Data Representation and Compression

Exam 1

Announcements

- The first lab exam is Monday during the lab session
  - Sample exam on web site
  - Practice problems (with soln) are on the web site
  - You can use your own laptop!
- PS6 due now!
- PA6 and OLI Data representation over the weekend

Review from last time (before exam)

Lingering questions…

- Data Structures

- Arrays
- Linked Lists
- Hash Tables
- Associative Arrays

Key Point:

- Data needs to be stored in physical memory
- How we organize data in memory has consequences
- In this class, you are not implementing data structures – you are taking advantage of python's implementations…
- ...but you still need to understand (and make decisions) about these data structures.

Recall Arrays and Linked Lists

	Advantages	**Disadvantages**
Arrays	Constant-time lookup (search) if you know the index	Requires a contiguous block of memory
Linked Lists	Flexible memory usage	Linear-time lookup (search)

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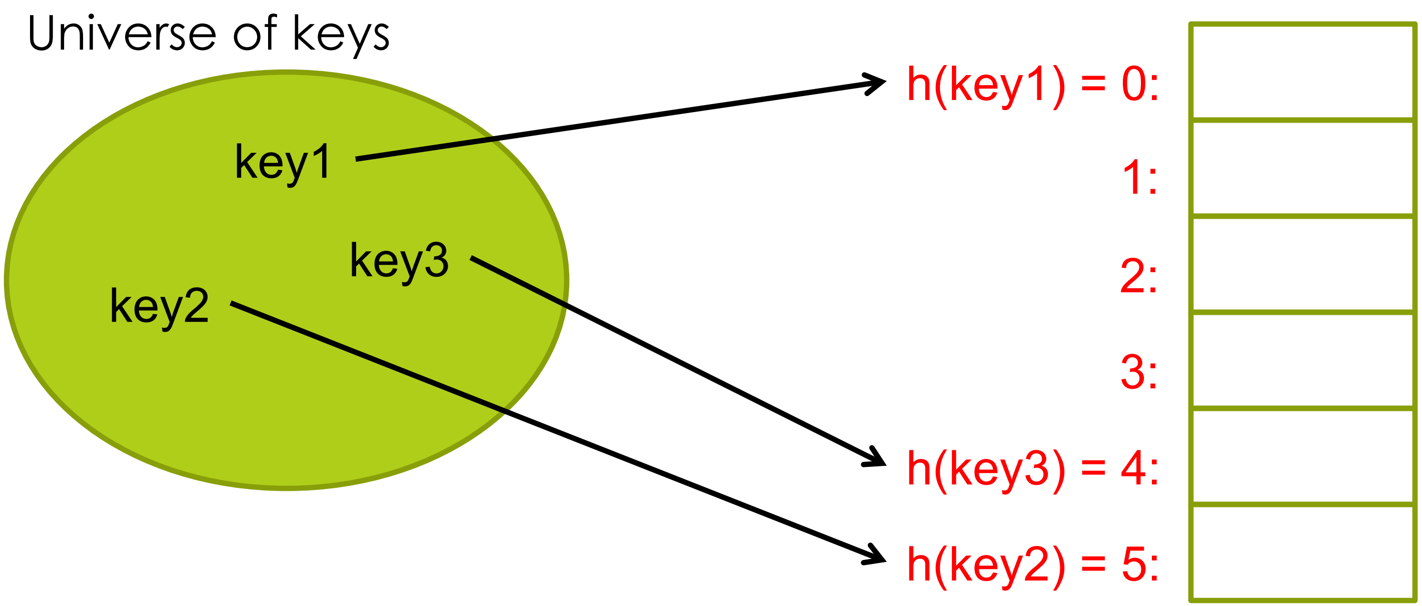
Hashing tables are one approach to exploit the advantages of arrays and linked lists

Hashing

- A "hash function" `h(key)` that maps a key to an array index in 0..*k*-1.
- To search the array `table` for that key, look in `table[h(key)]`



A hash function `h` is used to map keys to hash-table (array) slots. Table is an array bounded in size. The size of the universe for keys may be larger than the array size. We call the table slots buckets.

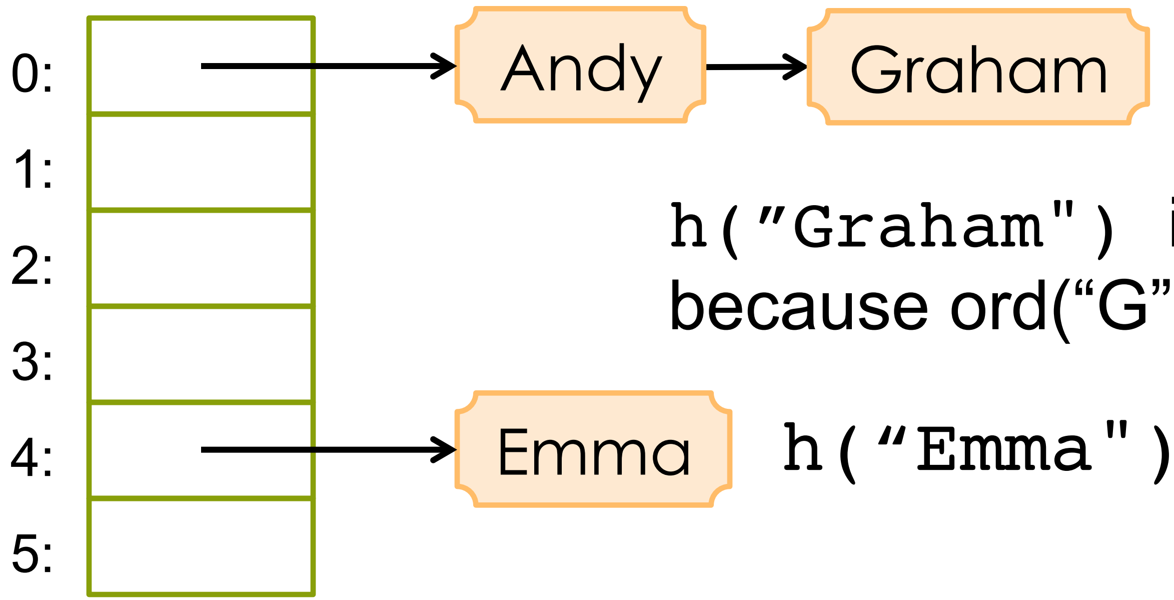
Example: Hash function

- Suppose we have (key,value) pairs where the key is a string such as (name, phone number) pairs and we want to store these key value pairs in an array.
- We could pick the array position where each string is stored based on the first letter of the string using this hash function:

```
def h(str):
   return (ord(str[0]) - 65) % 6
```

Note ord('A') = 65

Add Element “Graham”



`h(”Graham")` is also 0 because ord(“G”) is 71.

