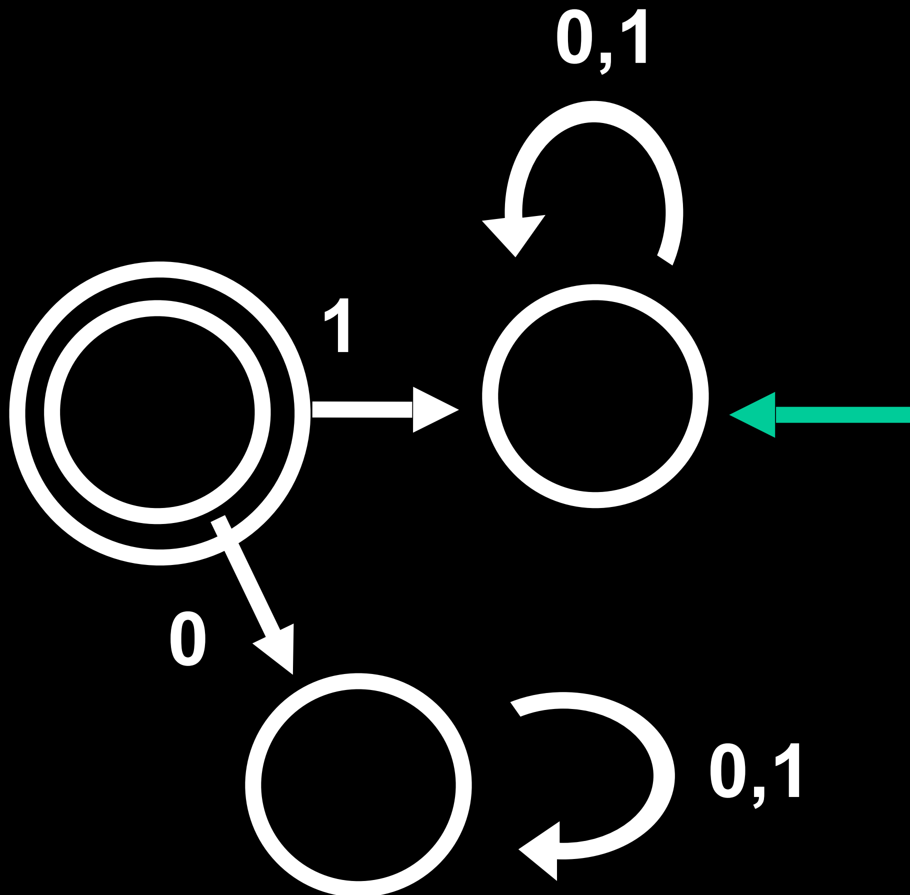
Is $\neg L(M)$ regular?



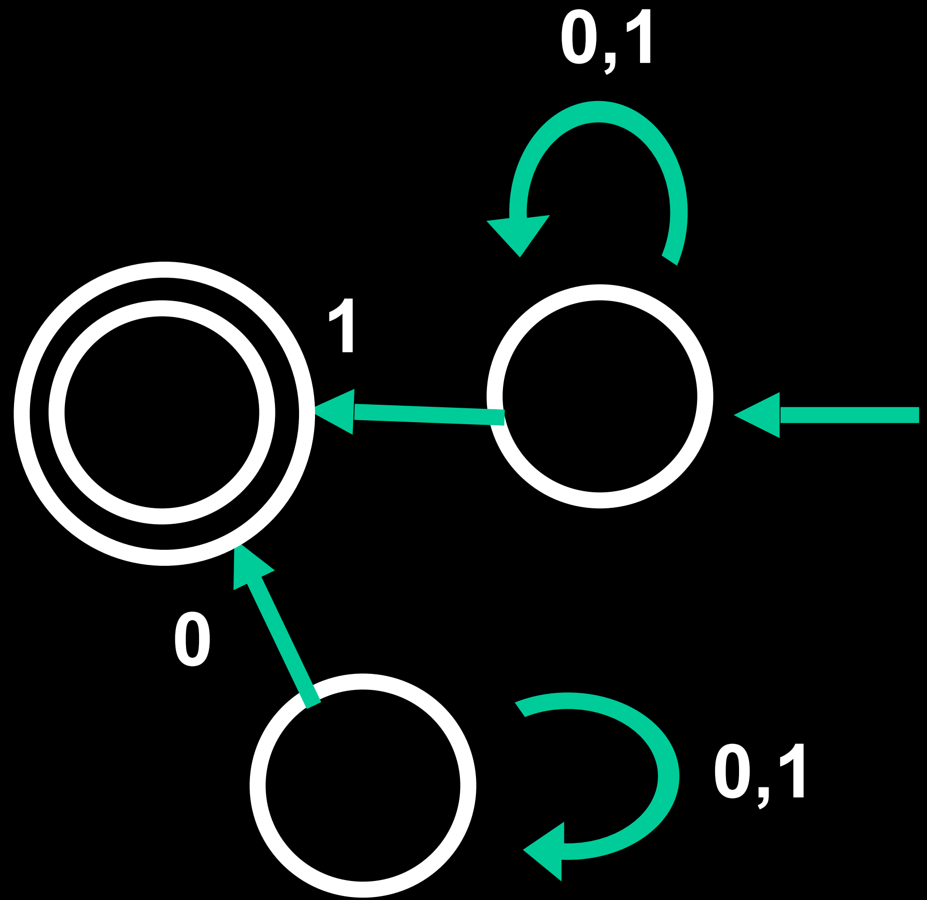
L(M) = { w | w begins with 1}

Suppose our machine reads strings from *right* to *left*…
What language would be recognized then?

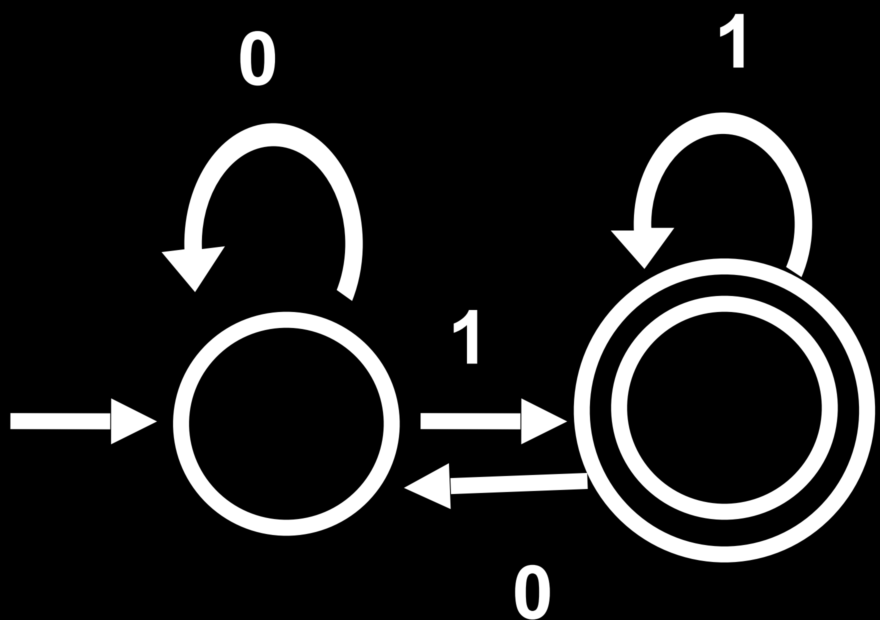
L^R = { w | w ends with 1} Is L^R regular?



L^R = { w | w ends with 1}   Is L^R regular?



L^R = { w | w ends with 1}   Is L^R regular?



$L^R = \{ w \mid w \text{ ends with } 1\}$ Is L^R regular?

THE REVERSE OF A LANGUAGE

Reverse: $L^R = \{ w_1 \ldots w_k \mid w_k \ldots w_1 \in L, w_i \in \Sigma \}$

If L is recognized by a normal DFA,
Then L^R is recognized by a DFA reading from right to left!

Can every "Right-to-Left DFA" be replaced by a normal DFA??

REVERSE **THEOREM**

The reverse of a regular language is also a regular language

``Regular Languages Are Closed Under Reverse”

If a language can be recognized by a DFA that reads strings from *right* to *left*,
then there is an “normal” DFA that accepts the same language

REVERSING DFAs

Assume L is a regular language.
Let M be a DFA that recognizes L

Task: Build a DFA M^R that accepts L^R

If M accepts w, then w describes a directed path in M from *start* to an *accept* state.

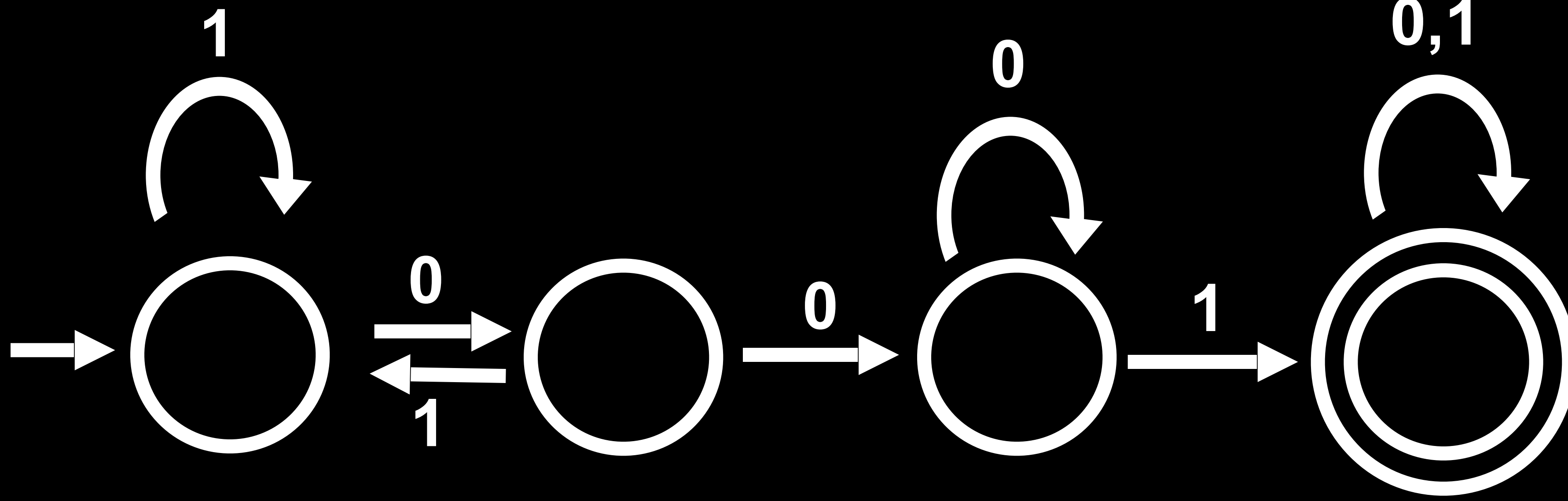
First Attempt:
Try to define M^R as M with the arrows reversed.
Turn **start state** into a **final state**.
Turn **final states** into **start states**.

M^R IS NOT ALWAYS A DFA!

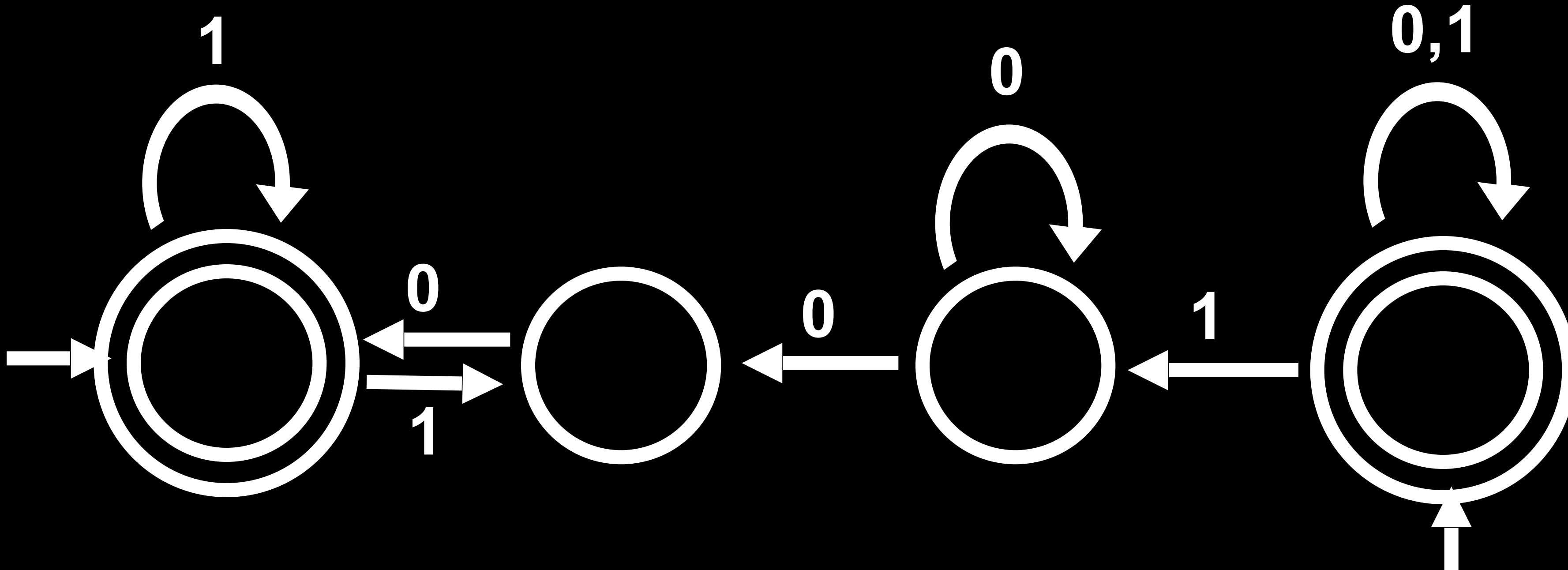
It could have many start states

Some states may have too many outgoing edges,

or none at all!


