15110 Summer 2018
Problem Set 5

Name: _____

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Instructions

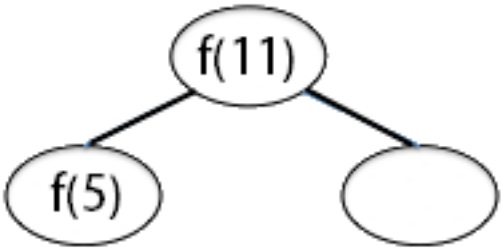
- Type or neatly write the answers to the following problems.
- Save or scan this file as a pdf and submit to Gradescope

Exercises

1. [2 points] You are given the following recursive function in Python:

```
def f(n):
    if n % 3 == 0 or n < 3:
        return n * (n - 1)
    else:
        return f(n // 2) + f(n - 2)
```

Trace the computation of `f(11)` by drawing a recursion tree (like the one for Fibonacci numbers on the slides for Lecture 9), showing **all** of the recursive calls that need to be computed to find the value for `f(11)`. Then using your recursion tree, compute the value of `f(11)`. We've started the tree for you below:



2. [2 points] This question is about simple recursions on lists. We saw in class that a standard way of splitting a list `items` is into its "head", using `items[0]`, and its "tail", using `items[1:]`
    a. In general, if `a` is a list of numbers, what does `g(a)` return?

```
def g(a):
    if a == []:
        return 0
    else:
        return a[0] + g(a[1:])
```

    b. We can split a list up in other ways, too. In Python, you can use negative index values when accessing list elements. These count backwards from the end of a list with -1 being an index for the last element in an array. The range notation for "slicing" lists can also use negative elements or a mix of positive and negative elements. For example:

```
>>> a = ['v', 'w', 'x', 'y', 'z']
>>> a[-1]
'z'
>>> a[-2]
'y'
>>> a[-3]
'x'
>>> a[-3:]
['x', 'y', 'z']
>>> a[-3,-1]
['x','y']
>>> a[:-1]
['v','w', 'x','y']
```

    Consider the following Python function. Given a list of numbers `a`, what does `h(a)` return? You may find it useful to trace what it does for a particular input.

```
def h(a):
    if  a == []:
        return 0
    else:
        return a[-1] + h(a[:-1])
```

3. For this question you will work with a recursive implementation of the binary search algorithm.
    a. [2 points] Do you believe that the recursive binary search algorithm we saw in class works in all cases? What about "edge cases", such as a key that precedes the first element in the list or follows the last element? What if the integer division `(lower + upper) // 2` causes problems when the list has an even number of elements? For this problem you will investigate this question. To help you, here is the Python code from lecture:

```
# main function
def bsearch(items, key):
    return bs_helper(items, key, -1, len(items))

# recursive helper function
def bs_helper(items, key, lower, upper):
    if lower + 1 == upper: # Base case: empty
        return None
    mid = (lower + upper) // 2 # Recursive case
    if key == items[mid]:
        return mid
    if key < items[mid]:          # Go left
        return bs_helper(items, key, lower, mid)
    else:                                  # Go right
        return bs_helper(items, key, mid, upper)
```

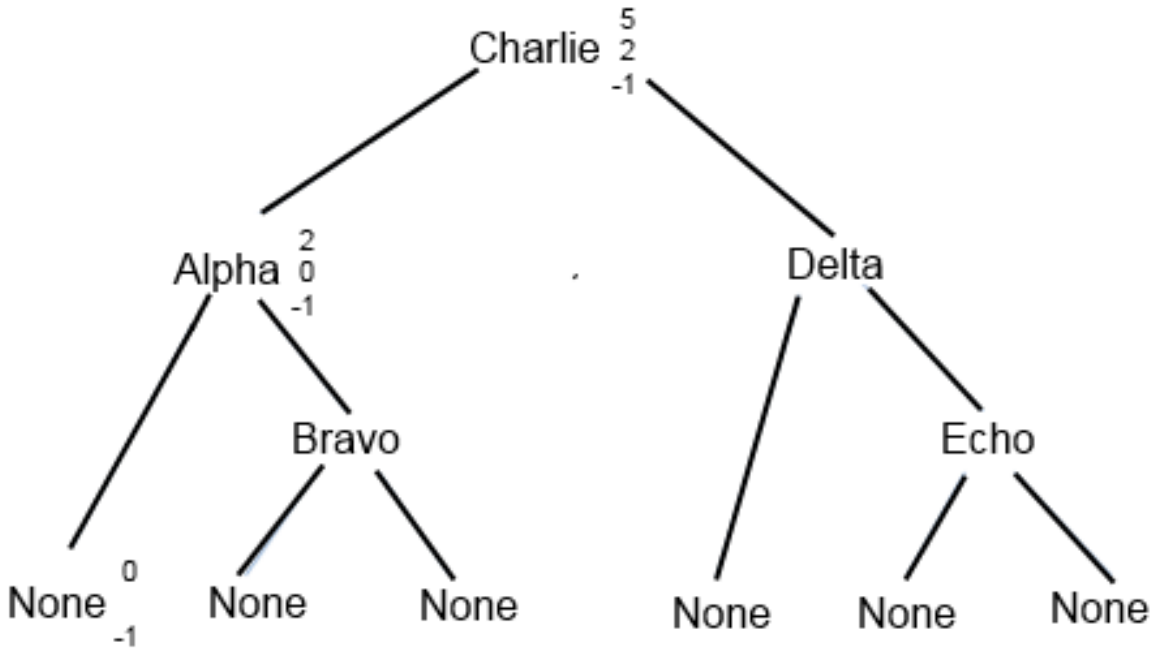
One way to convince yourself that the algorithm works is to show that every possible position in the ordered list, including both the values themselves (when the key is found) and the positions between values (when the key is not found and the binary search should return `None`), can be derived by following the algorithm for updating mid, lower, and upper. To do this, we're going to construct a complete binary search tree. First we introduce some notation for depicting nodes in the search tree. Each node corresponds either to a value in the list being searched, or to a base case where the search would fail. The **X** on the left below illustrates the first case, and the **None** on the right below illustrates the second case:

The above nodes are generated by starting with the `lower` and `upper` values. If `lower+1` is less than `upper`, then we compute `mid`, and compare the key against that element of the list, e.g., the value **X** as shown above. If they match we return `mid`, otherwise we take either the left or right branch. But if `lower+1` is equal to instead of less than `upper`, then we stop and return `None`, as in the node on the right.

We have started the binary search tree diagram below. You will complete it by filling in the numbers as appropriate for each node, showing that every node can be reached by the search function, and proving that if a key is in the list, the algorithm will find it.

In the diagram below, we've filled in the numeric values for three of the nodes for the case where the key is "Aardvark". The binary search always begins with `lower` = -1 and `upper` = 5, giving `mid` = 2, which puts us at "Charlie". So all searches of this list start at "Charlie". If the key is less than this value, the algorithm tells us to set `upper` equal to `mid`, so we have `upper` = 2 and `lower` = -1, which gives `mid` = 0, taking us to node "Alpha", as shown. If we go left from "Alpha" (because the key is less than "Alpha"), the next node has `upper` = 0 and `lower` = -1. Since now `lower+1` = `upper`, the algorithm returns `None`, making this node a terminal (leaf) node. Complet the tree by following the algorithm to fill in all the missing values, so that every node has either two or three numbers written next to it. For example, you might choose a key of "Bravo" and see how the algorithm gets to that node. Then choose a key of "Banana" and see at which leaf node the algorithm ends. And so on. Include your complete annotated tree in the pages you hand in.

```
items = ["Alpha", "Bravo", "Charlie", "Delta", "Echo"]
```



b. [1 point] Suppose that you have a "budget" of 5 key comparisons for searching for a key in a sorted list. That is, your search would terminate after comparing the key to *at most* 5 list elements. With this budget, what is the size of the largest sorted list that you could search using binary search? (Note: "key comparison" refers only to comparisons involving the key being searched for.)

4. [1 point] This question concerns Mergesort.

   Show how the merge step would behave by completing the trace below. Add a line to the trace for each element that is placed in the output, crossing the element off the input as shown.

First half	Second half	Output list
4, 18, 20, 55, 87, 90	1, 25, 27, 44, 88	
4, 18, 20, 55, 87, 90	1, 25, 27, 44, 88	1

5. [2 points] Quicksort is another algorithm for sorting data; there are many minor variations of it. We'll consider a particular version of Quicksort where we call the first element of the list the *pivot*. We compare all of the other elements in the list to the pivot; all of those that are less than the pivot go into one list and all of those that are greater than or equal to the pivot go into a second list. Then we sort these two sublists (recursively). The final sorted list is the first sorted sublist followed by the pivot followed by the second sorted sublist.

   For example, if we want to sort the list

```
list = [56, 42, 82, 75, 18, 58, 27, 61, 84, 41, 21, 15, 71, 90, 33]
```

   we create two lists, separating the elements based on the pivot 56:

```
list1 = [42, 18, 27, 41, 21, 15, 33]
list2 = [82, 75, 58, 61, 84, 71, 90]
```

   Each sublist is sorted recursively (final results shown for each sublist):

```
list1 = [15, 18, 21, 27, 33, 41, 42]
list2 = [58, 61, 71, 75, 82, 84, 90]
```

   and the final result is

```
[15, 18, 21, 27, 33, 41, 42] + [56] + [58, 61, 71, 75, 82, 84, 90]
=> [15, 18, 21, 27, 33, 41, 42, 56, 58, 61, 71, 75, 82, 84, 90]
```

a. Suppose we apply Quicksort to the list `[47, 55, 50, 20, 18, 47, 17, 25, 17]`. Show the two (unsorted) sublists that result from splitting based on the pivot.

b. For the best performance of Quicksort, would we rather have the two sublists of equal length, or would we rather have one be very short and the other very long? Explain briefly.