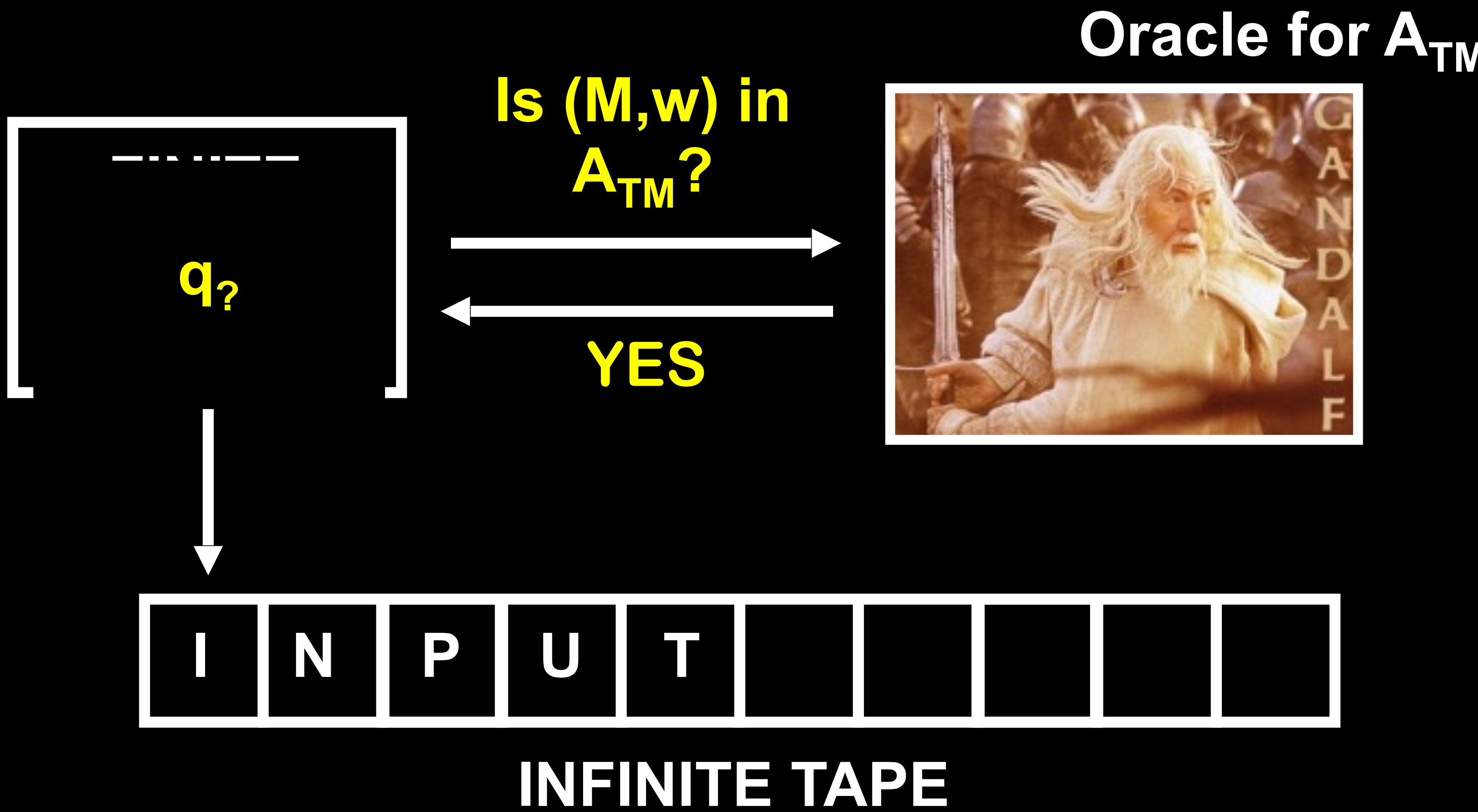
15-453

FORMAL LANGUAGES, AUTOMATA AND COMPUTABILITY

THE ARITHMETIC HIERARCHY

THURSDAY, MAR 6

ORACLE TMs



ORACLE MACHINES

An ORACLE is a set B to which the TM may pose membership questions "Is w in B?"
(formally: TM enters state $q_?$)
and the TM always receives a correct answer in one step
(formally: if the string on the "oracle tape" is in B, state $q_?$ is changed to q_{YES}, otherwise q_{NO})

This makes sense even if B is not decidable!
(We do not assume that the oracle B is a computable set!)

We say A is semi-decidable in B
if there is an oracle TM M with oracle B that semi-decides A

We say A is decidable in B
if there is an oracle TM M with oracle B that decides A

Language A "Turing Reduces" to Language B

if A is decidable in B, ie if there is an oracle TM M with oracle B that decides A

$$A \leq_T B$$

$\leq_T$ VERSUS $\leq_m$

Theorem: If A $\leq_m$ B then A $\leq_T$ B

Proof:

If A $\leq_m$ B then there is a computable function $f : \Sigma^* \rightarrow \Sigma^*$, where for every w,

$$w \in A \Leftrightarrow f(w) \in B$$

We can thus use an oracle for B to decide A

Theorem: $\neg AT_{TM} \leq_T AT_{TM}$

Theorem: $\neg AT_{TM} \not\leq_m AT_{TM}$

THE ARITHMETIC **HIERARCHY**

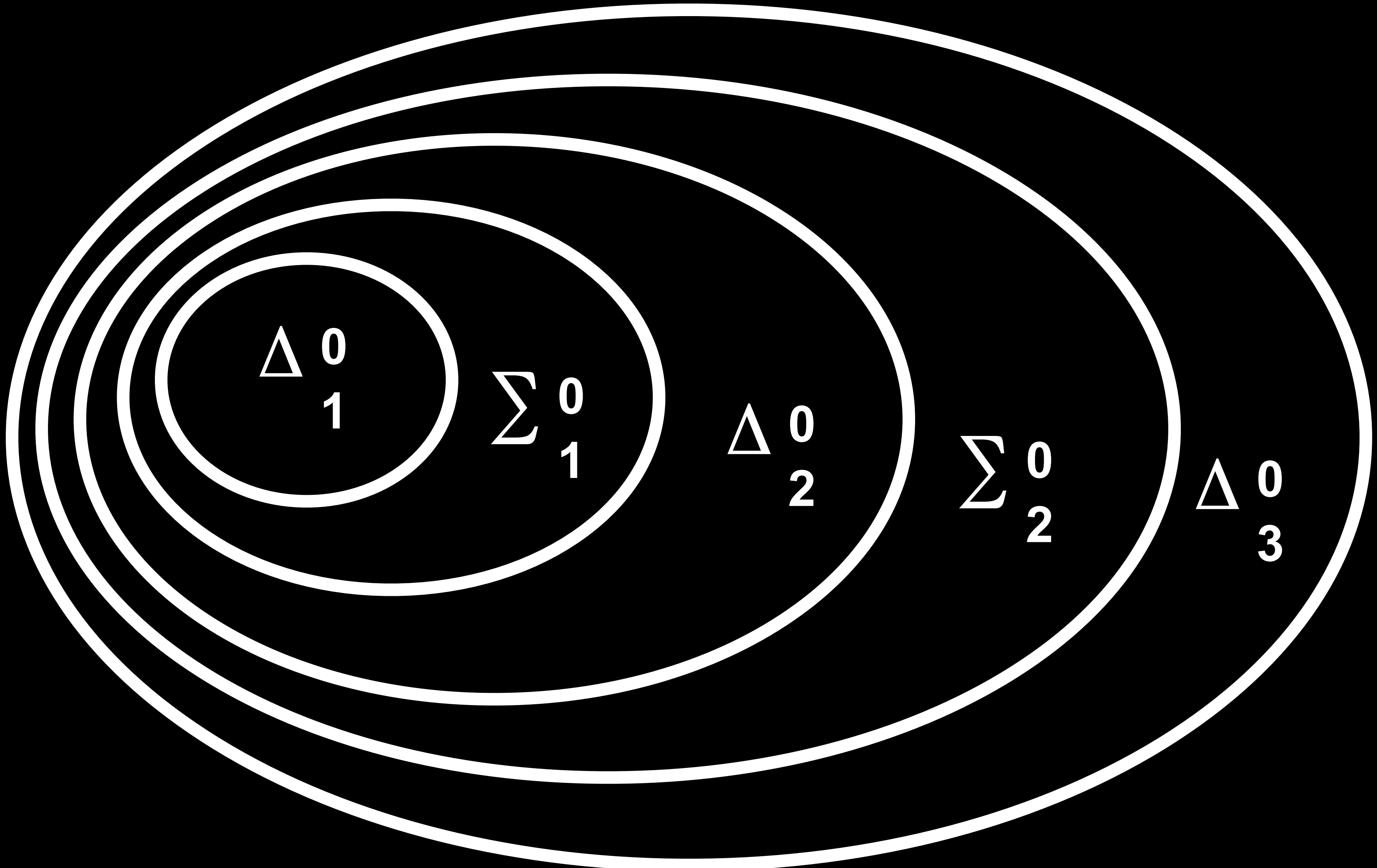
Δ^0_1 = { decidable sets } (sets = languages)