`h(“Emma")` is 4

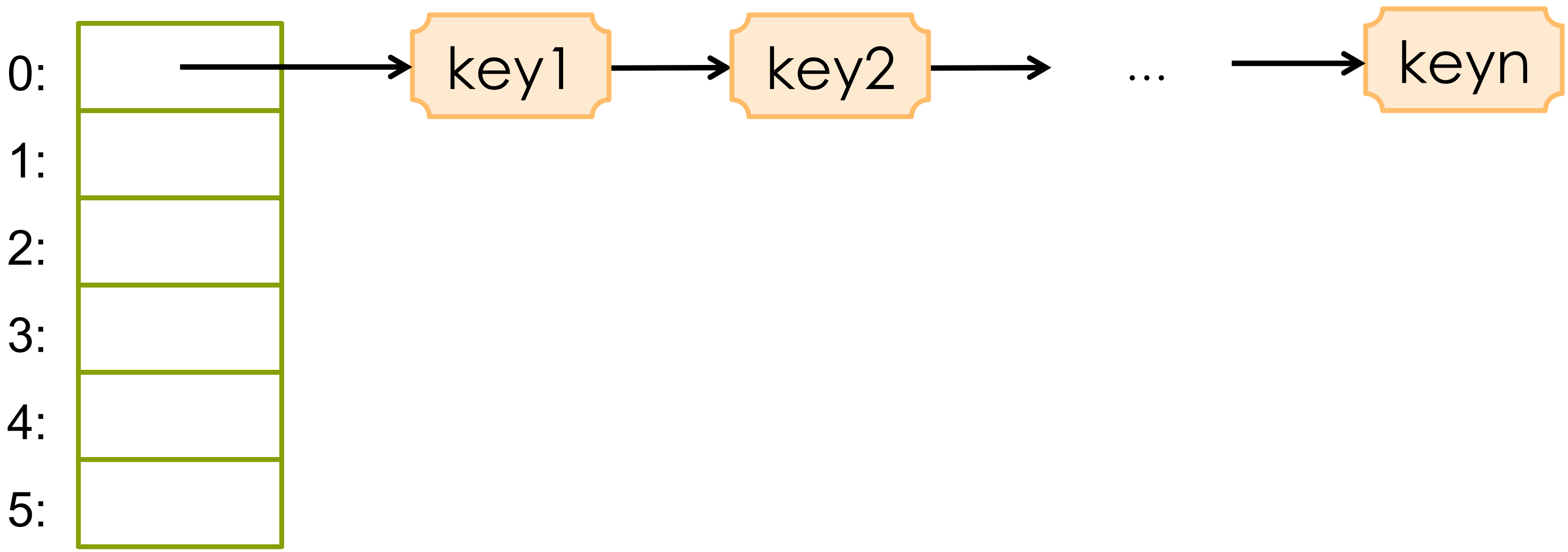
In order to add Graham’s information to the table we had to form a link list for bucket 0.

Requirements for the Hash Function h(x)

- Must be fast: O(1)
- Must distribute items roughly uniformly throughout the array, so everything doesn't end up in the same bucket.

Worst case for Hash Function h(x)

- Worst case: O(n)
- Every key maps to the same slot/bucket



What's A Good Hash Function?

- For strings:
  - Treat the characters in the string like digits in a base-256 number.
  - Divide this quantity by the number of buckets, *k*.
  - Take the remainder, which will be an integer in the range 0..*k*-1.

Summary of Search Techniques

Technique	Setup Cost	Search Cost
Linear search	0, since we're given the list	O(n)
Binary search	O(n log n) to sort the list	O(log n)
Hash table	O(n) to fill the buckets	O(1)

Associative Arrays

- Hashing is a method for implementing associative arrays.
- Some languages such as Python have associate arrays (**mapping** between keys and values) as a built-in data type.
- Examples:
  - Name in contacts list => Phone number
  - User name => Password
  - Product => Price

Dictionary Type in Python

This example maps car brands (*keys*) to prices (*values*).

```
>>> cars = {"Mercedes": 55000,
            "Bentley": 120000,
            "BMW":90000}
>>> cars["Mercedes"]
55000
```

Keys can be of any **immutable** data type.

Dictionaries are implemented using hashing.

Iteration over a Dictionary

```
>>> for i in cars:
        print(i)

BMW
Mercedes
Bentley

>>> for i in cars.items():
        print(i)

("BMW", 90000)
("Mercedes", 55000)

("Bentley", 120000)
```

Think what the loop variables are bound to in each case.

Note also that there is no notion of ordering in dictionaries. There is no such thing as the first element, second element of a dictionary.

```
>>> for k,v in cars.items():
        print(k, ":", v )

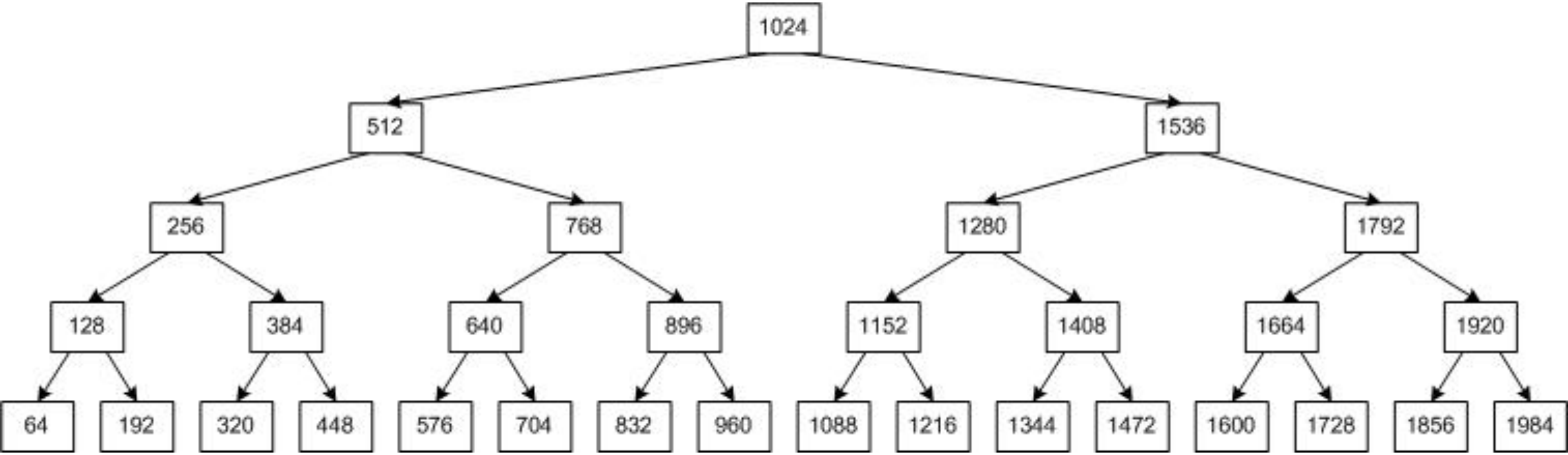
BMW : 90000
Mercedes 55000
Bentley : 120000
```