1
0
1
0
0
1
0,1

NONDETERMINISM is BORN!



What happens with 100?

We will say that this machine accepts a string if there is *some path* that reaches an accept state from a start state.

IBM JOURNAL APRIL 1959

Turing Award winning paper

M. O. Rabin*
D. Scott†

Finite Automata and Their Decision Problems‡

Abstract: Finite automata are considered in this paper as instruments for classifying finite tapes. Each one-tape automaton defines a set of tapes, a two-tape automaton defines a set of pairs of tapes, et cetera. The structure of the defined sets is studied. Various generalizations of the notion of an automaton are introduced and their relation to the classical automata is determined. Some decision problems concerning automata are shown to be solvable by effective algorithms; others turn out to be unsolvable by algorithms.

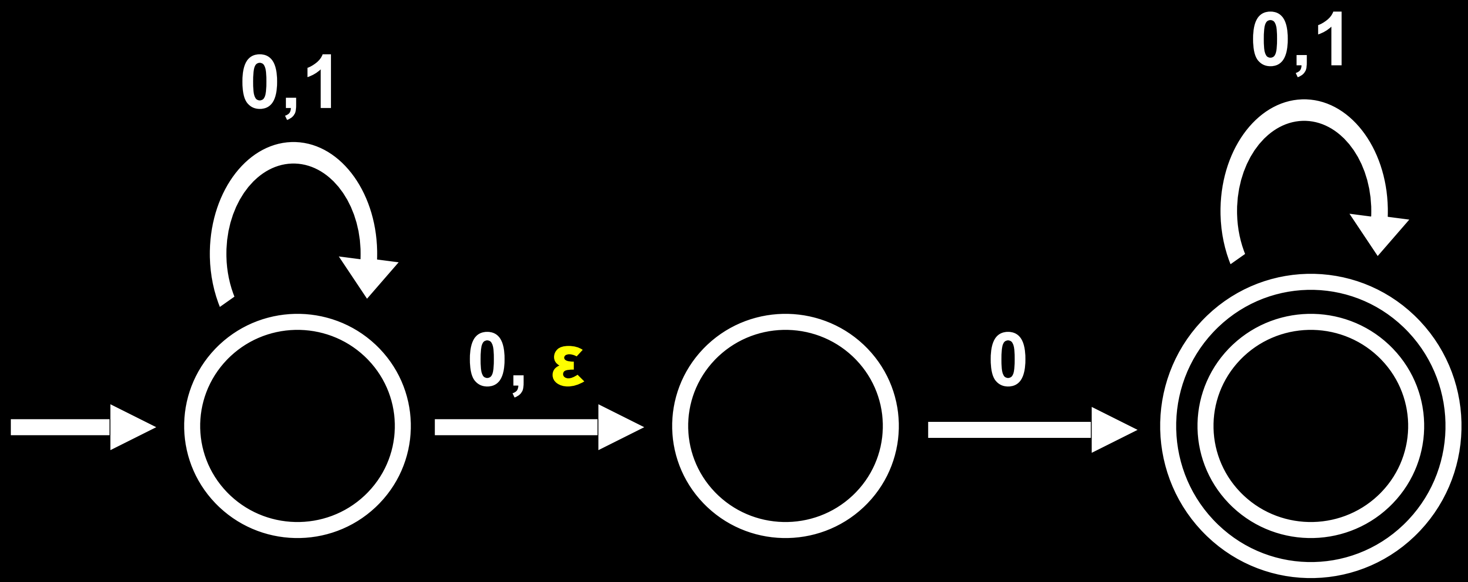
Introduction

Turing machines are widely considered to be the abstract prototype of digital computers; workers in the field, however, have felt more and more that the notion of a Turing machine is too general to serve as an accurate model of actual computers. It is well known that even for simple calculations it is impossible to give an *a priori* upper bound on the amount of tape a Turing machine will need for any given computation. It is precisely this feature that renders Turing's concept unrealistic.

In the last few years the idea of a *finite automaton* has appeared in the literature. These are machines having

a method of viewing automata but have retained throughout a machine-like formalism that permits direct comparison with Turing machines. A neat form of the definition of automata has been used by Burks and Wang[1] and by E. F. Moore,[4] and our point of view is closer to theirs than it is to the formalism of nerve-nets. However, we have adopted an even simpler form of the definition by doing away with a complicated output function and having our machines simply give "yes" or "no" answers. This was also used by Myhill, but our generalizations to the "nondeterministic," "two-way," and "many-tape"

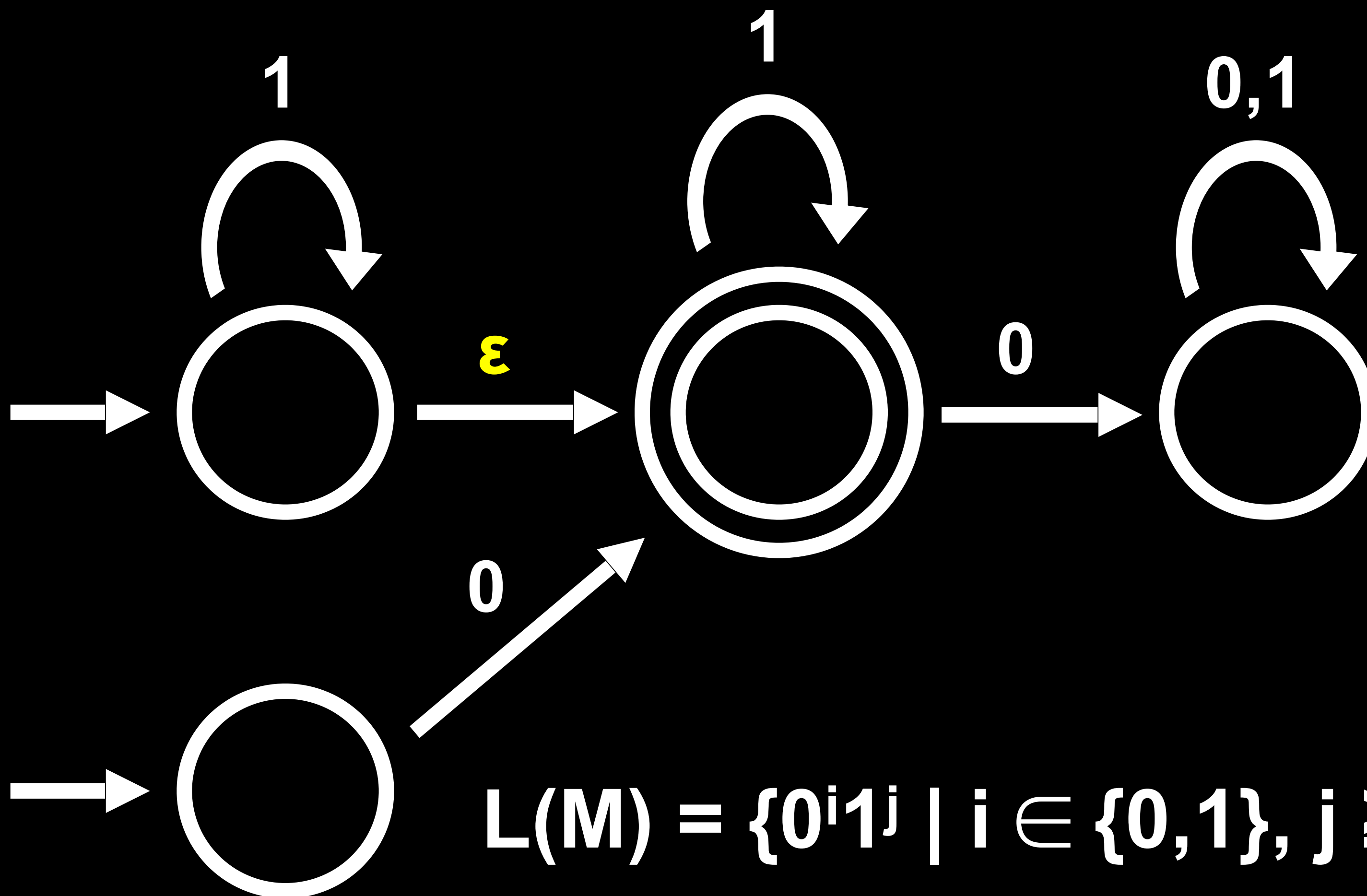
NFA EXAMPLES



L(M)={w | w contains a 0}

At each state, we can have ***any*** number of out arrows for each letter $\sigma \in \boldsymbol{\Sigma_\varepsilon} = \boldsymbol{\Sigma} \cup \boldsymbol{\{\varepsilon\}}$

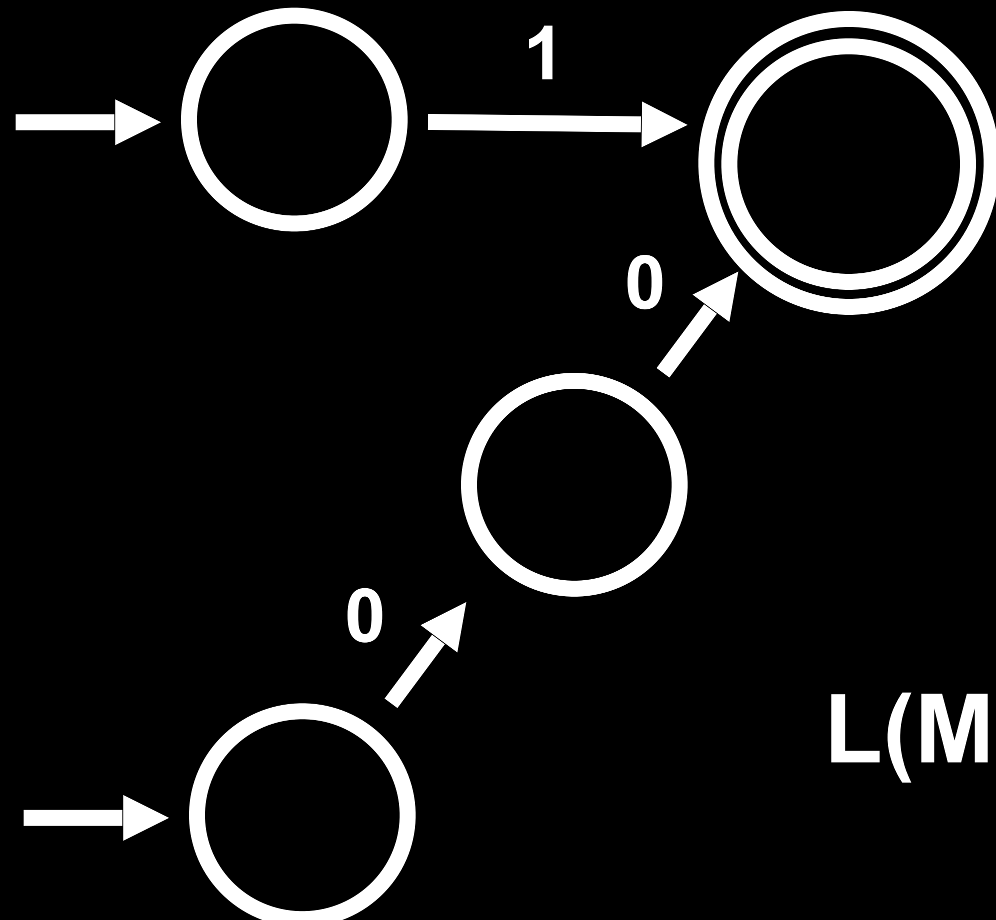
NFA EXAMPLES



$$L(M) = \{0^i 1^j \mid i \in \{0,1\},\ j \geq 0\}$$

Possibly many start states

NFA EXAMPLES



L(M)={1,00}

A *non-deterministic* finite automaton (**NFA**) is a 5-tuple $N = (Q, \Sigma, \delta, Q_0, F)$