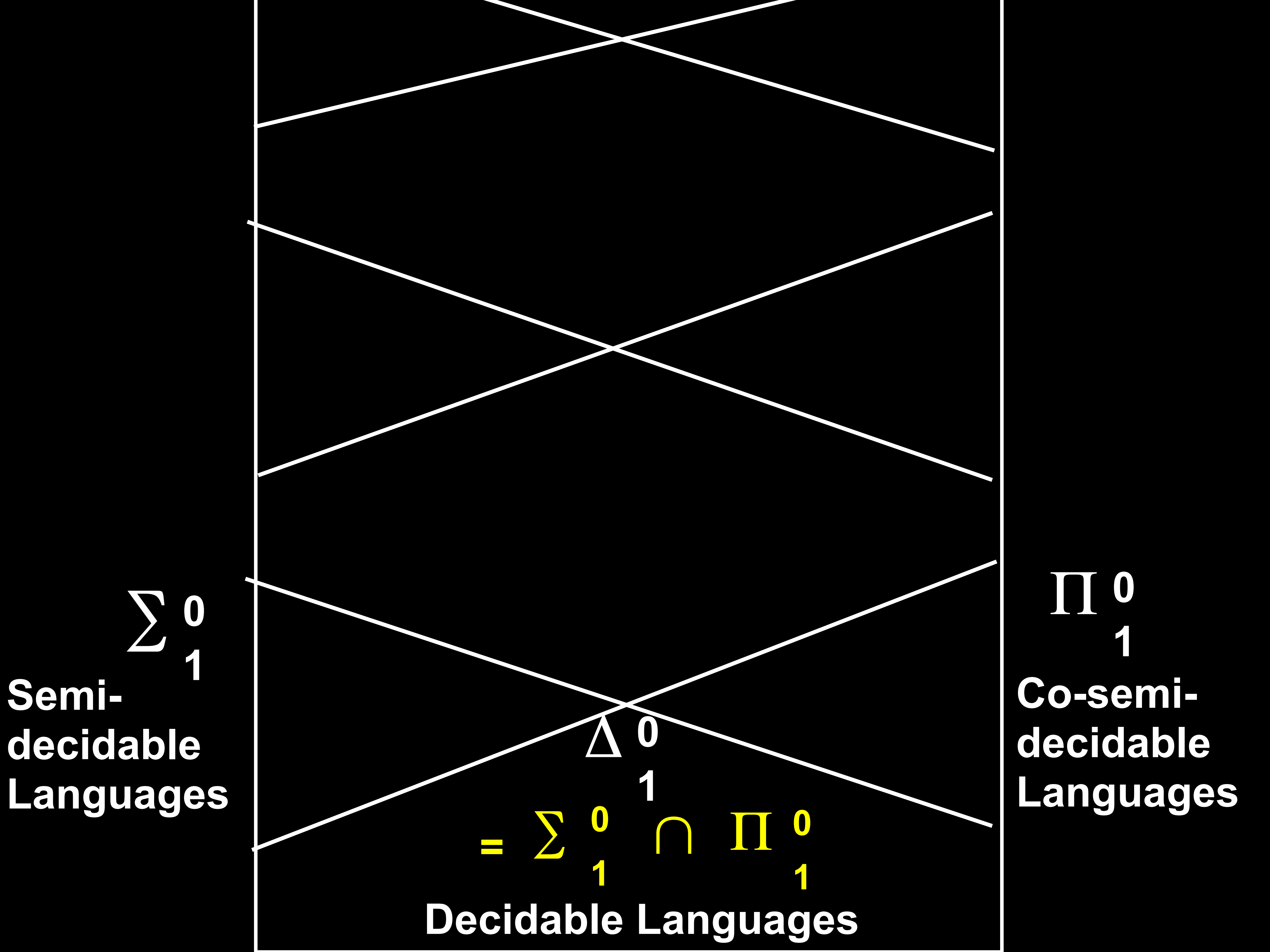
Σ^0_1 = { semi-decidable sets }

Σ^0_{n+1} = { sets semi-decidable in some $B \in \Sigma^0_n$ }

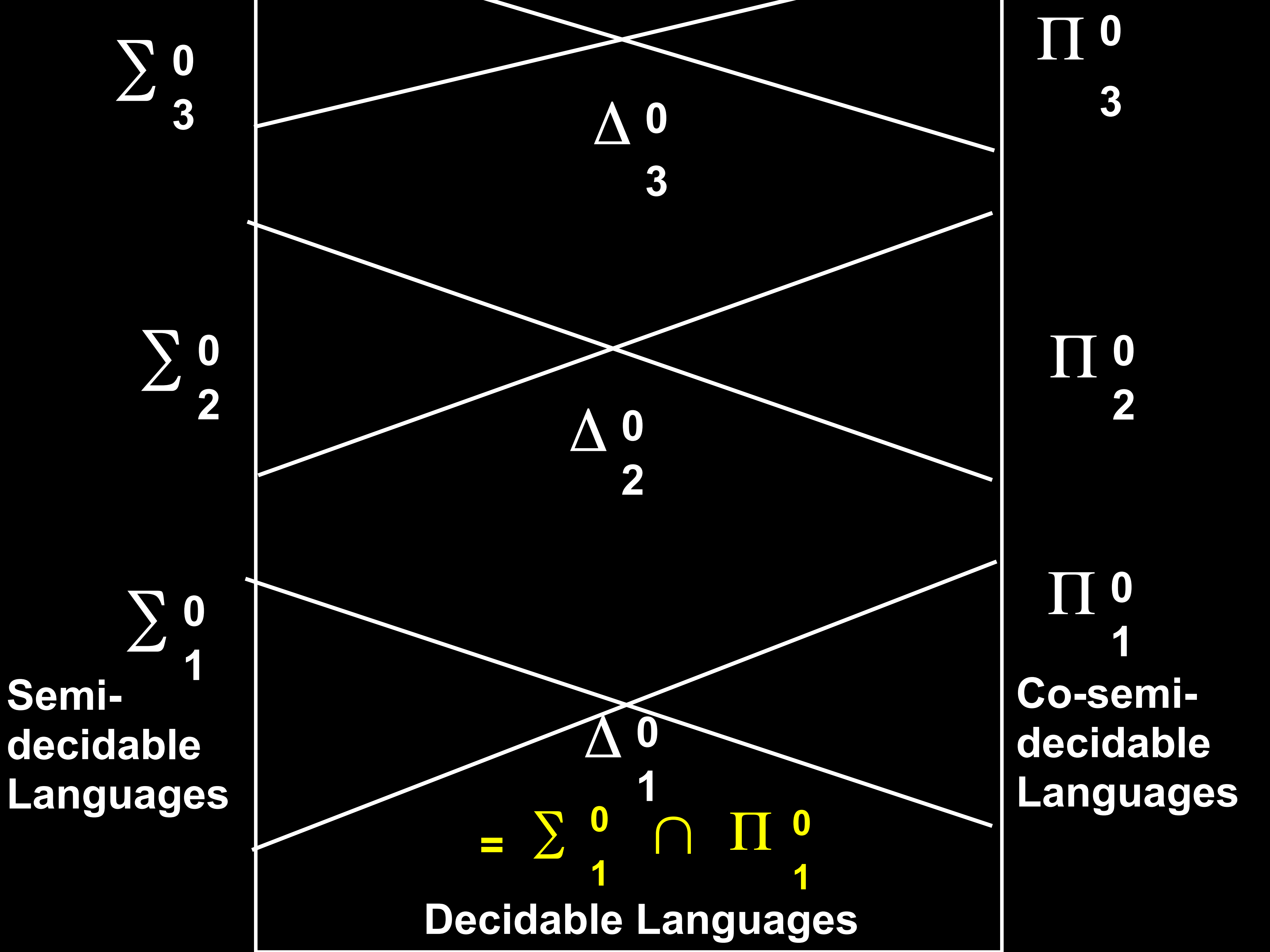
Δ^0_{n+1} = { sets decidable in some $B \in \Sigma^0_n$ }