Some Dictionary Operations

- `d[key] = value` -- Set `d[key]` to `value`.
- `del d[key]` -- Remove `d[key]` from `d`. Raises a an error if `key` is not in the map.
- `key in d` -- Return True if `d` has a key `key`, else False.
- `items()` -- Return a new view of the dictionary's items ((key, value) pairs).
- `keys()` -- Return a new view of the dictionary's keys.
- `pop(key[, default])` If `key` is in the dictionary, remove it and return its value, else return `default`. If `default` is not given and `key` is not in the dictionary, an error is raised.

Source: https://docs.python.org/

Left – Node - Right



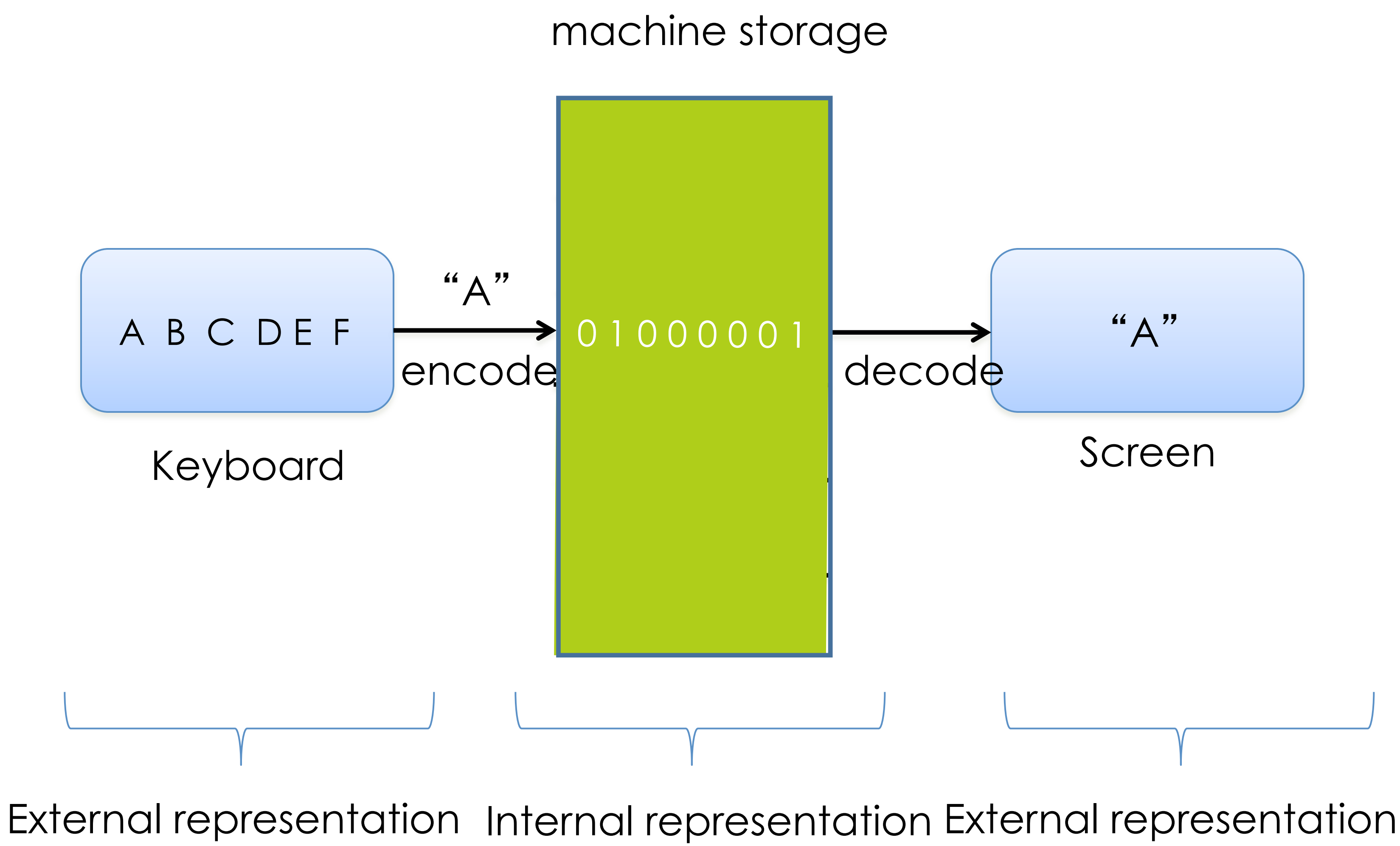
Representation

- We use computers to model i.e. *represent*, things in the real world:
  - Numbers, pictures, music, climate, markets…
- Three topics:
  - Representing numbers
  - Exploiting redundancy in representation (compression)
  - Representing images and sound

First, what do we mean by

Representation?