Q is the set of states

Σ is the alphabet

$\delta : Q \times \Sigma_\varepsilon \rightarrow 2^Q$ is the transition function

$Q_0 \subseteq Q$ is the set of start states

$F \subseteq Q$ is the set of accept states

2^Q is the set of all possible subsets of Q

$\Sigma_\varepsilon = \Sigma \cup \{\varepsilon\}$

Let $w \in \Sigma^*$ and suppose w can be written as $w_1 \dots w_n$ where $w_i \in \Sigma_\varepsilon$ (ε = empty string)

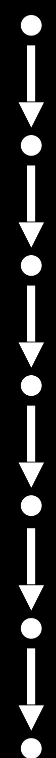
Then N accepts w if there are $r_0, r_1, \dots, r_n \in Q$ such that

1. $r_0 \in Q_0$
2. $r_{i+1} \in \delta(r_i, w_{i+1})$ for $i = 0, \dots, n-1$, and
3. $r_n \in F$

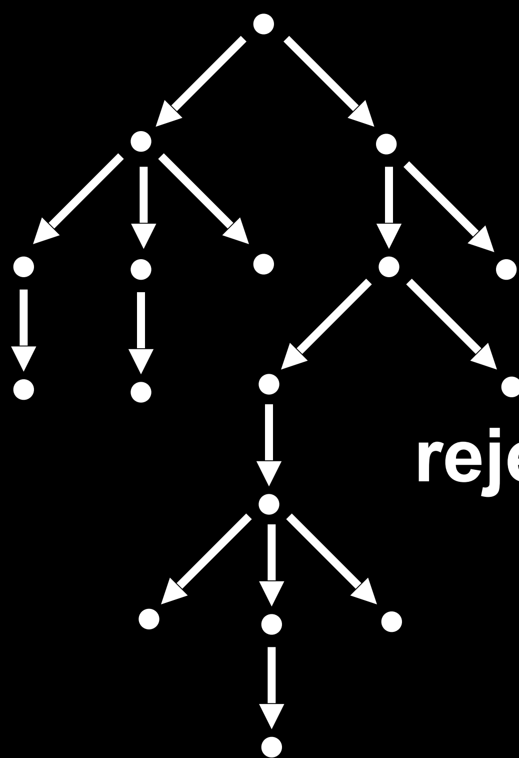
$L(N)$ = the language recognized by N
= set of all strings machine N accepts

A language L is recognized by an NFA N if $L = L(N)$.