Π^0_n = { complements of sets in Σ^0_n }


Δ^0_1
Σ^0_1
Δ^0_2
Σ^0_2
Δ^0_3



Σ^0_1
Semi-decidable Languages
Π^0_1
Co-semi-decidable Languages
Δ^0_1
$= \Sigma^0_1 \cap \Pi^0_1$
Decidable Languages



Σ^0_3
Δ^0_3
Π^0_3
Σ^0_2
Δ^0_2
Π^0_2
Σ^0_1
Semi-decidable Languages
Δ^0_1
$= \Sigma^0_1 \cap \Pi^0_1$
Decidable Languages
Π^0_1
Co-semi-decidable Languages

Theorem

Σ^0_1 = { semi-decidable sets }
= languages of the form { x | ∃y R(x,y) }

Π^0_1 = { complements of semi-decidable sets }
= languages of the form { x | ∀y R(x,y) }

Δ^0_1 = { decidable sets }
$= \Sigma^0_1 \cap \Pi^0_1$

Where R is a decidable predicate

Theorem

Σ^0_2 = { sets semi-decidable in some semi-dec. B }
= languages of the form $\{ x \mid \exists y_1 \forall y_2\, R(x,y_1,y_2) \}$

Π^0_2 = { complements of Σ^0_2 sets}
= languages of the form $\{ x \mid \forall y_1 \exists y_2\, R(x,y_1,y_2) \}$

$\Delta^0_2 = \Sigma^0_2 \cap \Pi^0_2$

Where R is a decidable predicate

Theorem

Σ^0_n = languages { x | $\exists y_1 \forall y_2 \exists y_3 \ldots Q y_n\ R(x,y_1,\ldots,y_n)$ }

Π^0_n = languages { x | $\forall y_1 \exists y_2 \forall y_3 \ldots Q y_n\ R(x,y_1,\ldots,y_n)$ }

$\Delta^0_n = \Sigma^0_n \cap \Pi^0_n$

Where R is a decidable predicate

Example

Decidable predicate

Σ^0_1 = languages of the form { x | ∃y R(x,y) }

We know that A_{TM} is in Σ^0_1 Why?

Show it can be described in this form:

A_{TM} = { <(M,w)> | ∃t [M accepts w in t steps] }

decidable predicate

A_{TM} = { <(M,w)> | ∃t T (<M>, w, t }

A_{TM} = { <(M,w)> | ∃v (v is an accepting computation history of M on w}

Definition: A **decidable predicate R(x,y)** is some proposition about **x** and **y**[1], where there is a TM **M** such that

for all x, y, R(x,y) is TRUE ⇒ M(x,y) accepts

R(x,y) is FALSE ⇒ M(x,y) rejects

We say **M** "decides" the predicate **R**.

EXAMPLES:

R(x,y) = "x + y is less than 100"

R(<N>,y) = "N halts on y in at most 100 steps"

Kleene's T predicate, T(<M>, x, y): M accepts x in y steps

1. x, y are positive integers or elements of Σ^*

Definition: A **decidable predicate R(x,y)** is some proposition about x and y[1], where there is a TM M such that

for all x, y, R(x,y) is TRUE $\Rightarrow$ M(x,y) accepts

R(x,y) is FALSE $\Rightarrow$ M(x,y) rejects

We say M "decides" the predicate R.

EXAMPLES:

R(x,y) = "x + y is less than 100"

R(<N>,y) = "N halts on y in at most 100 steps"

Kleene's T predicate, T(<M>, x, y): M accepts x in y steps

Note: A is decidable $\Leftrightarrow$ A = {x | R(x,ε)}, for some decidable predicate R.

Theorem

Σ^0_n = languages $\{ x \mid \exists y_1 \forall y_2 \exists y_3 \ldots Q y_n \, R(x,y_1,\ldots,y_n) \}$

Π^0_n = languages $\{ x \mid \forall y_1 \exists y_2 \forall y_3 \ldots Q y_n \, R(x,y_1,\ldots,y_n) \}$

$\Delta^0_n = \Sigma^0_n \cap \Pi^0_n$

Where R is a decidable predicate

Theorem: A language A is semi-decidable if and only if there is a **decidable predicate R(x, y)** such that: $A = \{ x \mid \exists y\, R(x,y) \}$

Proof:

(1) If $A = \{ x \mid \exists y\, R(x,y) \}$ then A is semi-decidable

Because we can enumerate over all y's

(2) If A is semi-decidable, then $A = \{ x \mid \exists y\, R(x,y) \}$

Let **M** semi-decide **A**

Then, $A = \{ x \mid \exists y\, T(\langle M \rangle, x, y) \}$ (Here **M** is fixed.)

where

Kleene's T predicate, T(<M>, x, y): M accepts x in y steps.

THE PAIRING FUNCTION

Theorem. There is a 1-1 and onto computable function $< , >: \Sigma^* \times \Sigma^* \rightarrow \Sigma^*$ and computable functions π_1 and $\pi_2 : \Sigma^* \rightarrow \Sigma^*$ such that

$$z = \langle w, t \rangle \Rightarrow \pi_1(z) = w \text{ and } \pi_2(z) = t$$

Proof: Let $w = w_1 \ldots w_n \in \Sigma^*$, $t \in \Sigma^*$.

Let $a, b \in \Sigma$, $a \neq b$.

$$\langle w, t \rangle := a\, w_1 \ldots a\, w_n\, b\, t$$

$\pi_1(z)$:= "if z has the form $a\, w_1 \ldots a\, w_n\, b\, t$, then output $w_1 \ldots w_n$, else output ε"

$\pi_2(z)$:= "if z has the form $a\, w_1 \ldots a\, w_n\, b\, t$, then output t, else output ε"

Theorem

Σ^0_1 = { semi-decidable sets }

= languages of the form { x | ∃y R(x,y) }

Π^0_1 = { complements of semi-decidable sets }

= languages of the form { x | ∀y R(x,y) }

Δ^0_1 = { decidable sets }

$= \Sigma^0_1 \cap \Pi^0_1$

Where R is a decidable predicate

Theorem

Σ^0_2 = { sets semi-decidable in some semi-dec. B }
= languages of the form $\{ x \mid \exists y_1 \forall y_2\, R(x,y_1,y_2) \}$

Π^0_2 = { complements of Σ^0_2 sets}
= languages of the form $\{ x \mid \forall y_1 \exists y_2\, R(x,y_1,y_2) \}$

$\Delta^0_2 = \Sigma^0_2 \cap \Pi^0_2$

Where R is a decidable predicate

Example

Decidable predicate

Σ^0_1 **= languages of the form { x | ∃y R(x,y) }**

We know that A_{TM} is in Σ^0_1 Why?