Representing Data

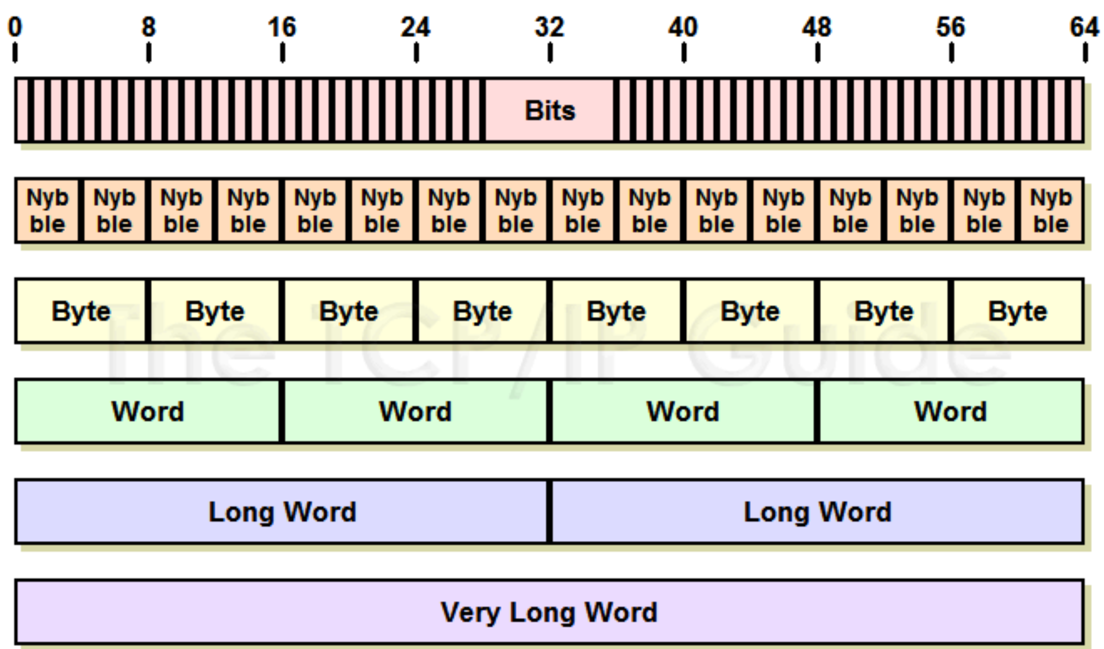


Digital Data

- 10010111110101010101110100010010
- Inside the digital machine it's all just
  - **binary** physical states (high or low voltages, etc.)
  - which we **interpret** as bits (1s and 0s)
- In turn we interpret these bits as representing data such as integers, real numbers, text, …

Machine storage

- is finite
- is divided into fixed-size chunks of bits
  - (smallest unit) bytes, usually 8 bits
  - (biggest chunk) words, usually 16, 32 or 64 bits
- machine storage capacity usually expressed as number of bytes or words
  - loosely speaking: “memory size”



Types interpret bits

- a 32-bit "word" might be
  1100 1100 1011 0111 0000 0000 0000 0000
- what this means depends on the machinery to interpret it

Type	Interpretation
“Raw” bits	1100 1100 1011 0111 0000 0000 0000 0000
Floating point number	6.59339 X 10^{-41}
String (Unicode UTF-16)	췋
RGB pixel color	
Little-endian integer	47052

Fundamental Issue: Information Capacity

# bits	Possible values								# possible values
1	0	1							2
2	00	01	10	11					4
3	000	001	010	011	100	101	110	111	8
4	0000 1000	0001 1001	0010 1010	0011 1011	0100 1100	0101 1101	0110 1110	0111 1111	16

$2^1 = 2, \ 2^2 = 4, \ 2^3 = 8, \ 2^4 = 16$

Hmmm…could it be?

Yes, k bits can represent 2^k different values.

numerals are not numbers!

don’t be drawn like moths to the flame of meaning*:

*Geoffrey Pullum

Numbers: semantics (quantities) versus syntax (numerals)

	Semantics	Syntax
What is it?	Our idea of quantity	How we write our idea of quantity
What is it good for?	Insight	Calculation, communication, computation
Example	 	II (Roman numeral) 2 (decimal Arabic numeral) 10 (binary numeral) – all with the same semantics!

machines don’t have ideas!

only syntax!

Numerals aren’t numbers, but

- …to communicate a number (quantity), I have to write *something*

- I will write numbers (quantities)
  - as ordinary base-10 numerals
  - (sometimes as words)

Non-negative integers

representing non-negative integers (0, 1, 2, 3, …)

Place-value numerals (base 10)

- The *numeral* we write: 15627
- What it means:
  $7 \times 10^0 + 2 \times 10^1 + 6 \times 10^2 + 5 \times 10^3 + 1 \times 10^4$

- **Problem**: electronic circuitry for base-10 arithmetic is slow.
- **Solution**: use place-value numerals, but in base 2–*binary notation*

Place-value numerals in general

- Choose a number *b* for the **base** or **radix**
- Choose list of **digits**, there must be *b* of them
  - **base 10 example: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9**
  - **base 2 example: 0, 1**
  - **base 16 example: 0, 1, …, 9, A, B, C, D, E, F**

Algorithm

- To represent a quantity *n* in base *b*
  - integer divide *n* by *b* with remainder *r* (a **digit**)
  - repeat until the quotient is zero
  - the remainders are the digits in reverse order

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n=2019 b=10

2019//10, r=9

201//10, r=1

20//10, r=0

2//10, r=2

Binary place-value example

- Base two, digits 0 and 1
- n=6 and b=2

- To represent "six":
  - 6 // 2 = 3, r=0

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remainder when dividing by 2 can only be 0 or 1

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  - 3 // 2 = 1, r=1
  - 1//2 = 0,   r=1

Binary place-value example

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- n=6 and b=2
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Read the remainders from bottom to top to get bits from left to right

Binary numeral: 110

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Binary numeral: 110

What it means:

$0 \times 2^0 + 1 \times 2^1 + 1 \times 2^2 =$ "six"

Numeral place-value example

- To represent n=6 with b=2:
  - 6 // 2 = 3, **r=0**
  - 3 // 2 = 1, **r=1**
  - 1//2 = 0, **r=1**

Binary numeral: 110

What it means:

$0 \times 2^0 + 1 \times 2^1 + 1 \times 2^2 =$ "six"

- To represent n=2019 with b=10
  - 2019//10, r=9
  - 201//10, r=1
  - 20//10, r=0
  - 2//10, r=2

Decimal numeral: 2019

What it means:

$9 \times 10^0 + 1 \times 10^1 + 0 \times 10^2 + 2 \times 10^3 =$ "twothousand and nineteen"

Algorithm

- To represent a quantity *n* in base *b*
  - integer divide *n* by *b* with remainder *r* (a **digit**)
  - repeat until the quotient is zero
  - the remainders are the digits in reverse order