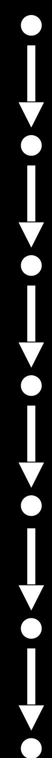
Deterministic Computation



Non-Deterministic Computation

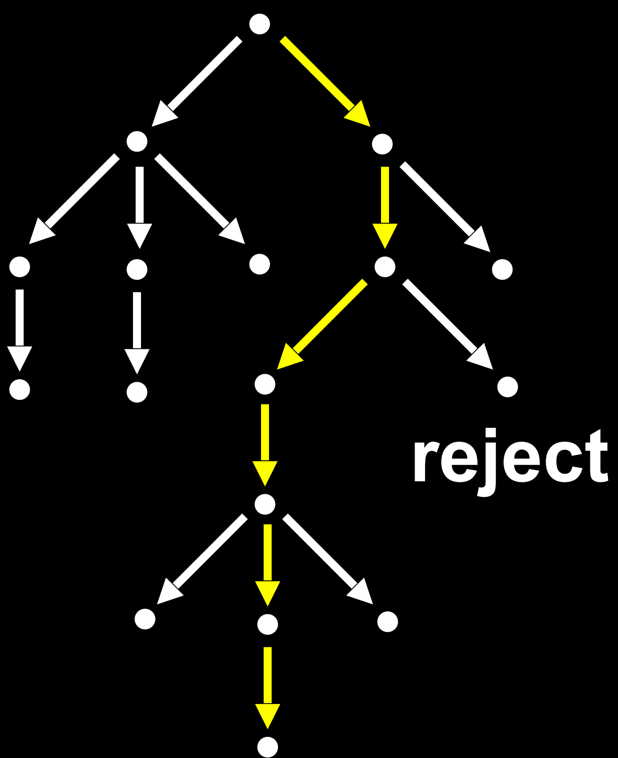


Deterministic Computation

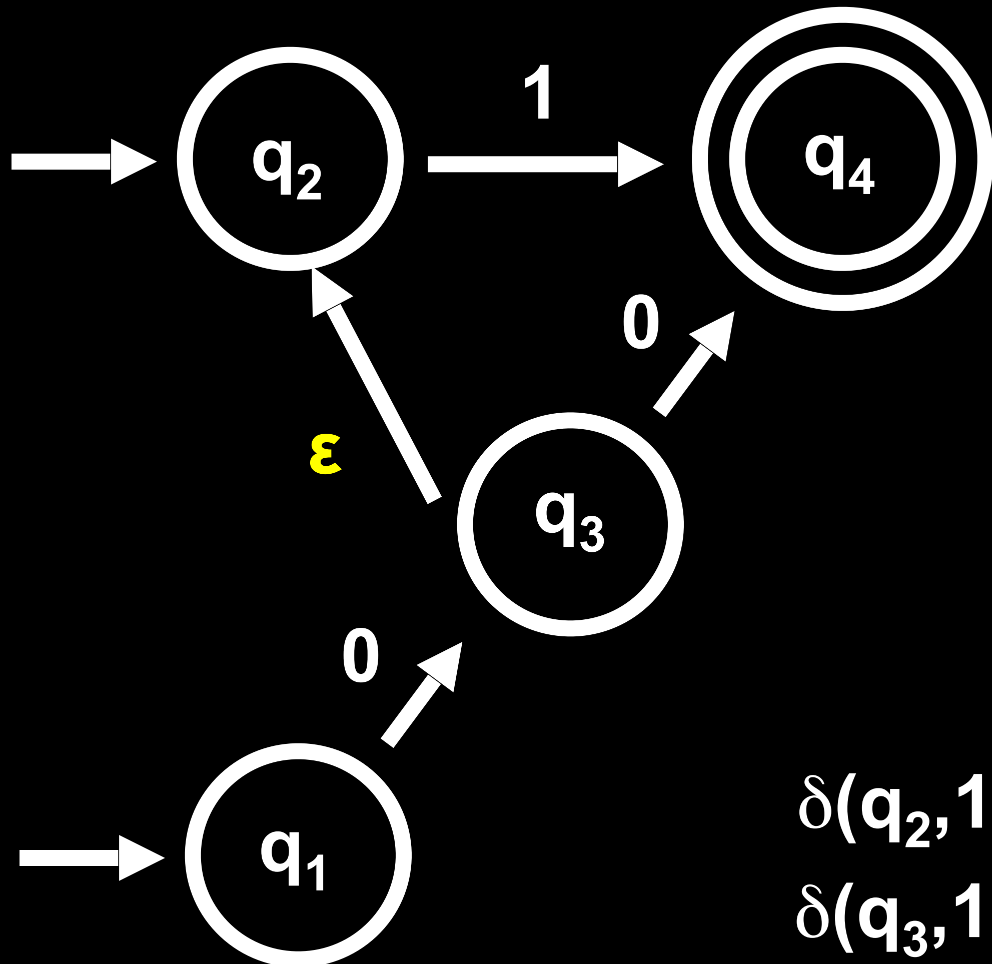


accept or reject

Non-Deterministic Computation



accept



$N = (Q, \Sigma, \delta, Q_0, F)$

$Q = \{q_1, q_2, q_3, q_4\}$

$\Sigma = \{0,1\}$

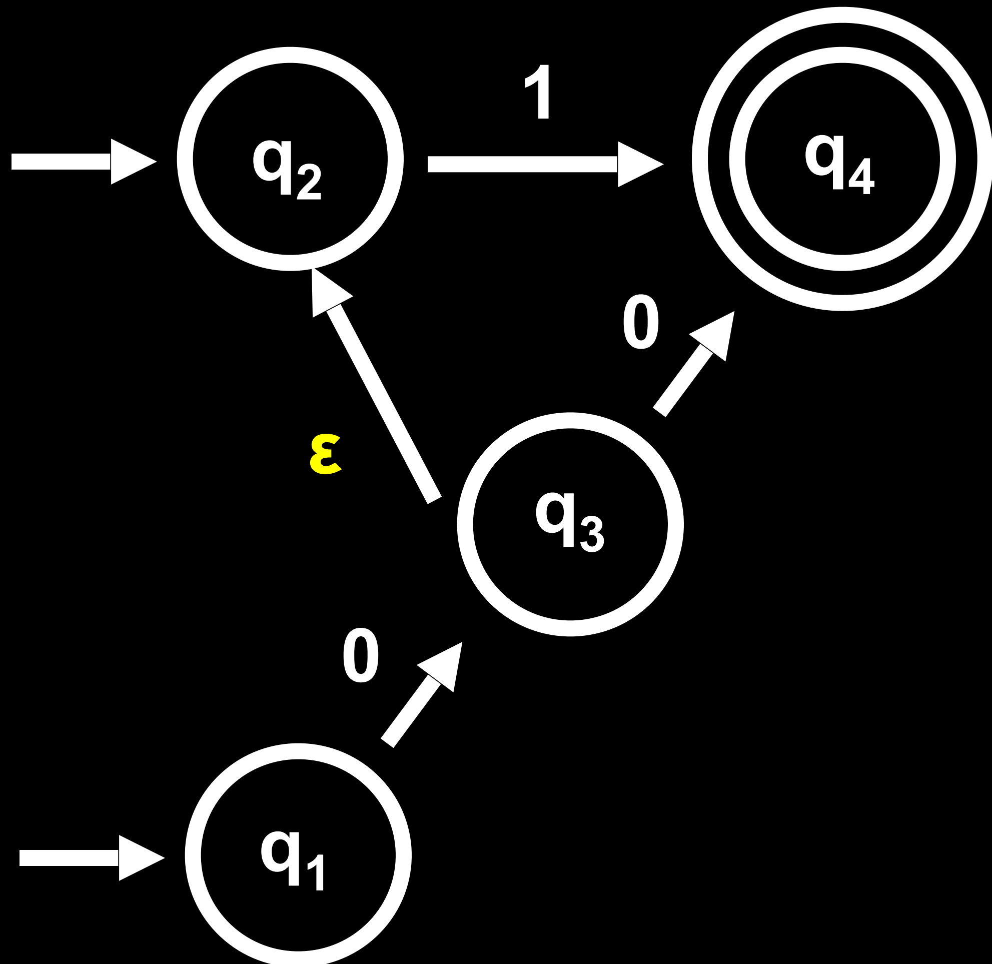
$Q_0 = \{q_1, q_2\}$

$F = \{q_4\} \subseteq Q$

$\delta(q_2,1) = \{q_4\}$

$\delta(q_3,1) = \emptyset$ $\quad \delta(q_3, \varepsilon) = \{ q_2\}$

$\delta(q_1,0) = \{ q_3\}$



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$\Sigma = \{0,1\}$

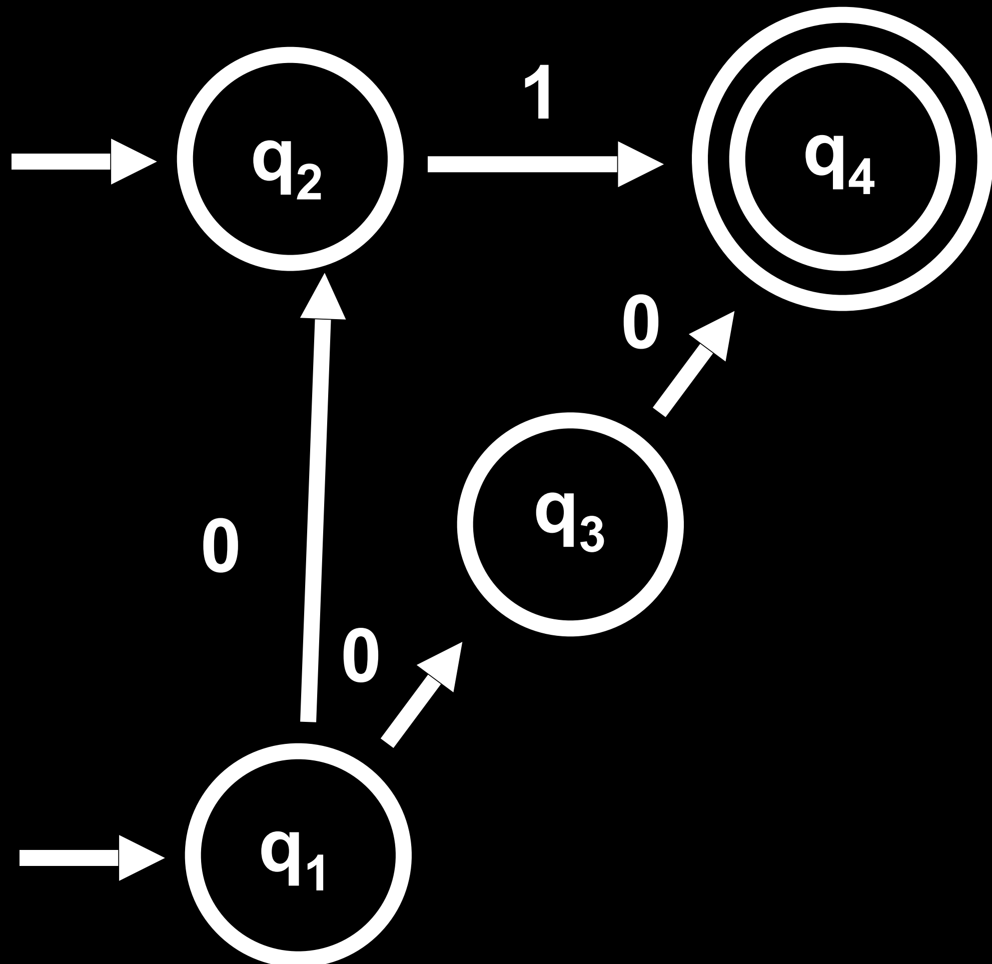
$Q_0 = \{q_1, q_2\}$

$F = \{q_4\} \subseteq Q$

δ	0	1	ε
q_1	$\{q_3\}$	$\varnothing$	$\varnothing$
q_2	$\varnothing$	$\{q_4\}$	$\varnothing$
q_3	$\{q_4\}$	$\varnothing$	$\{q_2\}$
q_4	$\varnothing$	$\varnothing$	$\varnothing$

$00 \in L(N)$?

$01 \in L(N)$?



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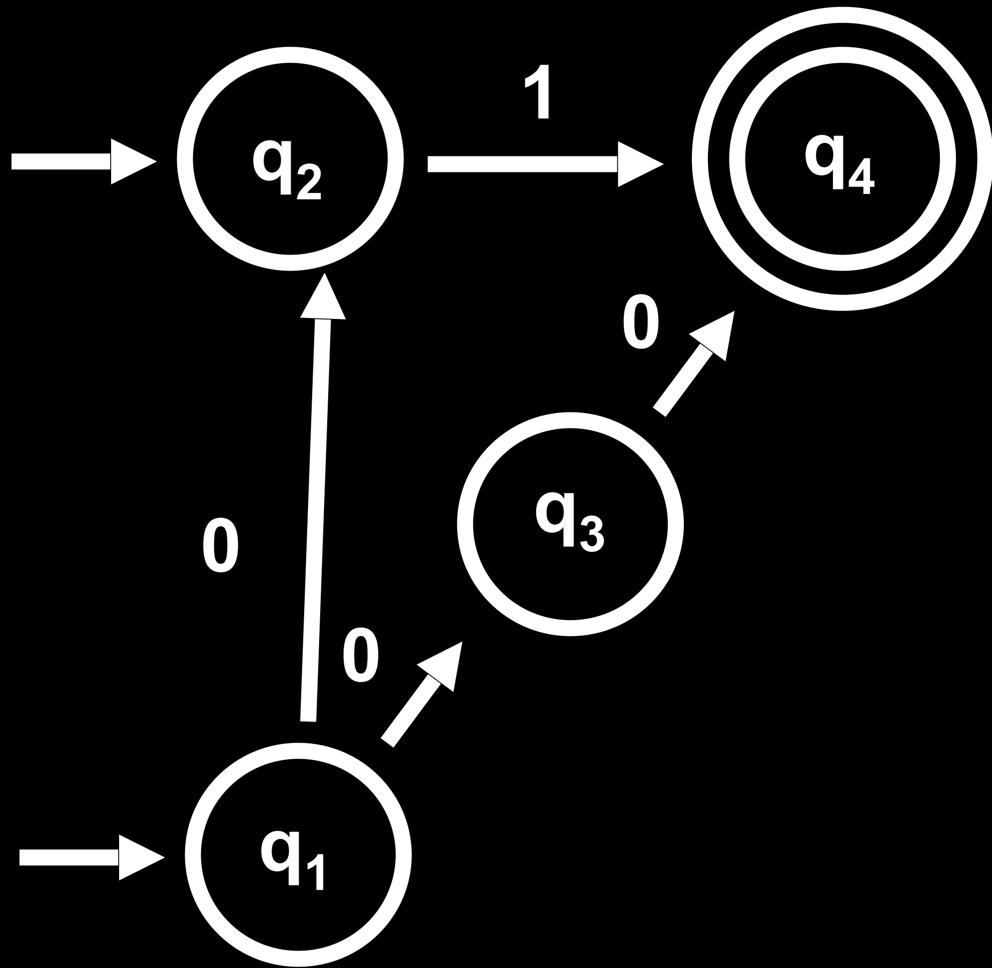
δ	0	1	ε
q_1	$\{q_2, q_3\}$	$\varnothing$	$\varnothing$
q_2	$\varnothing$	$\{q_4\}$	$\varnothing$
q_3	$\{q_4\}$	$\varnothing$	$\varnothing$
q_4	$\varnothing$	$\varnothing$	$\varnothing$

$00 \in L(N)$?

$01 \in L(N)$?



$N = (Q, \Sigma, \delta, Q_0, F)$

$Q = \{q_1, q_2, q_3, q_4\}$

$\Sigma = \{0,1\}$

$Q_0 = \{q_1, q_2\}$

$F = \{q_4\} \subseteq Q$

δ	0	1
q_1	$\{q_2, q_3\}$	$\varnothing$
q_2	$\varnothing$	$\{q_4\}$
q_3	$\{q_4\}$	$\varnothing$
q_4	$\varnothing$	$\varnothing$

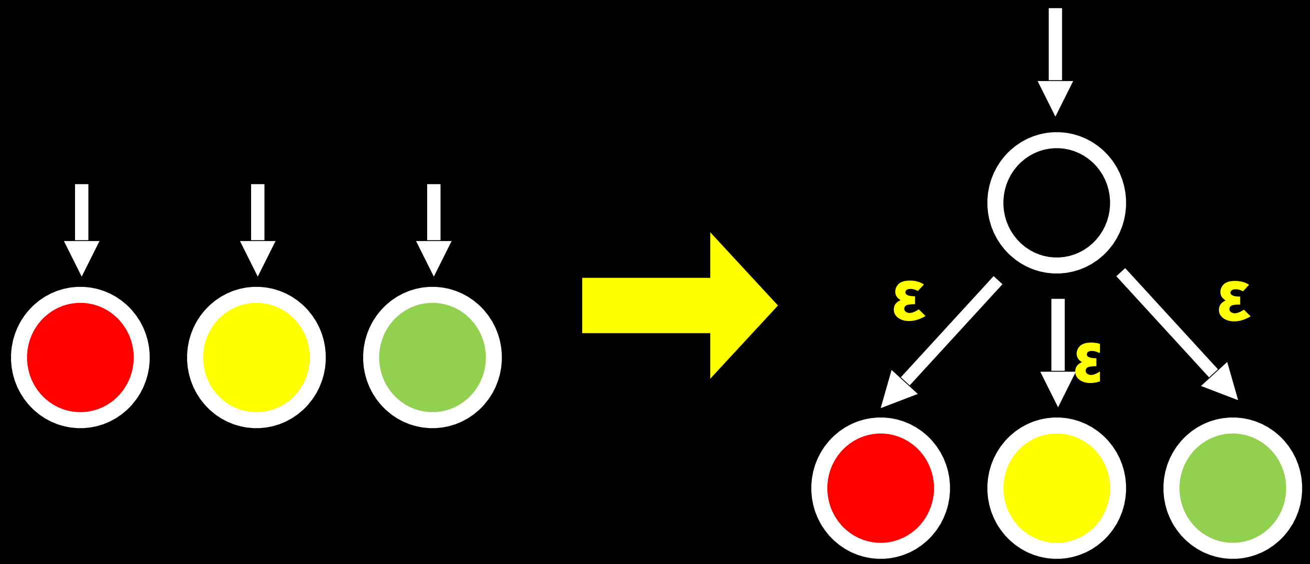
$00 \in L(N)$?

$01 \in L(N)$?

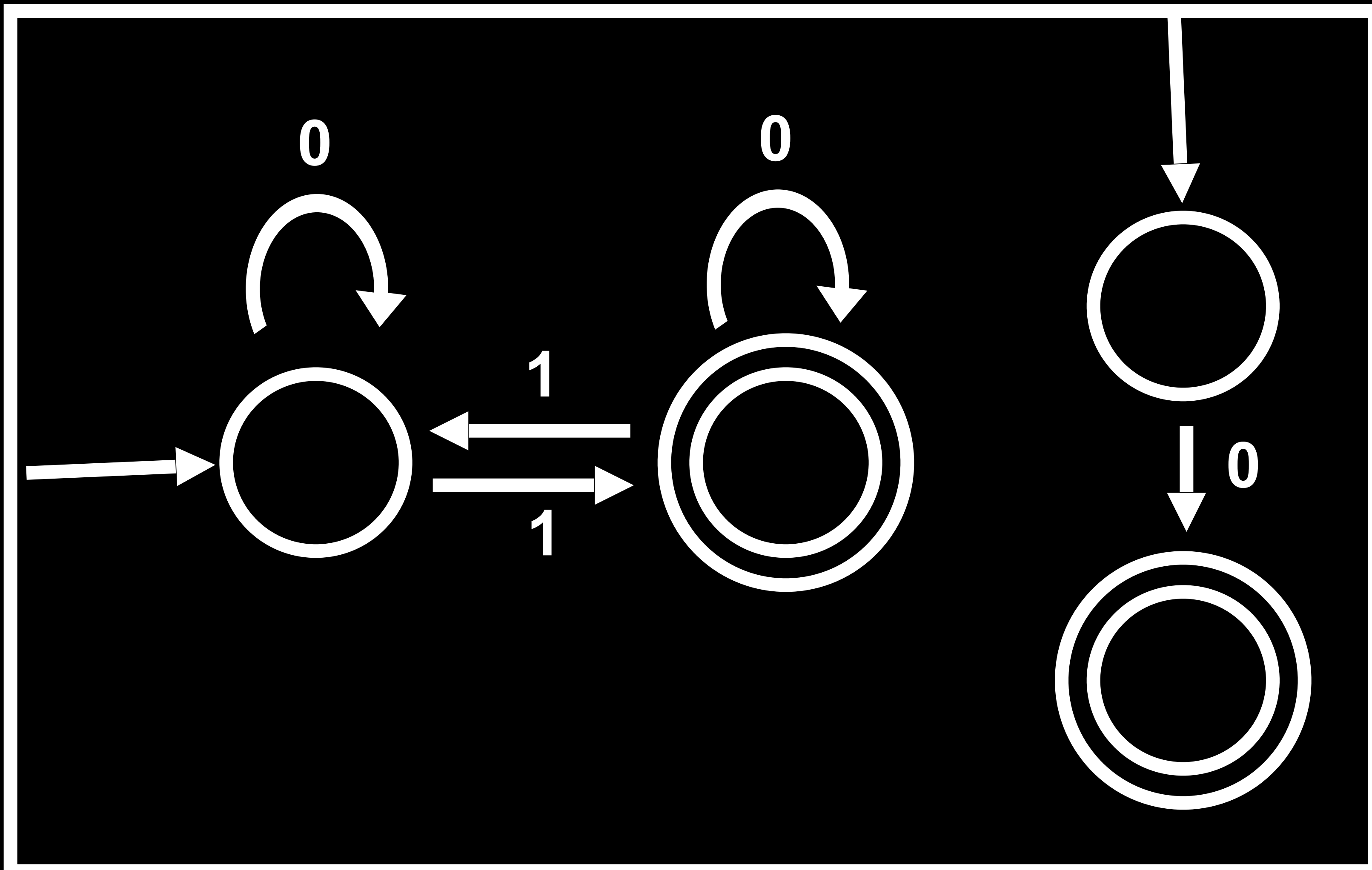
MULTIPLE START STATES

We allow *multiple* start states for NFAs, and Sipser allows only one

Can easily convert NFA with many start states into one with a single start state:

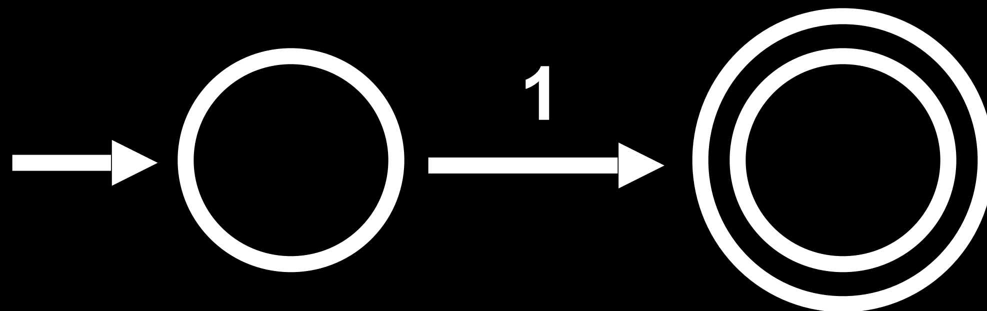


UNION THEOREM FOR NFAs?