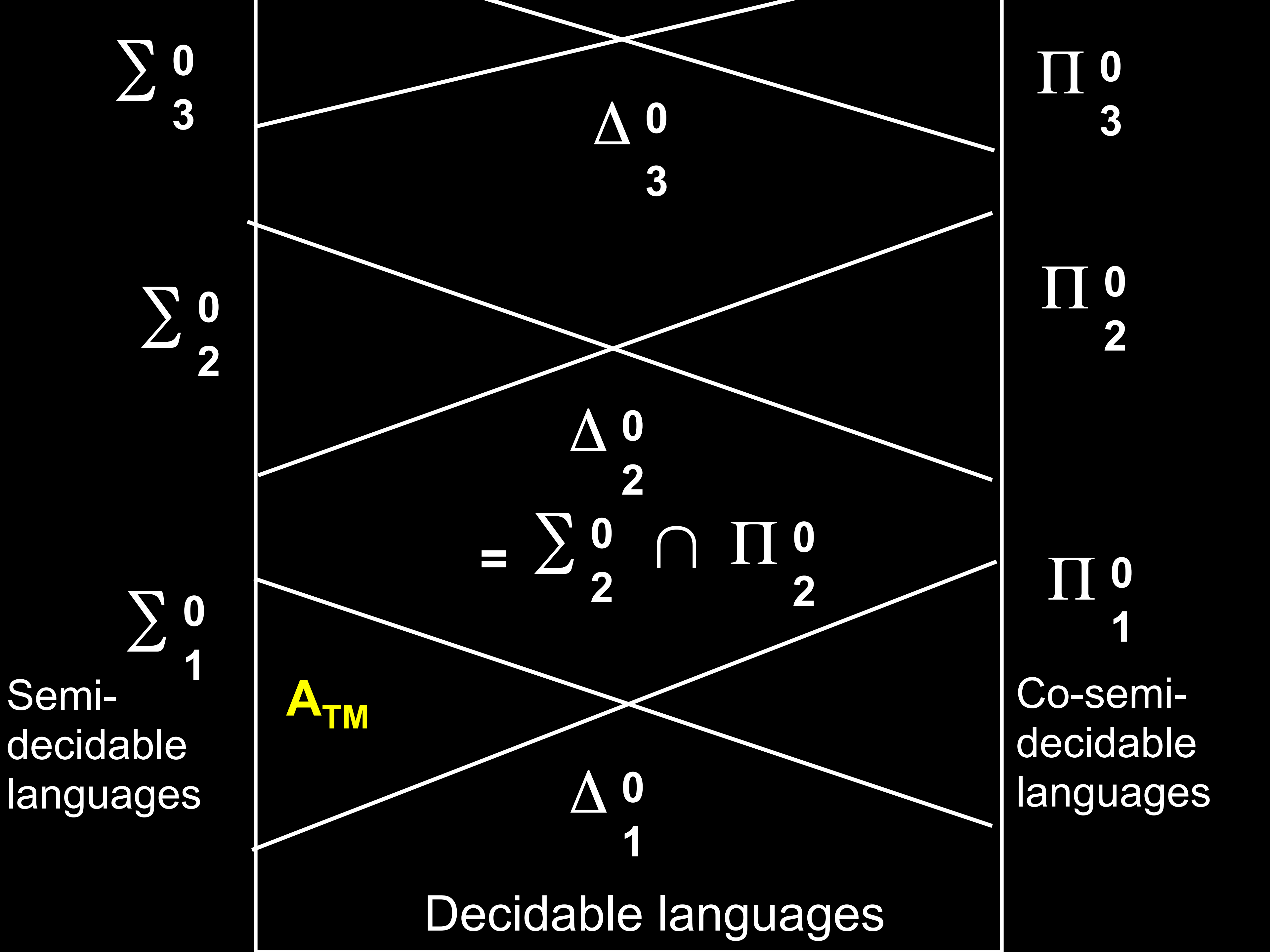
Show it can be described in this form:

A_{TM} = { <(M,w)> | ∃t [M accepts w in t steps] }

decidable predicate

A_{TM} = { <(M,w)> | ∃t T (<M>, w, t }

A_{TM} = { <(M,w)> | ∃v (v is an accepting computation history of M on w}


Σ^0_3
Π^0_3
Δ^0_3
Σ^0_2
Π^0_2
$\Delta^0_2 = \Sigma^0_2 \cap \Pi^0_2$
Σ^0_1
Π^0_1
Semi-decidable languages
A_{TM}
Co-semi-decidable languages
Δ^0_1
Decidable languages

Π^0_1 = languages of the form { x | $\forall$y R(x,y) }

Show that EMPTY (ie, E_{TM}) = { M | L(M) = $\varnothing$ } is in Π^0_1

EMPTY = { M | $\forall$w$\forall$t [M doesn't accept w in t steps] }

two quantifiers??

decidable predicate

Π^0_1 = languages of the form { x | ∀y R(x,y) }

Show that EMPTY (ie, E_{TM}) = { M | L(M) = ∅ } is in Π^0_1

EMPTY = { M | ∀w∀t [¬T(<M>, w, t)] }

two quantifiers??

decidable predicate

THE PAIRING FUNCTION

Theorem. There is a 1-1 and onto computable function $< , >: \Sigma^* \times \Sigma^* \rightarrow \Sigma^*$ and computable functions π_1 and $\pi_2 : \Sigma^* \rightarrow \Sigma^*$ such that

$$z = \langle w, t\rangle \Rightarrow \pi_1(z) = w \text{ and } \pi_2(z) = t$$

EMPTY = { M | $\forall w \forall t$ [M doesn't accept w in t steps] }

EMPTY = { M | $\forall z$ [M doesn't accept $\pi_1(z)$ in $\pi_2(z)$ steps]}

EMPTY = { M | $\forall z$ [$\neg T(\langle M\rangle, \pi_1(z), \pi_2(z))$] }

THE PAIRING FUNCTION

Theorem. There is a 1-1 and onto computable function $< , >: \Sigma^* \times \Sigma^* \to \Sigma^*$ and computable functions π_1 and $\pi_2 : \Sigma^* \to \Sigma^*$ such that

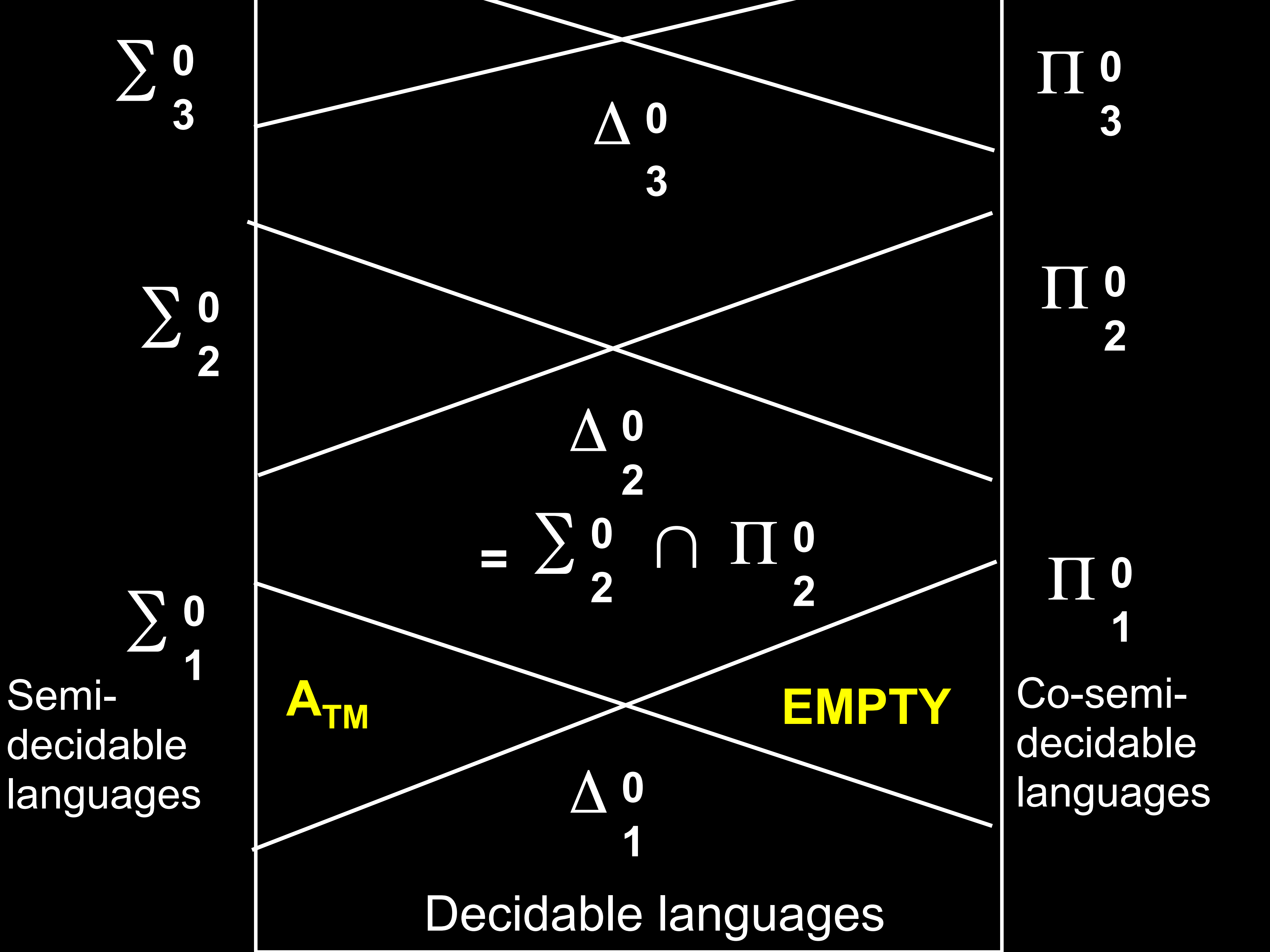
$$z = \langle w, t\rangle \Rightarrow \pi_1(z) = w \text{ and } \pi_2(z) = t$$