Information Capacity and Range

- Remember: ***k* bits can represent 2^k different things**
- So *k*-bit binary numerals represent $0 \ldots 2^k-1$
  - For $k = 3$,

000	001	010	011	100	101	110	111
0	1	2	3	4	5	6	7

Ranges for typical computer "word" sizes

bits	minimum	maximum	
8	0	$2^8 - 1$	(255)
16	0	$2^{16} - 1$	(65,535)
32	0	$2^{32} - 1$	(4,294,967,295)
64	0	$2^{64} - 1$	(18,446,744,073,709,551,615)

binary arithmetic

some familiar operations

Counting in binary

Binary numerals

- 0
- 1
- 10
- 11
- 100
- 101
- 110
- 111
- 1000
- 1001
- 1010
- 1011

Decimal equivalents

- 0
- 1
- 2
- 3
- 4
- 5
- 6
- 7
- 8
- 9
- 10
- 11

Counting in binary

Binary numerals

- **0** → $0*2^0$
- **1** → $1*2^0$
- **10** → $0*2^0 + 1*2^2$
- **11** → $1*2^0 + 1*2^2$
- 100
- 101
- 110
- 111
- 1000
- 1001
- 1010
- 1011

Decimal equivalents

- 0
- 1
- 2
- 3
- 4
- 5
- 6
- 7
- 8
- 9
- 10
- 11

Addition and Multiplication Tables

+	0	1
0	0	1
1	1	10

(which is 0 carry 1)

×	0	1
0	0	0
1	0	1

Binary Arithmetic

+	0	1
0	0	1
1	1	10

- All the familiar methods work, but with only 1 and 0 for digits
- 1 + 1 = 10, 10 - 1 = 1, 10 + 1 = 11, ...
- Example:

```
 1   1
  1010
 +1010
 -----
 10100
```

Notice: we need more bits for the answer than we did for the operands.

Overflow: the first difficulty

- Machine word only has *k* bits for some **fixed** *k*!
- If *k* is 4, then we have **overflow** in the following:

```
 1  1
  1010
 +1010
 -----
 10100
```

- The machine retains only 0100 (the "least significant" bits), so $(n+n) - n$ **not** always equal to $n + (n - n)$

Modular Arithmetic

- Dropping the overflow bit is **modular arithmetic**
- We can carry out any arithmetic operation modulo 2^k for the precision *k*. The example again for precision 4:

binary	**decimal**
1 0 1 0	= 10
+ 1 0 1 0	= 10
(1) 0 1 0 0	= 20 = 4 (20 mod 16)

overflow can be ignored or signaled as an error

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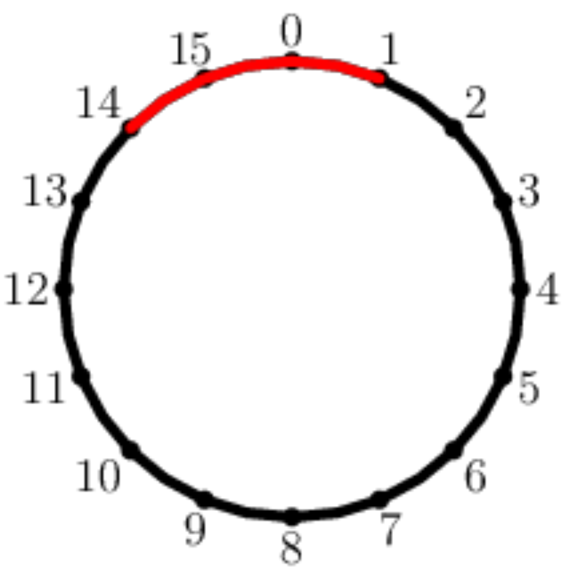

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negative integers

representing all the integers…

Different ways of representing negative numbers

Decimal	Signed Magnitude	Signed One's Complement	Signed Two's Complement
+7	0111	0111	0111
+6	0110	0110	0110
+5	0101	0101	0101
+4	0100	0100	0100
+3	0011	0011	0011
+2	0010	0010	0010
+1	0001	0001	0001
+0	0000	0000	0000
-0	1000	1111	–
-1	1001	1110	1111
-2	1010	1101	1110
-3	1011	1100	1101
-4	1100	1011	1100
-5	1101	1010	1011
-6	1110	1001	1010
-7	1111	1000	1001

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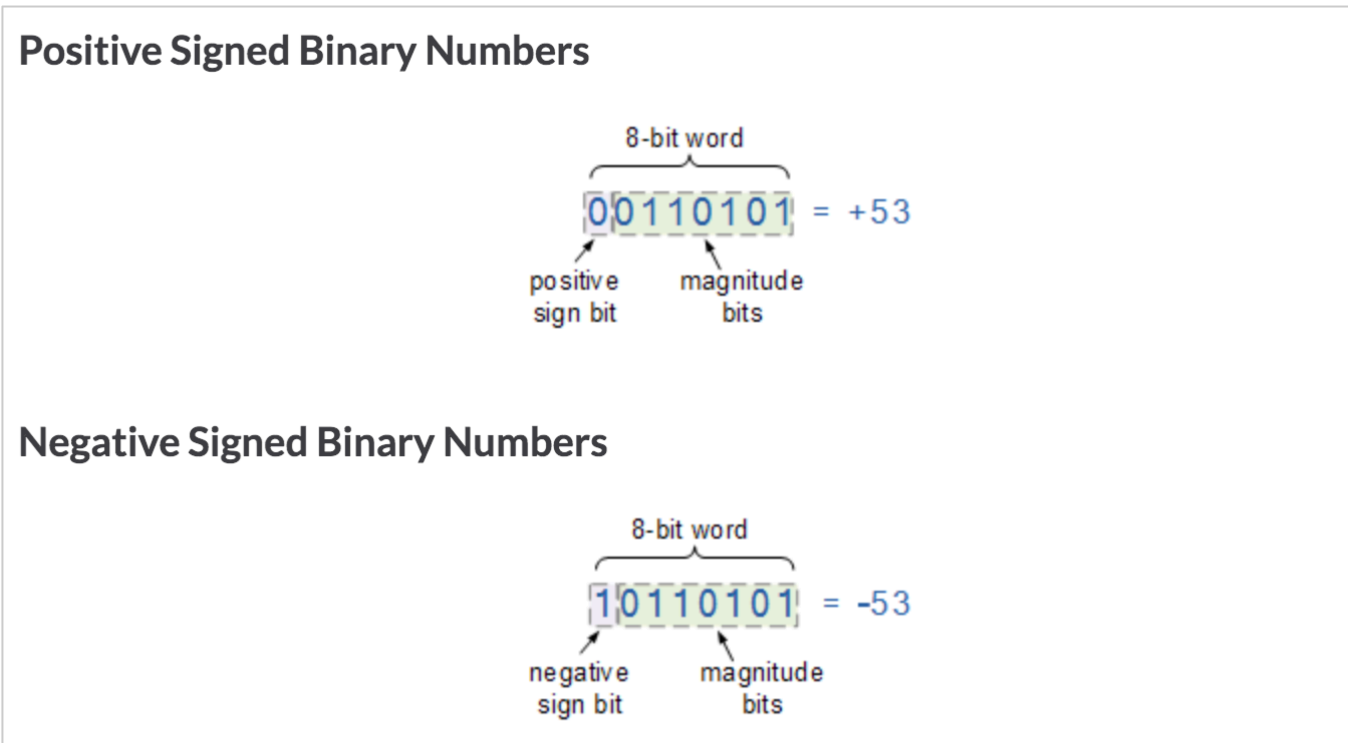
+1	0001	0001	0001
+0	0000	0000	0000
-0	1000	1111	–
-1	1001	1110	1111
-2	1010	1101	1110
-3	1011	1100	1101
-4	1100	1011	1100
-5	1101	1010	1011
-6	1110	1001	1010
-7	1111	1000	1001