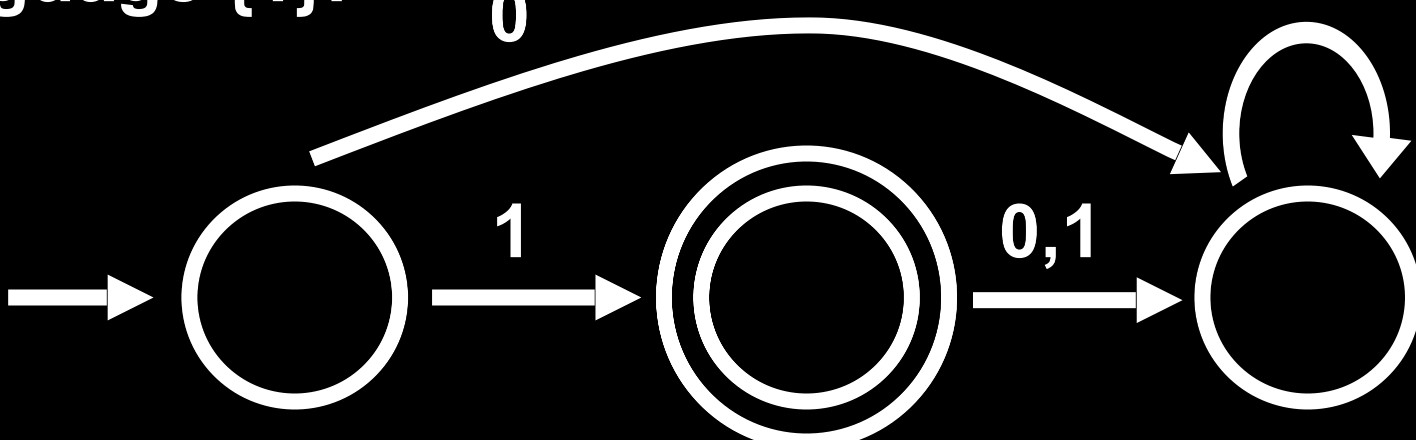


NFAs ARE SIMPLER THAN DFAs

An NFA that recognizes the language {1}:



A DFA that recognizes the language {1}:



BUT DFAs CAN **SIMULATE** NFAs!

Theorem: Every NFA has an equivalent* DFA

Corollary: A language is regular iff it is recognized by an NFA

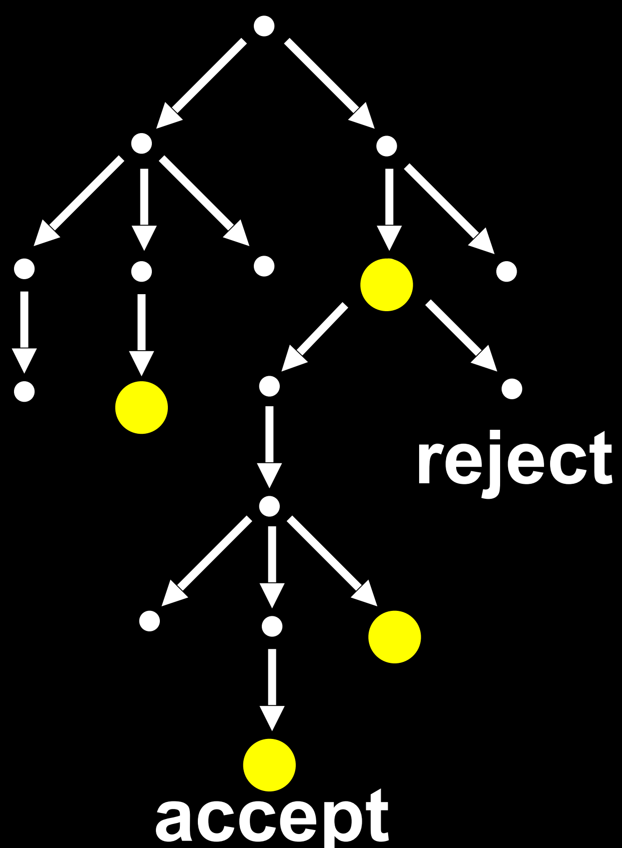
Corollary: L is regular iff L^R is regular

*** N is equivalent to M if L(N) = L (M)**

FROM NFA TO DFA

Input: NFA N = $(Q, \Sigma, \delta, Q_0, F)$

Output: DFA M = $(Q', \Sigma, \delta', q_0', F')$



To see if NFA accepts, we could do the computation in parallel, maintaining the set of all possible states that can be reached

Idea:

$$Q' = 2^Q$$

FROM NFA TO DFA

Input: NFA N = $(Q, \Sigma, \delta, Q_0, F)$

Output: DFA M = $(Q', \Sigma, \delta', q_0', F')$

$$Q' = 2^Q$$

$$\delta' : Q' \times \Sigma \rightarrow Q'$$

$$\delta'(R,\sigma) = \bigcup_{r \in R} \varepsilon(\delta(r,\sigma)) \quad *$$

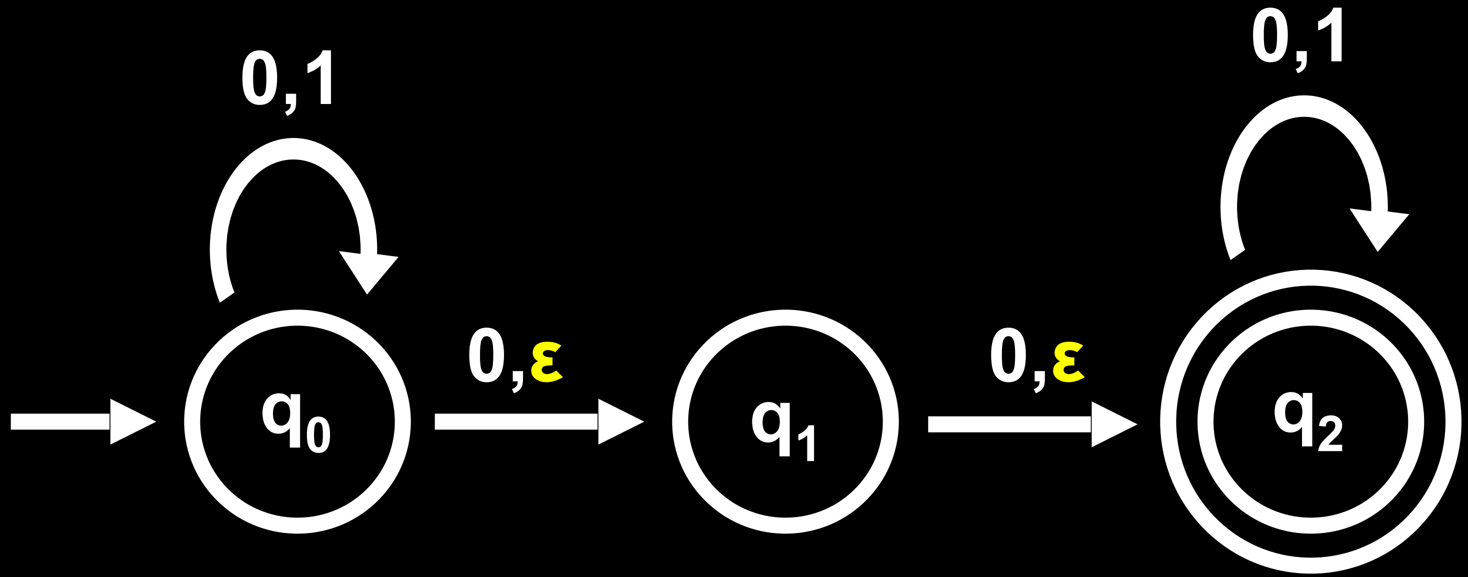
$$q_0' = \varepsilon(Q_0)$$

$$F' = \{ R \in Q' \mid f \in R \text{ for some } f \in F \}$$

*

For $R \subseteq Q$, the **ε-closure** of R, $\varepsilon(R)$ = {q that can be reached from some $r \in R$ by traveling along zero or more ε arrows}

EXAMPLE OF ε-CLOSURE



$$\varepsilon(\{q_0\}) = \{q_0, q_1, q_2\}$$

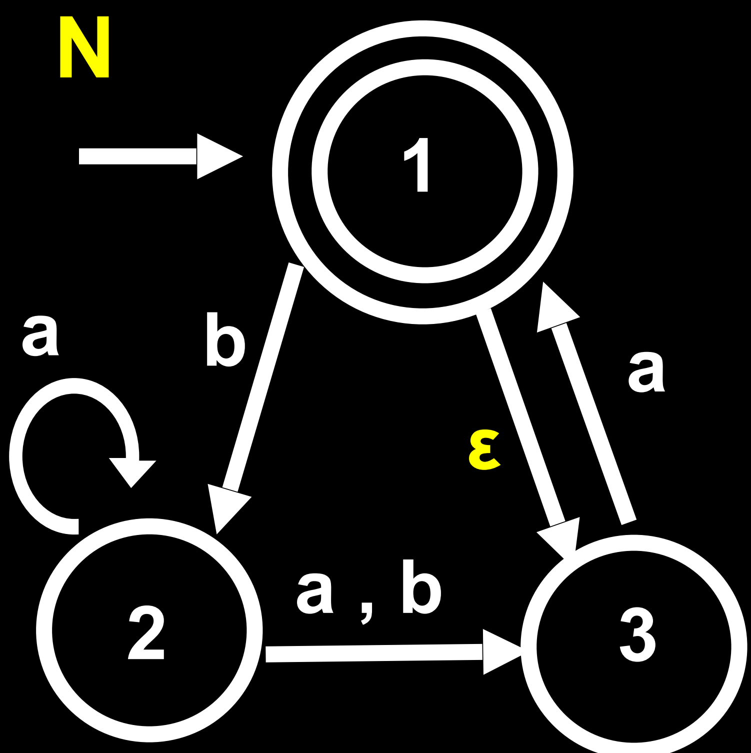
$$\varepsilon(\{q_1\}) = \{q_1, q_2\}$$

$$\varepsilon(\{q_2\}) = \{q_2\}$$

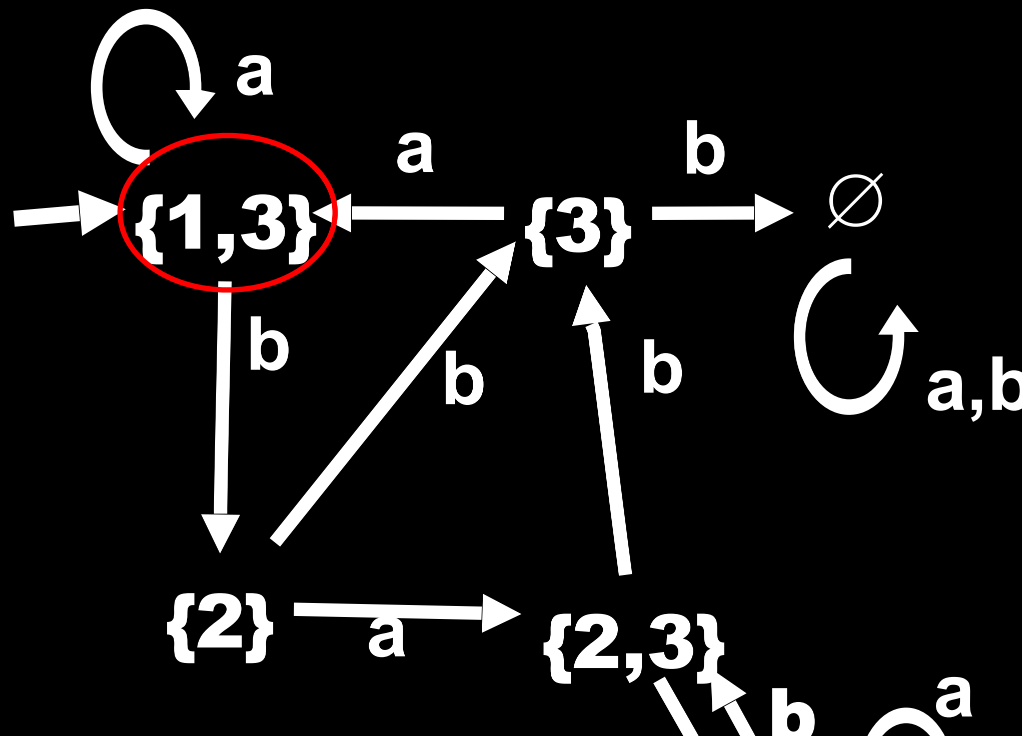
Given: NFA N = ({1,2,3}, {a,b}, δ , {1}, {1})