Proof: Let $w = w_1 \ldots w_n \in \Sigma^*$, $t \in \Sigma^*$.
Let $a, b \in \Sigma$, $a \neq b$.

$$\langle w, t\rangle := a\, w_1 \ldots a\, w_n\, b\, t$$

$\pi_1(z) :=$ "if z has the form $a\, w_1 \ldots a\, w_n\, b\, t$, then output $w_1 \ldots w_n$, else output ε"

$\pi_2(z) :=$ "if z has the form $a\, w_1 \ldots a\, w_n\, b\, t$, then output t, else output ε"


Σ^0_3
Δ^0_3
Π^0_3
Σ^0_2
Π^0_2
Δ^0_2
$= \Sigma^0_2 \cap \Pi^0_2$
Σ^0_1
Π^0_1
Semi-decidable languages
A_{TM}
EMPTY
Co-semi-decidable languages
Δ^0_1
Decidable languages

Π^0_2 = languages of the form { x | ∀y∃z R(x,y,z) }

Show that TOTAL = { M | M halts on all inputs } is in Π^0_2

TOTAL = { M | ∀w ∃t [M halts on w in t steps] }

decidable predicate

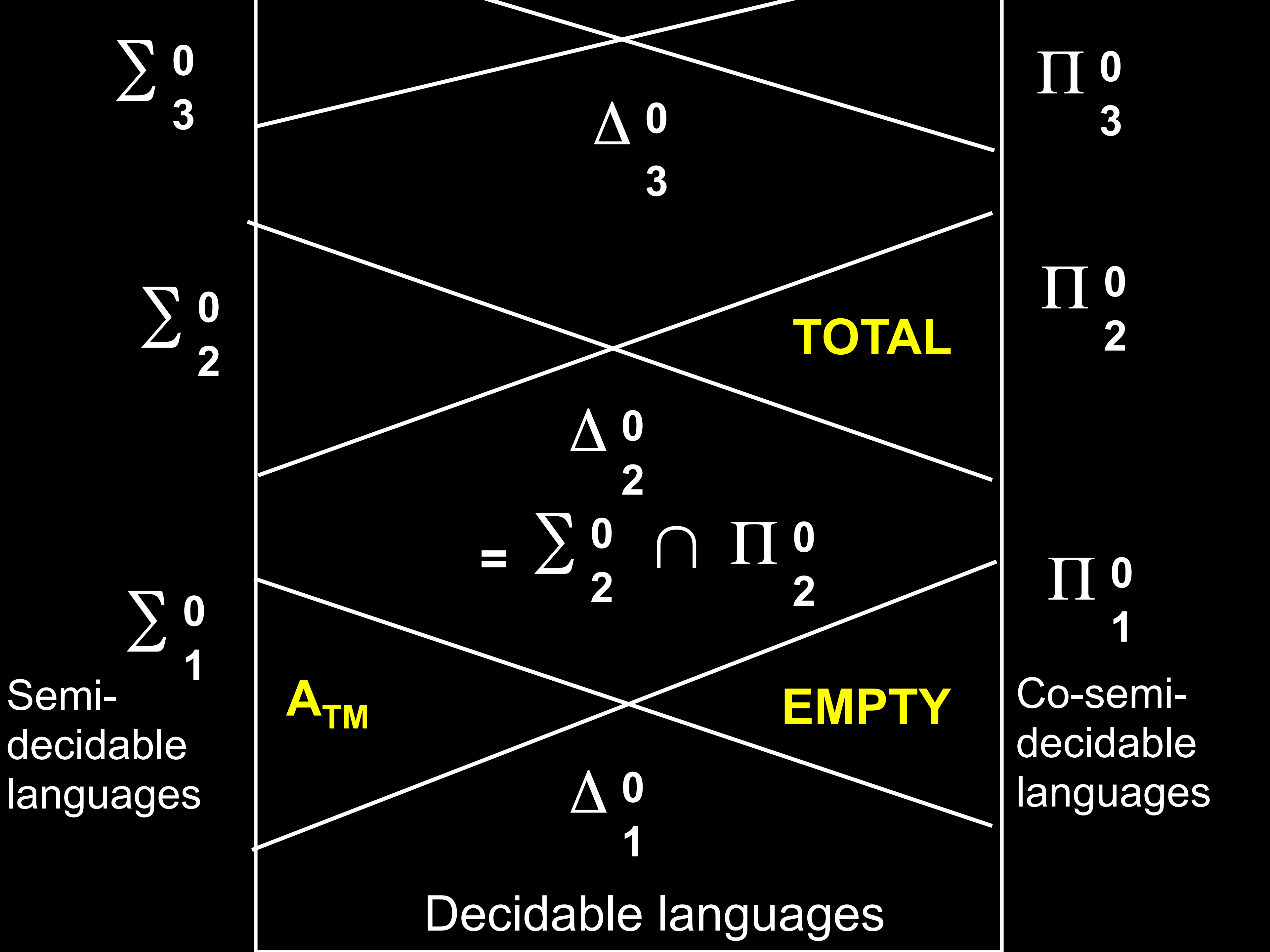
Π^0_2 = languages of the form { x | ∀y∃z R(x,y,z) }

Show that TOTAL = { M | M halts on all inputs }

is in Π^0_2

TOTAL = { M | ∀w ∃t [T(<M>, w, t)] }

decidable predicate


Σ^0_3
Δ^0_3
Π^0_3
Σ^0_2
TOTAL
Π^0_2
Δ^0_2
$= \Sigma^0_2 \cap \Pi^0_2$
Σ^0_1
Π^0_1
Semi-decidable languages
A_{TM}
EMPTY
Co-semi-decidable languages
Δ^0_1
Decidable languages