One way of Representing a sign +/-

- A natural idea: reserve one of the bits to stand for a sign.
- E.g., 0 could stand for + and 1 could stand for –
  - unsigned "ten" is 1010
  - so "negative ten" would be 11010



Another way of
Representing a sign +/-

- Two's (2's) complement
- Someone had a cleverer idea…
  - first, we'd like to avoid "two zeroes": +0 and -0
  - second, we'd like the same machinery to work for addition and subtraction

Two's Complement Negative Numbers

+	0	1
0	0	1
1	1	10

- A clever approach based on modular arithmetic
- Positive numbers → same as before
- Negative numbers →
  - *additive inverse*: -x is the number *y* such that x + *y* = 0
  - this is the **two's complement of x**
- *Example with 4 bits:* if 1 is 0001, what is -1?

```
carry bits
    0001
  + ????
    ----
    0000
```

Two's Complement Negative Numbers

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representation for -1

```
carry bits      1       11     111    1111
      0001     0001    0001   0001   0001     0001
   + ????     +???1   +??11  +?111  +1111    +1111
     ----      ----    ----   ----   ----     ----
      0000     ???0    ??00   ?000   0000    10000
```

modular arithmetic discards overflow

Two’s complement property

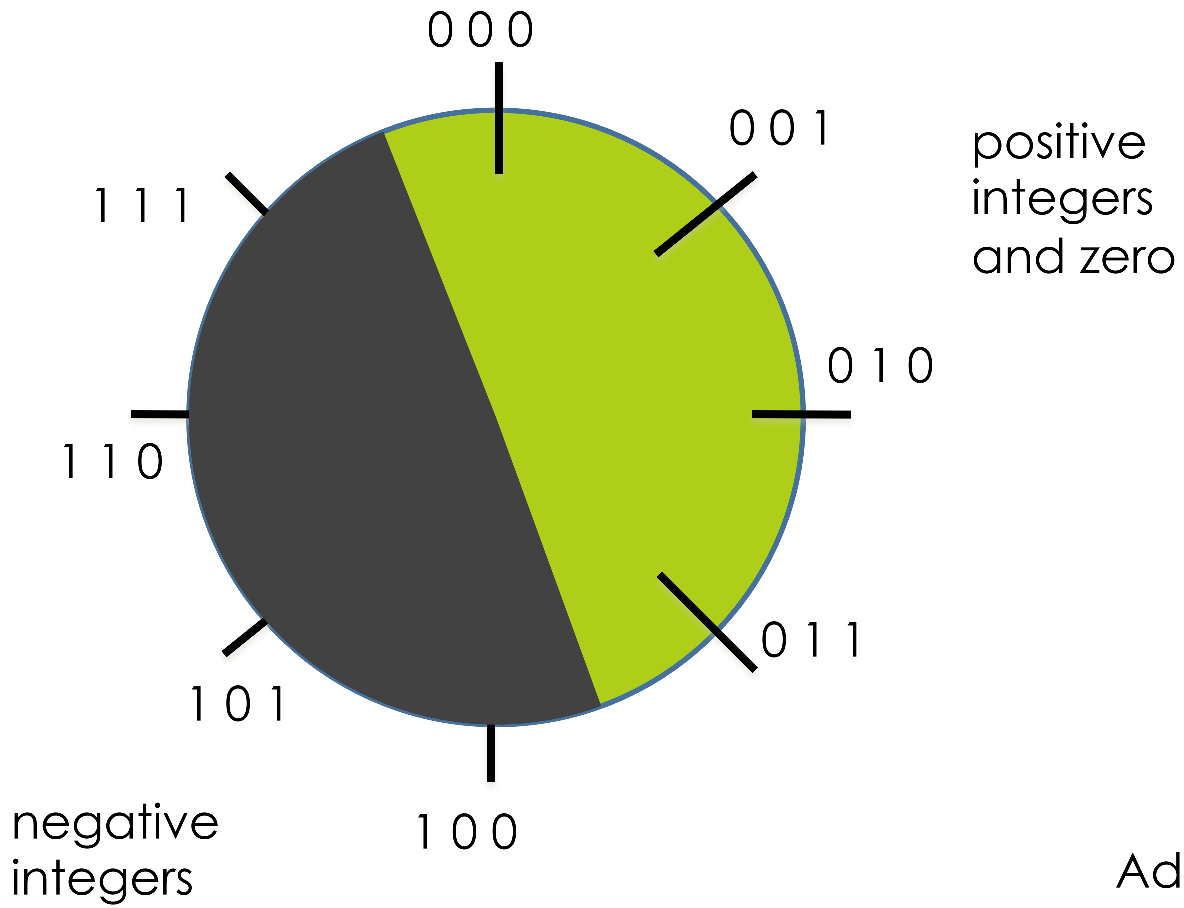
- When you add a number to its two’s complement you always get 0.
  - That’s why we use it to represent negative numbers!
  - Remember, you’re using base 2 arithmetic.