Construct: Equivalent DFA M

$M = (2^{\{1,2,3\}}, \{a,b\}, \delta', \{1,3\}, \{ \{1\}, \ldots \})$



$\varepsilon(\{1\}) = \{1,3\}$

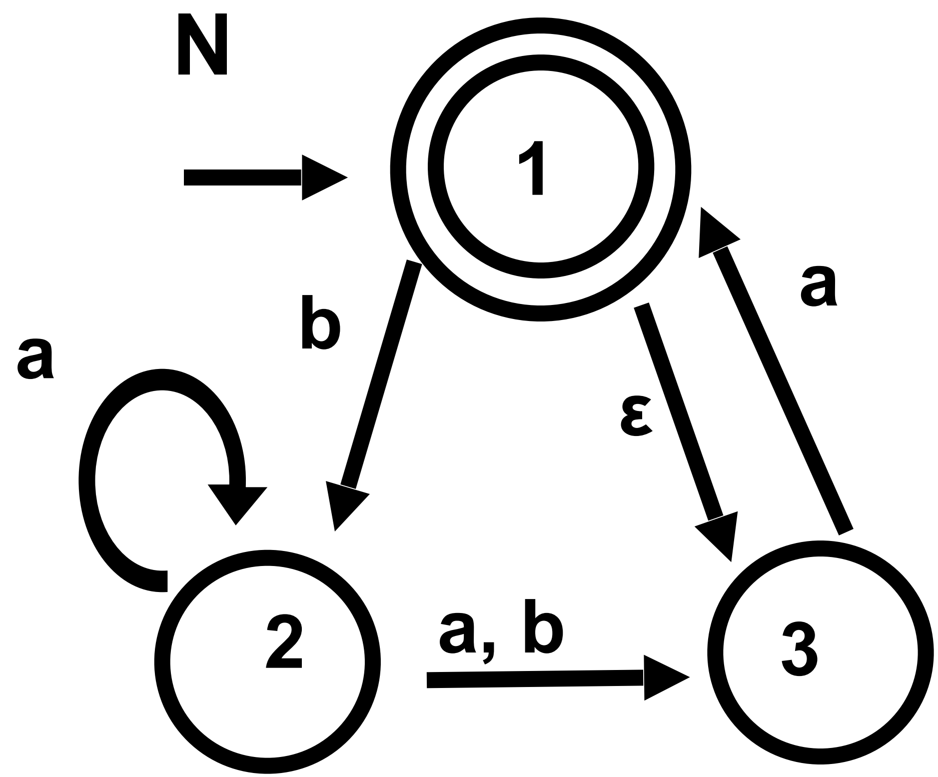


{1}, {1,2} ?

$N = (Q, \Sigma, \delta, Q_0, F)$

Given: NFA $N = (\{1,2,3\}, \{a,b\}, \delta, \{1\}, \{1\})$

Construct: equivalent DFA M $= (Q', \Sigma, \delta', q_0', F')$



$\varepsilon(\{1\}) = \{1,3\}$

δ'	a	b
$\varnothing$		
{1}		
{2}		
{3}		

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Given: NFA $N = (\{1,2,3\}, \{a,b\}, \delta, \{1\}, \{1\})$

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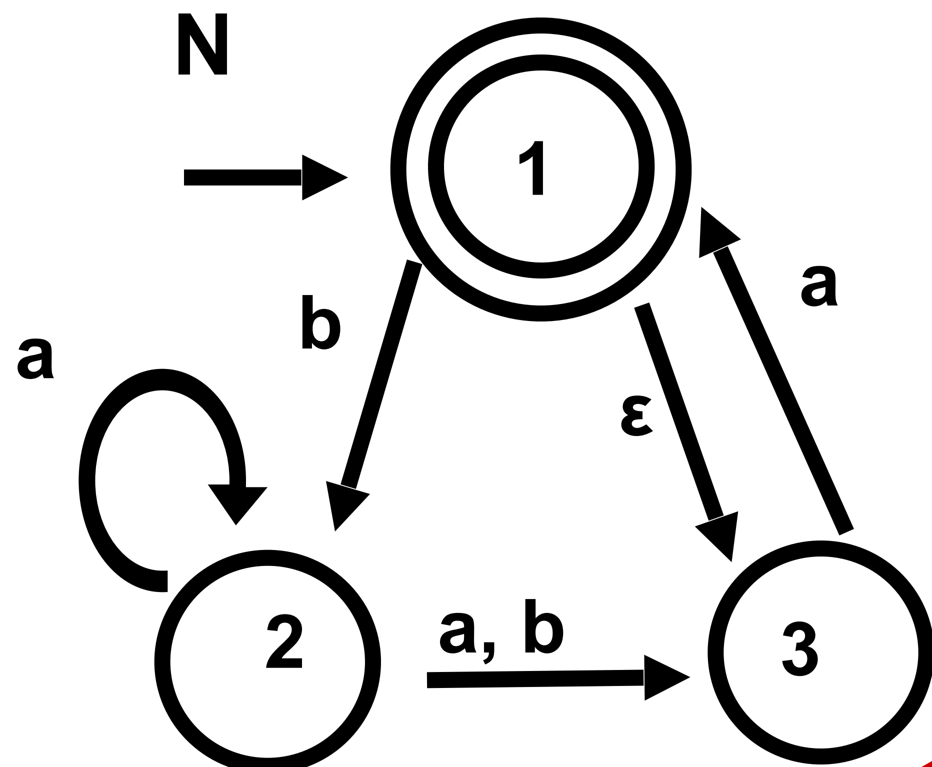

$q_0' = \varepsilon(\{1\}) = \{1,3\}$

δ'	a	b
$\varnothing$		
{1}		
{2}		
{3}		
{1,2}		
{1,3}		
{2,3}		
{1,2,3}		

$N = (Q, \quad \Sigma, \quad \delta, \quad Q_0, \quad F)$

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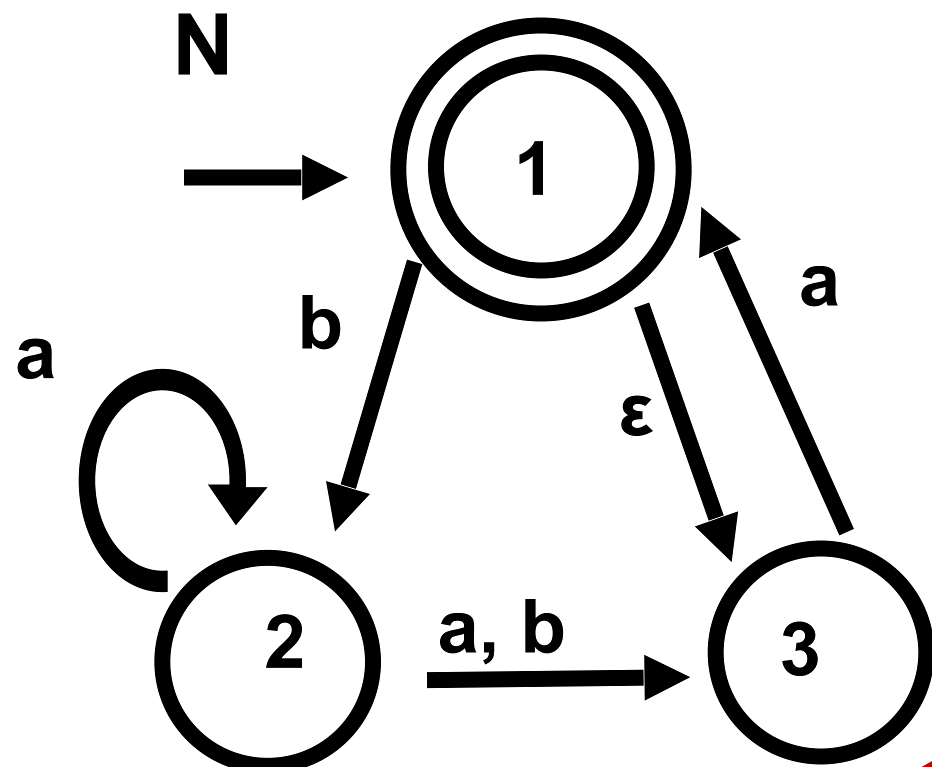

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δ'	a	b
$\varnothing$		
{1}		
{2}		
{3}		
{1,2}		
{1,3}		
{2,3}		
{1,2,3}		

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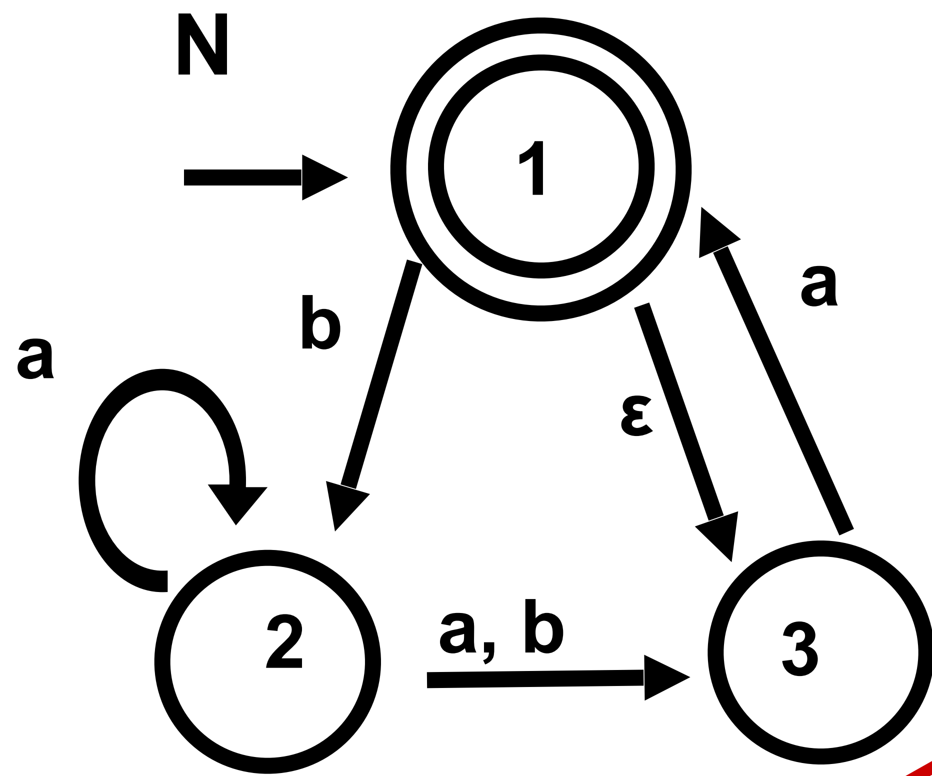

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δ'	a	b
$\varnothing$	$\varnothing$	$\varnothing$
{1}		
{2}		
{3}		
{1,2}		
{1,3}		
{2,3}		
{1,2,3}		