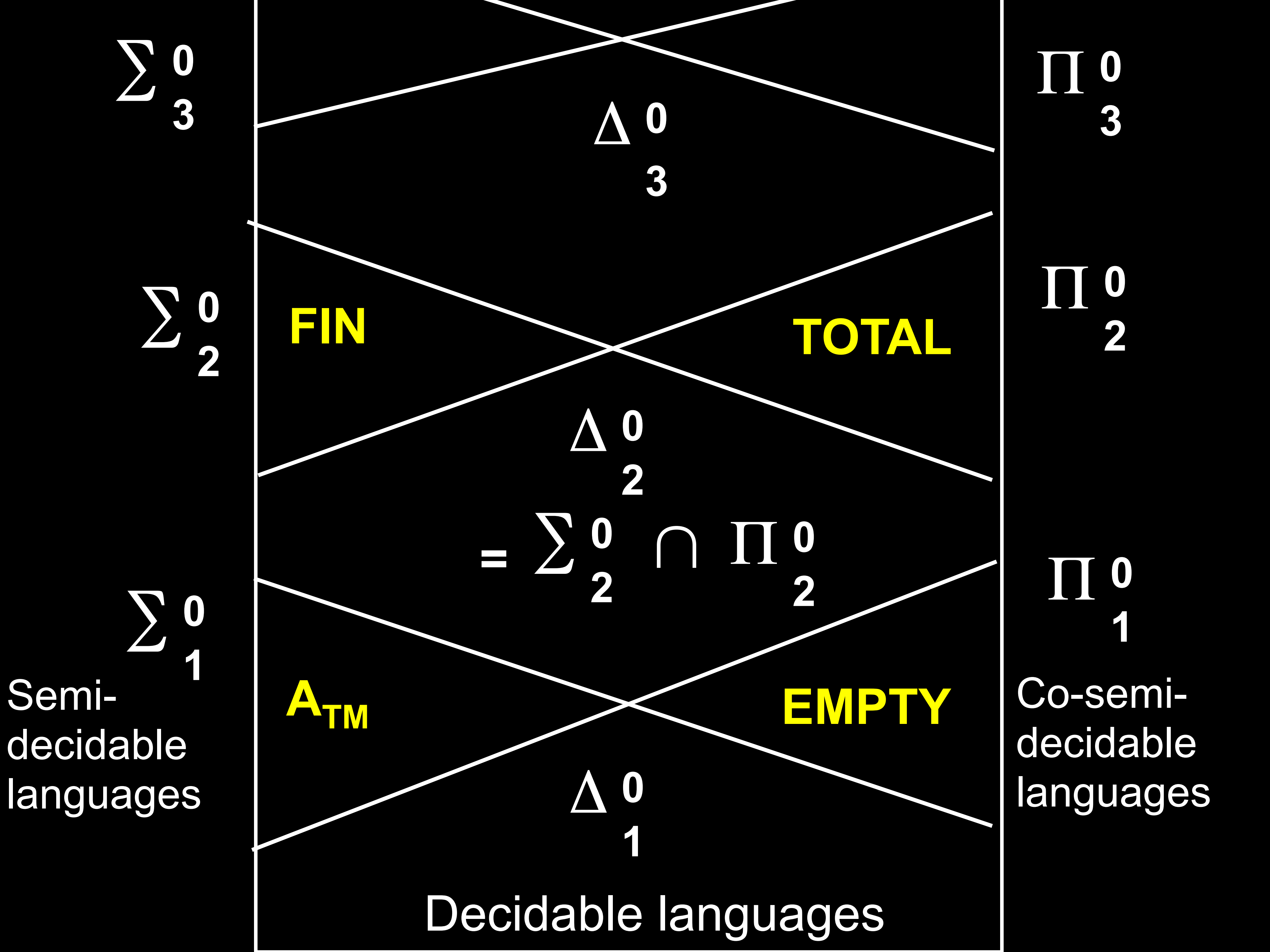
Σ^0_2 = languages of the form $\{ x \mid \exists y \forall z \, R(x,y,z) \}$

Show that FIN = { M | L(M) is finite } is in Σ^0_2

FIN = { M | ∃n∀w∀t [Either |w| < n, or
M doesn't accept w in t steps] }

$$FIN = \{ M \mid \exists n \forall w \forall t \, (|w| < n \vee \neg T(<M>, w, t)) \}$$

decidable predicate



Σ^0_3
Π^0_3
Δ^0_3
Σ^0_2
FIN
TOTAL
Π^0_2
Δ^0_2
$= \Sigma^0_2 \cap \Pi^0_2$
Σ^0_1
Π^0_1
Semi-decidable languages
A_{TM}
EMPTY
Co-semi-decidable languages
Δ^0_1
Decidable languages

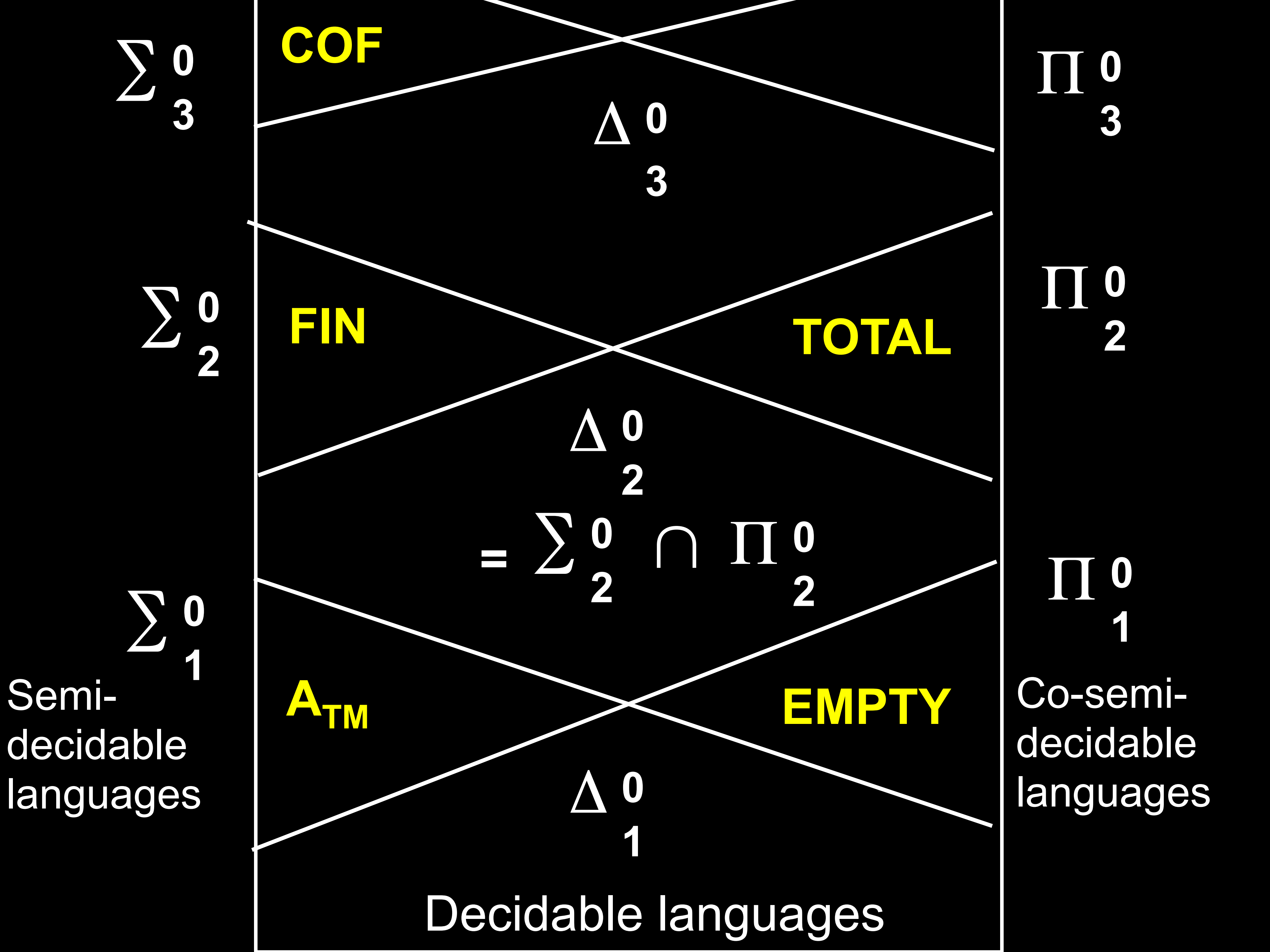
$\sum_{3}^{0}$ = languages of the form $\{ x \mid \exists y \forall z \exists u \; R(x,y,z,u) \}$