- Example (using 3 bits):

```
    011     (+3 in decimal)
+   101     (-3 in decimal)
---------------------------
 (1)000      0
```

modular arithmetic discards

All two's complement integers using 3 bits, arithmetic mod 8



Bit pattern	Decimal value
0 0 0	0
0 0 1	+ 1
0 1 0	+ 2
0 1 1	+ 3
1 0 0	- 4
1 0 1	- 3
1 1 0	- 2
1 1 1	- 1

Adding + n to – n gives 0
For example: 011 + 101 = 000

Decoding signed integers

negative numbers with two’s complement

Great! but how do we "read" two's complement integers?

- **What does 1010 represent??**

Great! but how do we "read" two's complement integers?

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- **Sign:** look at leftmost bit
  - **1 means negative, 0 means positive**
    e.g. with four bits 1010 represents a negative number

Great! but how do we “read” two’s complement integers?

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    e.g. flip 1010 to get 0101
  - then add 1
    e.g. 0101 + 0001 = 0110, or
    $0 \times 2^0 + 1 \times 2^1 + 1 \times 2^2 + 0 \times 2^3 = 6$

Great! but how do we "read" two's complement integers?

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  - then add 1
    e.g. 0101 + 0001 = 0110, or
    $0 \times 2^0 + 1 \times 2^1 + 1 \times 2^2 + 0 \times 2^3 = 6$
  - **voilà!** 1010 represents negative six

Recap of Two's complement

- Two's complement is an approach for representing negative integers
- Define negative by addition: -x is value added to x to get 0
- Process (**only for negative numbers!!**):
  1. Write out the number in binary
  2. Invert the bits
  3. Add 1
- From and To two's complement use an identical process
- How does this work? Overflow…

Another Example

What value is this ?-bit signed integer?

sign bit

	1	1	0	0	1	1	0	0
	0	0	1	1	0	0	1	1
+	0	0	0	0	0	0	0	1
	0	0	1	1	0	1	0	0

Flip each bit

Add one

two's complement

2^5 2^4 2^2

32+16 + 4 = 52

So 11001100 represents −52

so we can “decode” binary signed integers, now for

encoding signed integers

Signed Integers: encoding negative values

- Example: How do you store -52 in 8 bits?
- Start by encoding +52:

Signed Integers: encoding negative values

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- Start by encoding +52:

One way to do it: by repeated integer division

52 // 2 = 26 r 0
26 // 2 = 13 r 0
13 // 2 = 6 r 1
6 // 2 = 3 r 0
3 // 2 = 1 r 1
1 // 2 = 0 r 1

00110100

Signed Integers: encoding negative values

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00110100

Another way: find the powers of two that add up to 52:

52 = 32 + 16 + 4

2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0
0	0	1	1	0	1	0	0

Signed Integers: encoding negative values

Example continued: How do you store -52 in 8 bits?

We've encoded +52 like this:

52 =			**32 +**	**16 +**		**4**		
	2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0
	0	0	1	1	0	1	0	0

Signed Integers: encoding negative values

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Flip each bit (one's complement):

1	1	0	0	1	0	1	1

Signed Integers: encoding negative values

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Flip each bit (one's complement):

1	1	0	0	1	0	1	1

Add 00000001:

1	1	0	0	1	1	0	0	= -52

Signed Integers: encoding negative values

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Flip each bit (one's complement):

1	1	0	0	1	0	1	1

Add 00000001:

1	1	0	0	1	1	0	0	= -52

The same steps convert positive to negative and vice-versa! (try it and see)

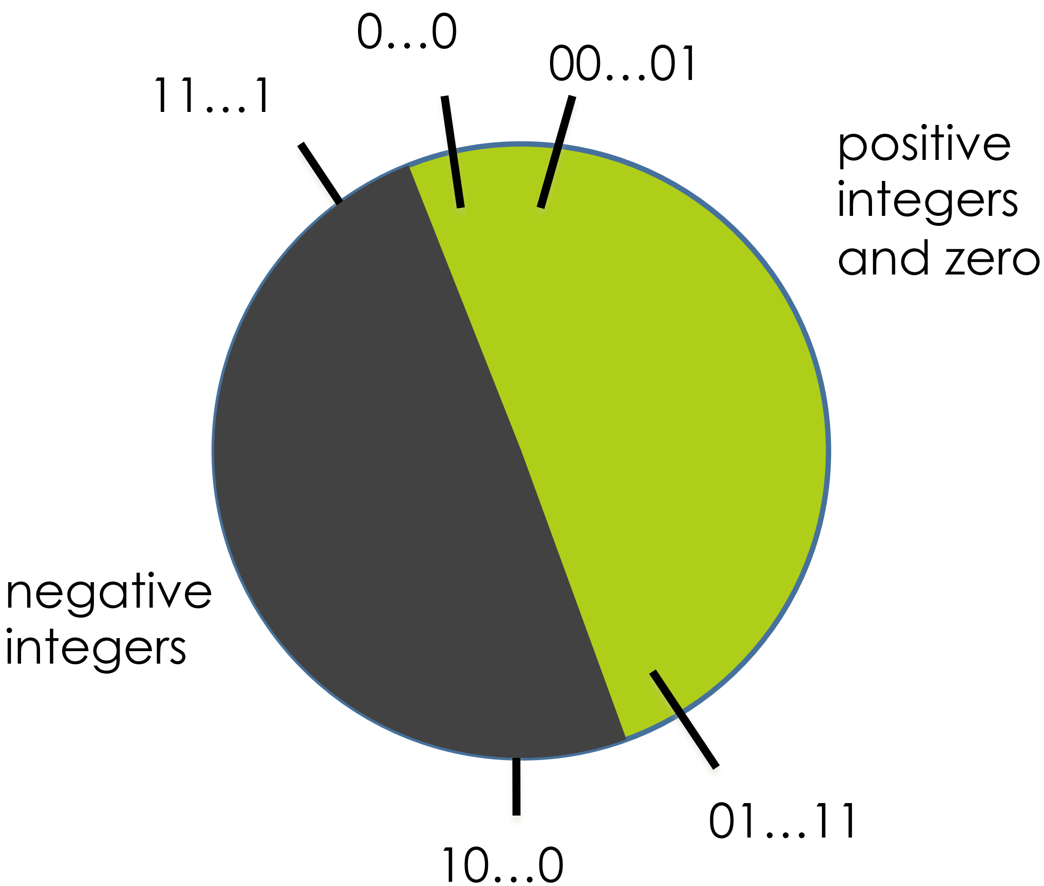
Decoding Algorithm for negative integers (decode 1010, 4 bits)