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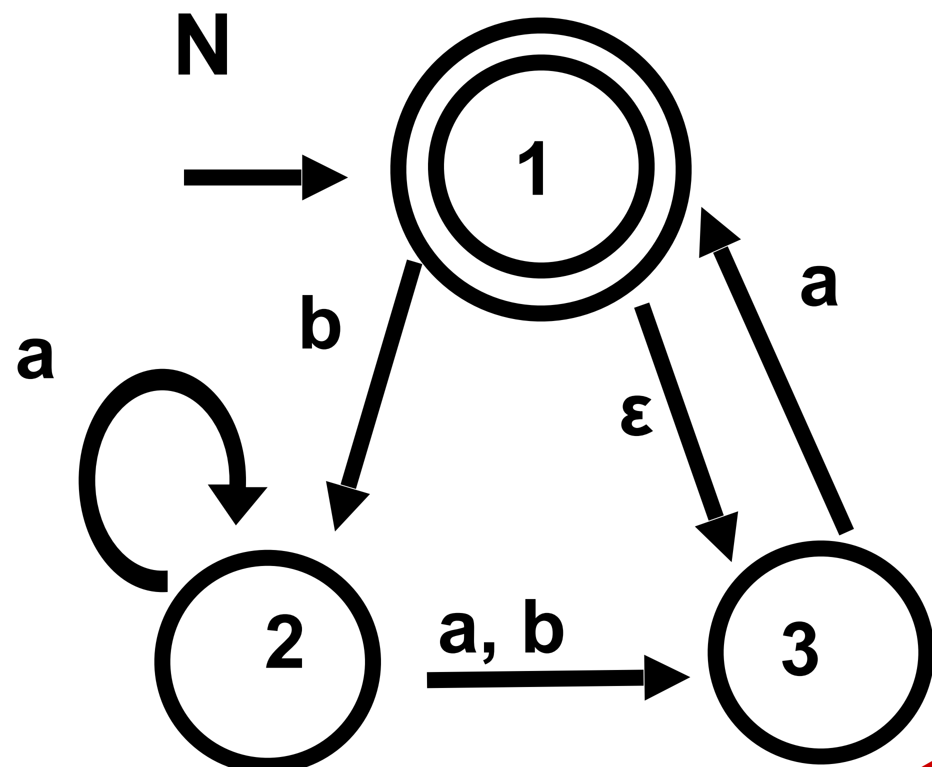

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$\varnothing$	$\varnothing$	$\varnothing$
{1}	$\varnothing$	{2}
{2}		
{3}		
{1,2}		
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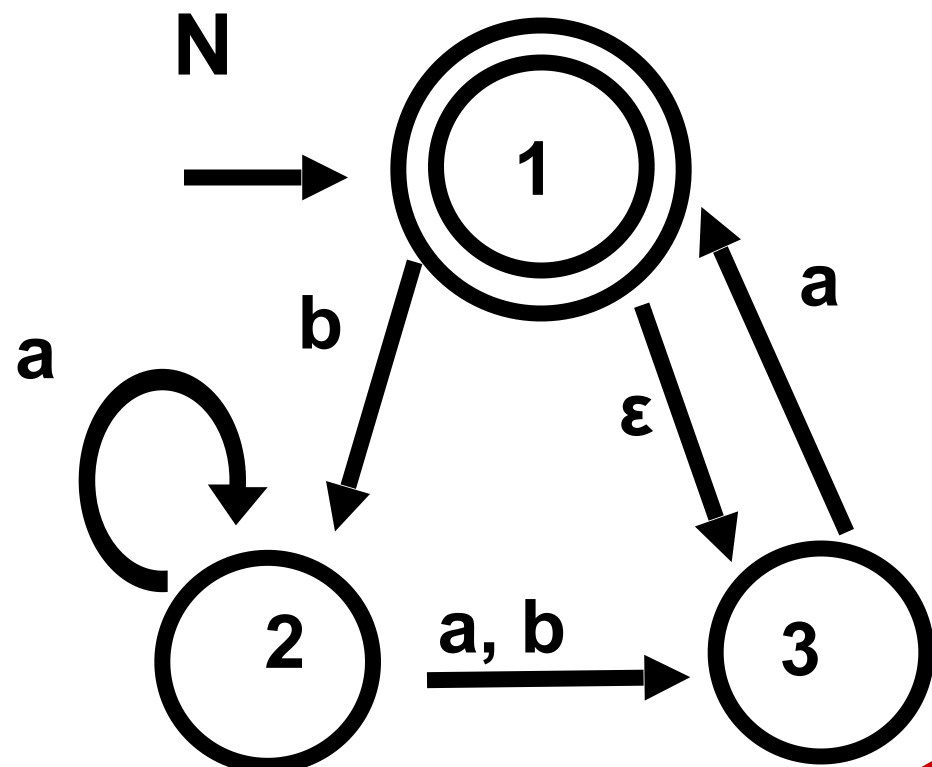

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$\varnothing$	$\varnothing$	$\varnothing$
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{2}	{2,3}	{3}
{3}		
{1,2}		
{1,3}		
{2,3}		
{1,2,3}		

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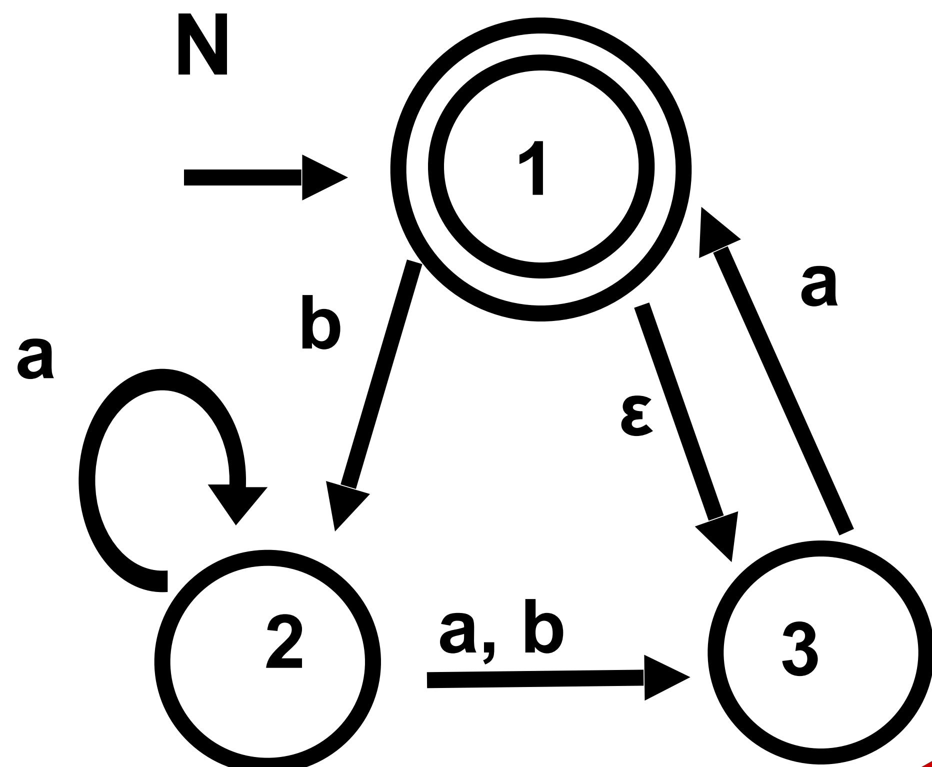

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δ'	a	b
$\varnothing$	$\varnothing$	$\varnothing$
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{2}	{2,3}	{3}
{3}	{1,3}	$\varnothing$
{1,2}	{2,3}	{2,3}
{1,3}	{1,3}	{2}
{2,3}	{1,2,3}	{3}
{1,2,3}	{1,2,3}	{2,3}

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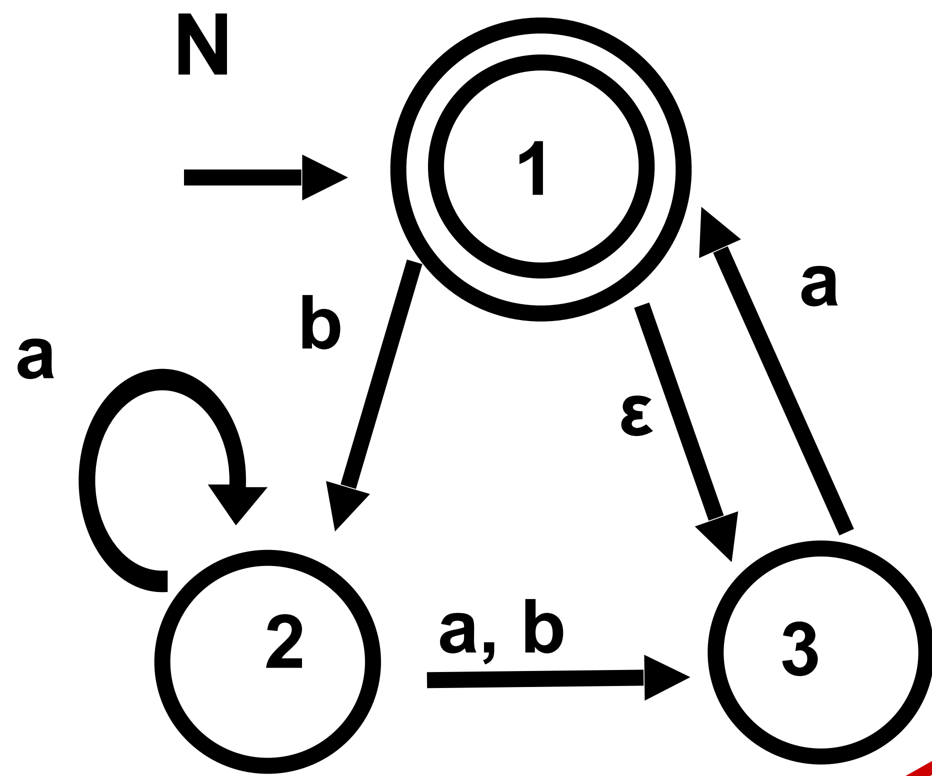
Given: NFA $N = (\{1,2,3\}, \{a,b\}, \delta, \{1\}, \{1\})$

Construct: equivalent DFA $M = (Q', \Sigma, \delta', q_0', F')$



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$\varnothing$	$\varnothing$	$\varnothing$
{1}	$\varnothing$	{2}
{2}	{2,3}	{3}
{3}	{1,3}	$\varnothing$
{1,2}	{2,3}	{2,3}
{1,3}	{1,3}	{2}
{2,3}	{1,2,3}	{3}
{1,2,3}	{1,2,3}	{2,3}

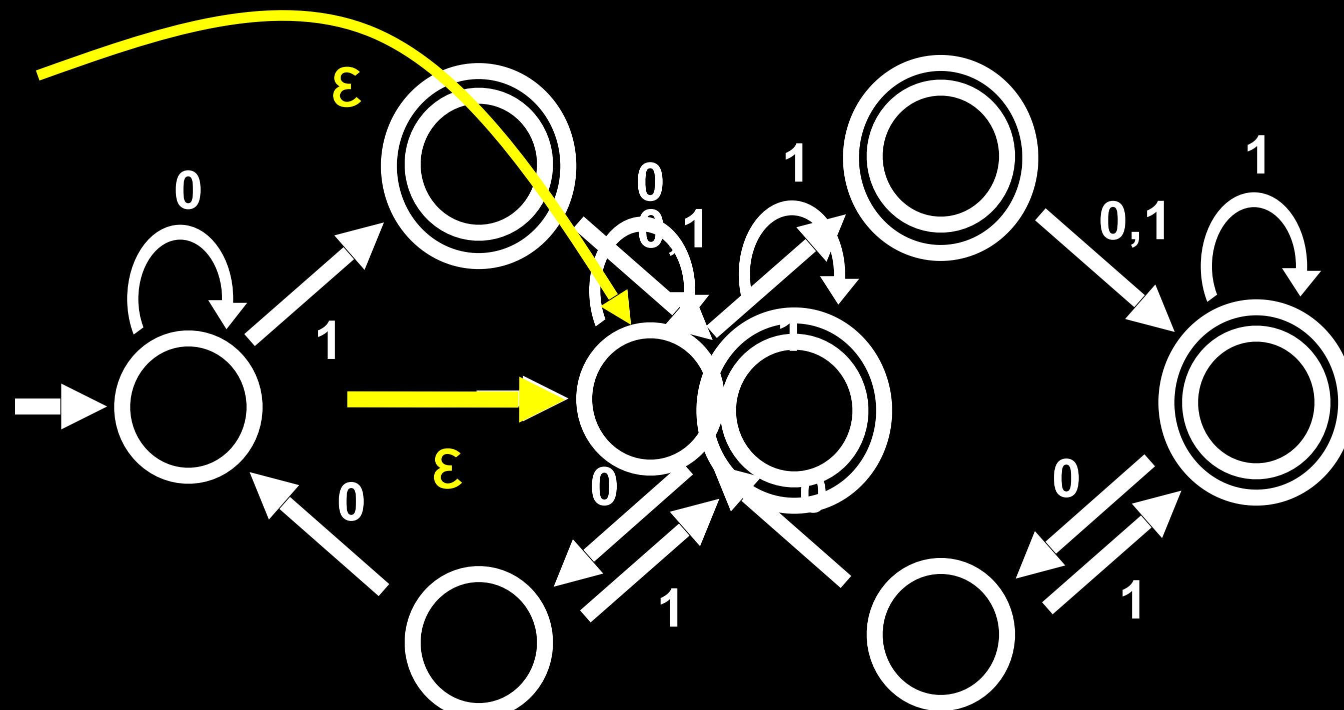
NFAs CAN MAKE PROOFS MUCH EASIER!