Show that COF = { M | L(M) is cofinite } is in $\sum_{2}^{0}$

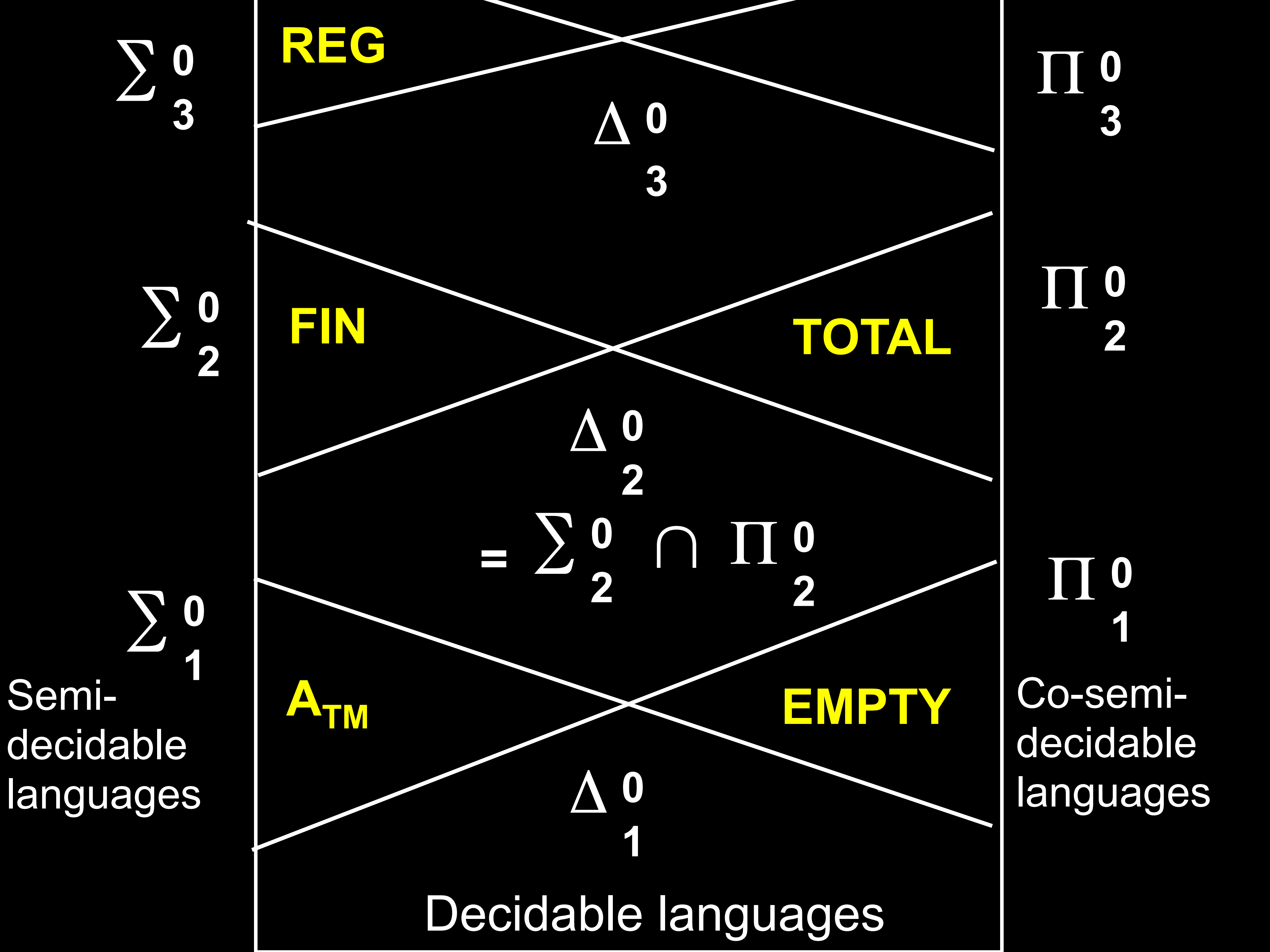
COF = $\{ M \mid \exists n \forall w \exists t \; [\, |w| > n \Rightarrow$ M accept w in t steps$] \}$

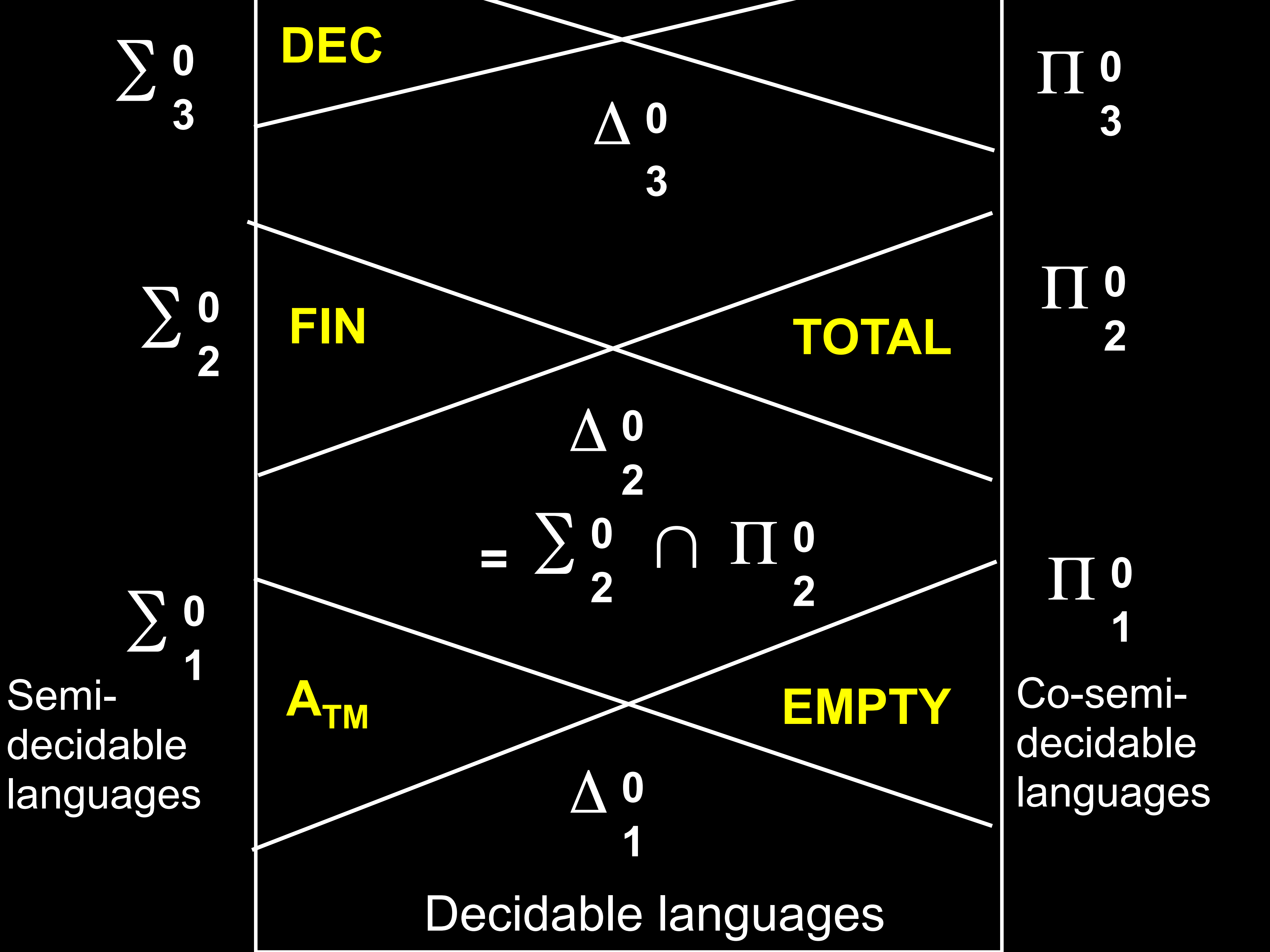
COF = $\{ M \mid \exists n \forall w \exists t \; (|w| \leq n \vee T(<M>, w, t)) \}$