- **Sign:** look at leftmost bit
  - **1 means negative, 0 means positive**
    e.g. with four bits 1010 represents a negative number
- **Magnitude:** if negative, compute the two's complement
  - flip each bit (one's complement)
    e.g. flip 1010 to get 0101
  - then add 1 (in base 2!!)
    e.g. 0101 + 0001 = 0110, or
    $0 \times 2^0 + 1 \times 2^1 + 1 \times 2^2 + 0 \times 2^3 = 6$
  - **voilà!** 1010 represents negative six

Encoding Algorithm for negative integers (encode -52 in 8 bits)

- Start by encoding +52
  - **52 = 00110100**
- Flip each bit (one's complement):
  - flip **00110100** to get **11001011**
- Add 00000001
  - **11001011 +** 00000001
  - **= 11001100**
  - = −52

Range of Two's Complement Representations (for *k* bits)



Bit pattern	Decimal value
00…00	0
00…01	+1
…	
01…11	$+2^{k-1}-1$
10…00	-2^{k-1}
…	
11…11	-1

Range Examples

bits(k)	minimum value	maximum value
8	$-2^7 = -128$ 10000000	$2^7 - 1 = +127$ 01111111
16	$-2^{15} = -32,768$	$2^{15} - 1 = +32,767$
32	-2^{31} $= -2,147,483,648$	$2^{31} - 1$ $= +2,147,483,647$
64	-2^{63} = -9,223,372,036,854,775,808	$2^{63} - 1$ = +9,223,372,036,854,775,807

From whole numbers to rational numbers

Rounding in binary

```
>>> x = 1/10
>>> x
0.1
>>> y = 2/10
>>> y
0.2
```

python prints a rounded value

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>>> x = 1/10
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Ack!
Whyyyy?

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python prints a rounded value

Ack! Whyyyy?

most decimal fractions cannot be represented exactly as binary fractions!!

Rounding in binary

```
>>> x = 1/10
>>> x
0.1
>>> y = 2/10
>>> y
0.2
>>> x + y
0.30000000000000004
>>> from decimal import Decimal
>>> Decimal(x)
Decimal('0.1000000000000000055511151231257827021181583404541015625')
>>> Decimal(y)
Decimal('0.200000000000000011102230246251565404236316680908203125')
>>> Decimal(x+y)
Decimal('0.3000000000000000444089209850062616169452667236328125')
>>>
```

python prints a rounded value

Ack!
Whyyyy?

the actual value looks like this (in decimal)!

Why is 1/10 not exactly .1?

Let's compute 1/10 using binary long division:

```
         .000110011...
       ________________
1010 ) 1.000000000...
         1010
         ----
          1100
          1010
          ----
            10000
             1010
             ----
              1100
              1010
              ----
                10...
```

we get a repeating series of digits 11001100…

same

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          .000110011...
1010 ) 1.000000000...
         1010
         ----
          1100
          1010
          ----
           10000
            1010
            ----
             1100
             1010
             ----
               10...
```

we get a repeating series of digits 11001100…

same

Similar in decimal to:
1/3 = 0.33333…

Real Numbers in the Machine?

- Real numbers measure **continuous** quantities; can we represent them exactly in the machine?
- Not possible with a fixed number of bits
- Can only approximate by rational numbers using **floating point representations**
- e.g. $\pi \approx 3.14159$

Floating point is based on scientific notation

Age of the Universe in years:

$$+ \underline{1.37} \times 10^{10}$$

sign

significand
or *mantissa*

exponent

Floating point is based on scientific notation

Age of the Universe in years:

$+ \underline{1.37} \times 10^{10}$

sign	significand or *mantissa*	exponent
(1 bit)	(23 bits)	(8 bits)

Idea: use same method, but with a binary number for each part (and remember, a fixed number of bits)

Binary and fractions

- Decimal 5.75 can be represented in binary as follows, because .75 = ½ + ¼ = $2^{-1} + 2^{-2}$

  5.75 = 5 + 0.75

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  = 101 + 0.11 (i.e. $2^{-1} + 2^{-2}$)

decimal

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= 101 + 0.11 (i.e. $2^{-1} + 2^{-2}$)

= 101.11 = 1.0111×10^{10}

decimal

binary

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$$5.75 = 5 + 0.75$$
$$= 101 + 0.11 \text{ (i.e. } 2^{-1} + 2^{-2}\text{)}$$
$$= 101.11 = 1.0111 \times 10^{10}$$

decimal

binary

In binary floating point:

- the significand is a binary fraction
- exponent is a binary integer, and the base of the exponent is always 2

101.11 has *significand* 1.0111 and *exponent* 10

Some Floating Point Anomalies

- Rounding error
  >>> round(2.675, 2)
  2.67

  - remember, floating point with a fixed number of digits is an ***approximation****, no matter what base is used!*
  - in addition, there is no finite base two representation for 1/10

- Accumulation of errors: repeated operations may get further and further from the “true” value

Rounding in any base

- Floating point works with a finite fixed number of digits
- No matter what the base, some numbers can only be approximated
  - π, e, other irrationals
  - but also rationals needing more digits than we have in a machine word

Resolution

- Tiny example: *suppose we use a binary floating point notation like this (4 bits):*

  $d_1.d_2d_3 \times 2^e$, where $-1 \le e \le 2$ and $d_1 = 1$ unless $e=0$

- Representable values get sparser as we go to bigger and bigger numbers!

Image source: "What Every Computer Scientist Should Know About Floating-Point Arithmetic", by David Goldberg. *Computing Surveys, 1991*

Floating point: the bottom line

For serious work like simulating the weather or the economy, hire an expert! (or be an expert)

You should be able to

- Count in unsigned binary
  0, 1, 10, 11, 100, …
- Add in binary and know what overflow is
- Determine the sign and magnitude of an integer represented in two's complement binary
- Determine the two's complement binary representation of a positive or negative integer

Some Helpful Python functions

```
>>> bin(10)
'0b1010'
>>> hex(10)
'0xa'
>>> from decimal import Decimal
>>> Decimal(.2)
Decimal('0.200000000000000011102230246251565404236316680908203125')
```