Remember this on your Homework!

REGULAR LANGUAGES CLOSED UNDER **CONCATENATION**

Concatenation: $A \cdot B = \{ vw \mid v \in A \text{ and } w \in B \}$

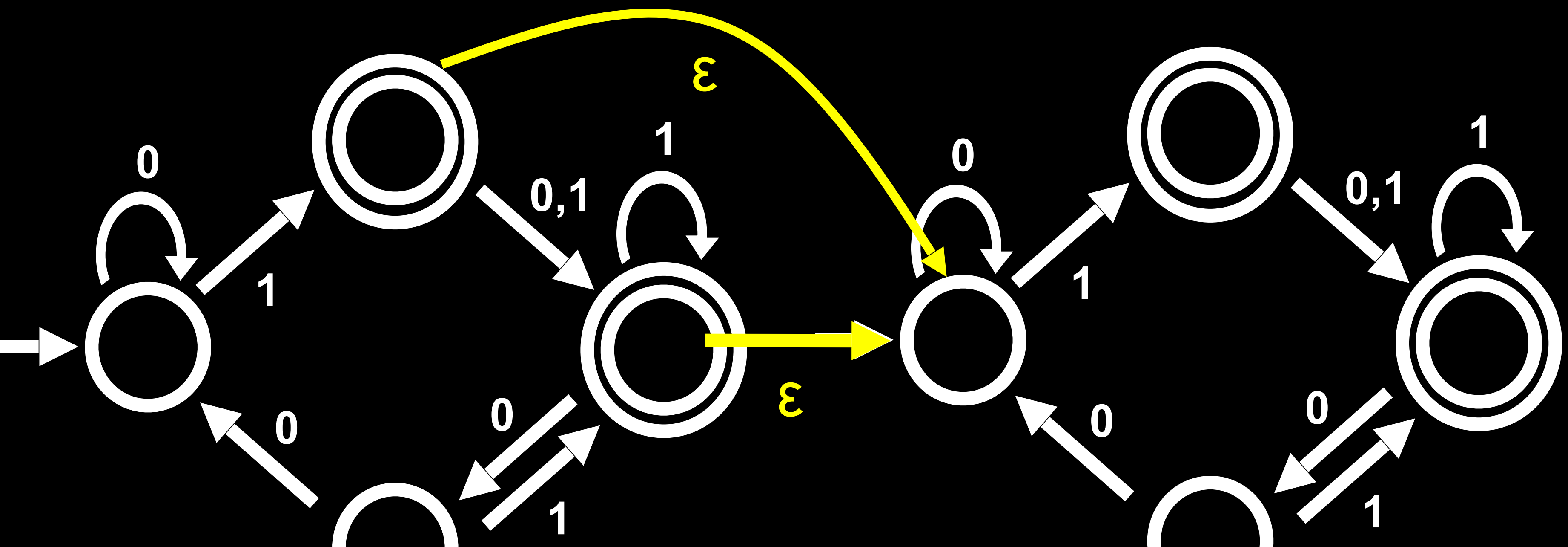
Given DFAs M_1 and M_2, connect accept states in M_1 to start states in M_2



REGULAR LANGUAGES CLOSED UNDER **CONCATENATION**

Concatenation: A · B = { vw | v ∈ A and w ∈ B }

Given DFAs M_1 and M_2, connect accept states in M_1 to start states in M_2



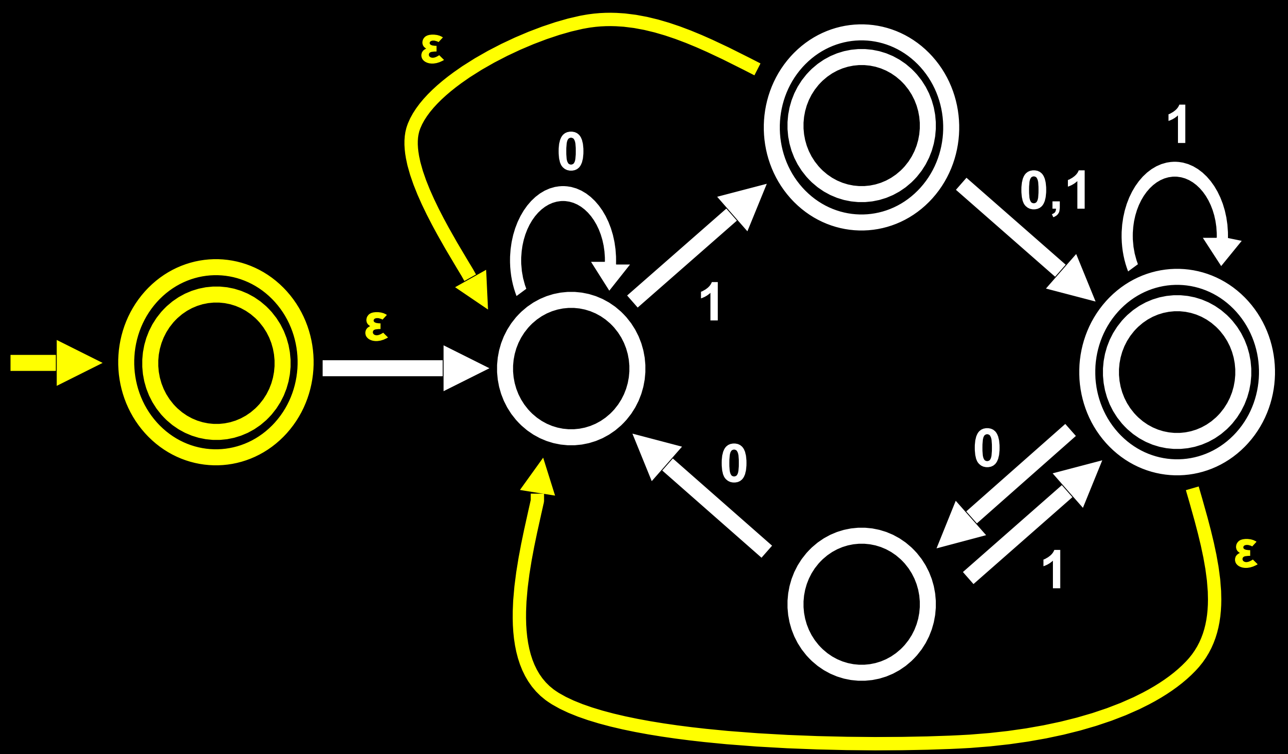
$L(N) = L(M_1) \cdot L(M_2)$

RLs ARE CLOSED UNDER **STAR**

Star: $A^* = \{ s_1 \ldots s_k \mid k \geq 0 \text{ and each } s_i \in A \}$

Let M be a DFA, and let L = L(M)

Can construct an NFA N that recognizes L*



Formally:

Input: **M** = $(Q, \Sigma, \delta, q_1, F)$

Output: **N** = $(Q', \Sigma, \delta', \{q_0\}, F')$

$$Q' = Q \cup \{q_0\}$$

$$F' = F \cup \{q_0\}$$

$$\delta'(q,a) = \begin{cases} \{\delta(q,a)\} & \text{if } q \in Q \text{ and } a \neq \varepsilon \\ \{q_1\} & \text{if } q \in F \text{ and } a = \varepsilon \\ \{q_1\} & \text{if } q = q_0 \text{ and } a = \varepsilon \\ \varnothing & \text{if } q = q_0 \text{ and } a \neq \varepsilon \\ \varnothing & \text{else} \end{cases}$$

Show: L(N) = L* where L = L(M)

1. $L(N) \supseteq L^*$

2. $L(N) \subseteq L^*$

1. $L(N) \supseteq L^*$ (where $L = L(M)$)

Assume $w = w_1 \ldots w_k$ is in L^*, where $w_1, \ldots, w_k \in L$

We show N accepts w by induction on k

Base Cases:

- ✓ $k = 0$ ($w = \varepsilon$)
- ✓ $k = 1$ ($w \in L$)

Inductive Step:

Assume N accepts all strings $v = v_1 \ldots v_k \in L^*$, $v_i \in L$
and let $u = u_1 \ldots u_k u_{k+1} \in L^*$, $u_j \in L$

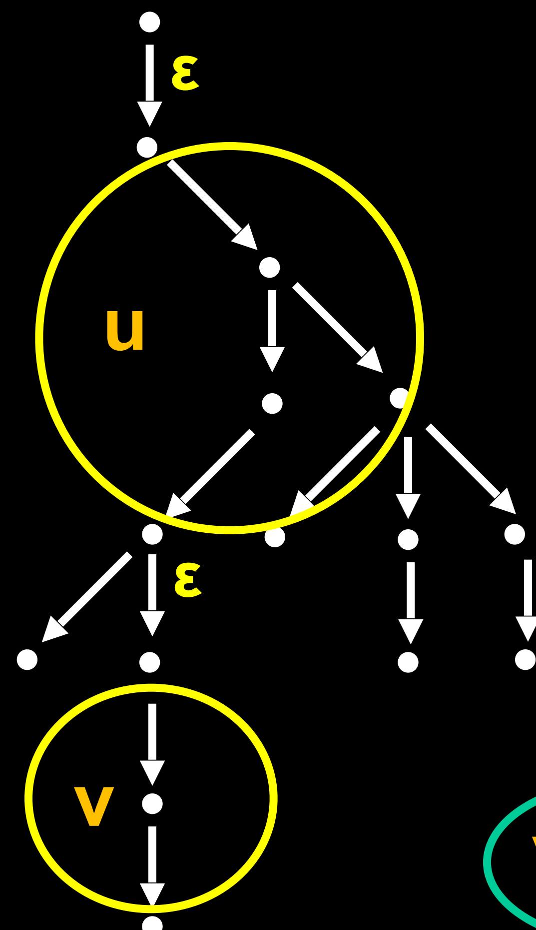
Since N accepts $u_1 \ldots u_k$ (by induction) and
M accepts u_{k+1}, N must accept u

2. L(N) $\subseteq$ L* (where L = L(M))

Assume w is accepted by N, we show w $\in$ L*

If w = ε, then w $\in$ L*

If w ≠ ε,
write w as w=uv,
where v is the
substring read
after the *last*
ε-transition



accept

u $\in$ L*

By induction

w = uv $\in$ L*

v $\in$ L

REGULAR LANGUAGES ARE CLOSED UNDER **THE REGULAR OPERATIONS**

- **Union:** $A \cup B = \{ w \mid w \in A \text{ or } w \in B \}$
- **Intersection:** $A \cap B = \{ w \mid w \in A \text{ and } w \in B \}$
- **Negation:** $\neg A = \{ w \in \Sigma^* \mid w \notin A \}$
- **Reverse:** $A^R = \{ w_1 \ldots w_k \mid w_k \ldots w_1 \in A \}$
- **Concatenation:** $A \cdot B = \{ vw \mid v \in A \text{ and } w \in B \}$
- **Star:** $A^* = \{ w_1 \ldots w_k \mid k \geq 0 \text{ and each } w_i \in A \}$

SOME LANGUAGES **ARE**

NOT REGULAR

B = $\{0^n1^n \mid n \geq 0\}$ is NOT regular!

WHICH OF THESE ARE REGULAR

C = { w | w has equal number of occurrences of 01 and 10 }

REGULAR!!!

D = { w | w has equal number of 1s and 0s}

NOT REGULAR

Read Chapters 1.3 and 1.4 of the book for next time