decidable predicate

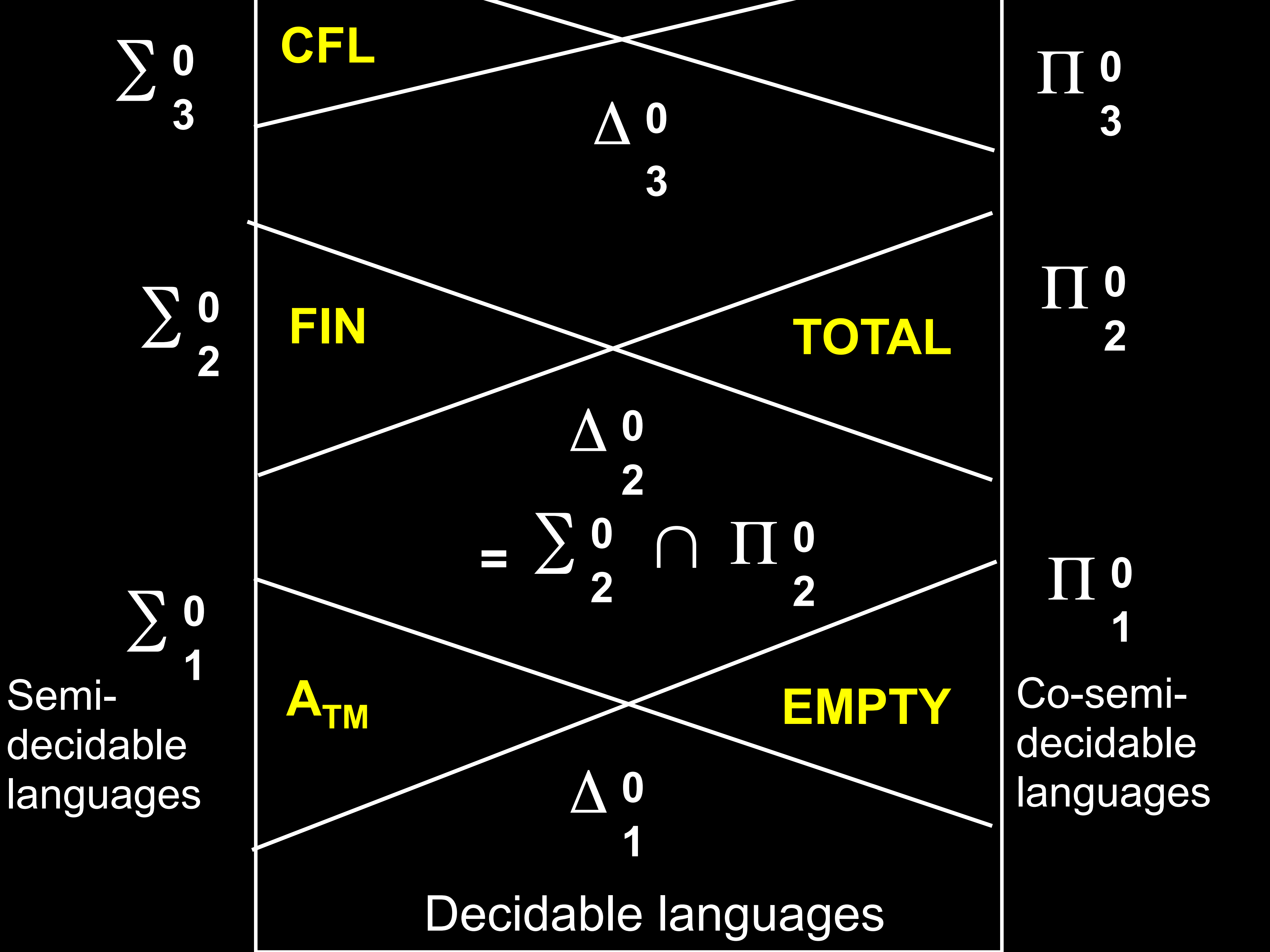


COF
Σ^0_3
Π^0_3
Δ^0_3
Σ^0_2
FIN
TOTAL
Π^0_2
Δ^0_2
$= \Sigma^0_2 \cap \Pi^0_2$
Σ^0_1
Π^0_1
Semi-decidable languages
A_{TM}
EMPTY
Co-semi-decidable languages
Δ^0_1
Decidable languages


Σ^0_3
REG
Π^0_3
Δ^0_3
Σ^0_2
FIN
TOTAL
Π^0_2
$\Delta^0_2 = \Sigma^0_2 \cap \Pi^0_2$
Σ^0_1
Π^0_1
Semi-decidable languages
A_{TM}
EMPTY
Co-semi-decidable languages
Δ^0_1
Decidable languages



DEC
Σ^0_3
Δ^0_3
Π^0_3
Σ^0_2
FIN
TOTAL
Π^0_2
Δ^0_2
$= \Sigma^0_2 \cap \Pi^0_2$
Σ^0_1
Π^0_1
Semi-decidable languages
A_{TM}
EMPTY
Co-semi-decidable languages
Δ^0_1
Decidable languages



Σ^0_3
CFL
Π^0_3
Δ^0_3
Σ^0_2
FIN
TOTAL
Π^0_2
Δ^0_2
$= \Sigma^0_2 \cap \Pi^0_2$
Σ^0_1
Π^0_1
Semi-decidable languages
A_{TM}
EMPTY
Co-semi-decidable languages
Δ^0_1
Decidable languages

Each is m-complete for its level in hierarchy and cannot go lower (by next Theorem, which shows the hierarchy does not collapse).

L is m-complete for class C if

i) L $\in$ C and

ii) L is m-hard for C,

ie, for all L' $\in$ C , L' $\leq_m$ L

A_{TM} is m-complete for class $C = \Sigma^0_1$

i) $A_{TM} \in C$

ii) A_{TM} is m-hard for C,

Suppose $L \in C$. Show: $L \leq_m A_{TM}$

Let M semi-decide L. Then Map

$\Sigma^* \to \Sigma^*$

where $w \to (M, w)$.

Then, $w \in L \Leftrightarrow (M,w) \in A_{TM}$ QED

FIN is **m-complete** for class $C = \sum_2^0$

i) FIN $\in$ C
ii) FIN is m-hard for C,

Suppose L $\in$ C . Show: L $\leq_m$ FIN

Suppose L= { w | $\exists y \forall z$ R(w,y,z) }
where R is decided by some TM D

Map $\Sigma^* \rightarrow \Sigma^*$
where w $\rightarrow$ $N_{D,w}$

Supose $L \in \Sigma_2^0$ ie $L = \{ w \mid \exists y \forall z\, R(w,y,z) \}$
where R is decided by some TM D

Show: $L \le_m FIN$

Map $\Sigma^* \rightarrow \Sigma^*$
where $w \rightarrow N_{D,w}$

Define $N_{D,w}$ On input s:

1. Write down all strings y of length |s|
2. For each y, try to find a z such that $\neg R(w, y, z)$ and accept if all are successful (here use D and w)

So, $w \in L \Leftrightarrow N_{D,w} \in FIN$

ORACLES not all powerful

The following problem cannot be decided, **even by a TM with an oracle for the Halting Problem:**

SUPERHALT = { (M,x) | M, with an oracle for the Halting Problem, halts on x}

Can use diagonalization here!

Suppose H decides SUPERHALT (with oracle)

Define **D(X) = "if H(X,X) accepts (with oracle) then LOOP, else ACCEPT."**

D(D) halts ⇔ H(D,D) accepts ⇔ D(D) loops…

ORACLES not all powerful

Theorem: The arithmetic hierarchy is strict. That is, the nth level contains a language that isn't in any of the levels below n.

Proof IDEA: Same idea as the previous slide.

$SUPERHALT^0$ = HALT = { (M,x) | M halts on x}.

$SUPERHALT^1$ = { (M,x) | M, with an oracle for the Halting Problem, halts on x}

$SUPERHALT^n$ = { (M,x) | M, with an oracle for $SUPERHALT^{n-1}$, halts on x}

WWW.FLAC.WS

Read Chapter 6.4 for next time