Optimal Rating Design

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ASU Annual Economic Theory Conference
December 2022

Introduction

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  - eBay, college grades, security rating, Google Ranking

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    - Ratings are often used to incentivize quality provision
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- Key Elements:
    - Ratings are often used to incentivize quality provision
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        - Potentially incentivizes design of securities
    - Ratings often involve manipulation: USNews, ESG
- **Questions:**
    - How should we think about rating design when it provides incentives?
    - What to do about manipulation?

What do we do?

- Rating design with moral hazard
  - DM takes action that leads to an outcome
  - Market cares about action and/or outcome
  - Intermediary observes outcome and designs a disclosure policy
  - Market pays expected value to the DM

Main Findings ...

- Map this mechanism design problem without transfers into a problem with transfers (interim prices)
- Key mathematical result: provide a simple characterization of feasible transfers
  - Interim prices are mean-preserving contraction of market values conditional on the outcome
- Rating design $\equiv$ Mechanism design with transfer and majorization constraints
- Study various applications (with productive effort and manipulation):
  - Highlights the importance of rating uncertainty

Related Literature

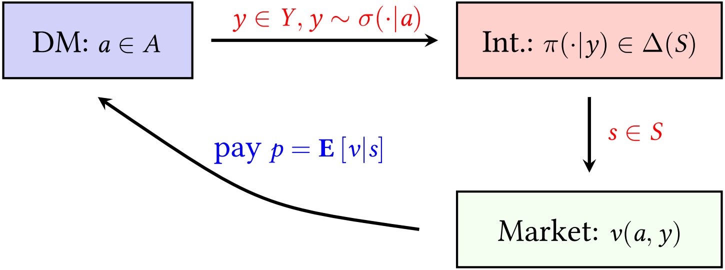
- Bayesian Persuasion: Kamenica and Gentzkow (2011), Rayo and Segal (2010), Gentzkow and Kamenica (2016), Dworczak and Martini (2019), Mathevet, Perego and Taneva (2019), ...
  - Characterize second order expectations + endogenous state; no incentives for receiver
- Certification and disclosure: Lizzeri (1999), Ostrovsky and Schwartz (2010), Harbough and Rasmusen (2018), Hopenhayn and Saeedi (2019), Vellodi (2019), ...
  - Information design as mechanism design
- Falsification and muddled information: Perez-Richet and Skreta (2020), Frankel and Kartik (2020), Ball (2020)
  - General characterization of feasible mechanisms under moral hazard

Roadmap

- The Model
- Characterization for arbitrary rating system
- Two Applications – more in the paper:
    - Optimal ratings absent input manipulation
    - Optimal ratings with input manipulation

The Model

- DM chooses an action $a \in A \subset \mathbb{R}^N$
- Induces $y \in Y \subset \mathbb{R}^M$ with $\sigma(\cdot|a) \in \Delta(Y)$
- Market value: $v(a, y)$; paid to the DM – conditional on available information
- Intermediary observes y and sends a signal to the market: $(S, \pi(\cdot|y))$ with $\pi(\cdot|y) \in \Delta(S)$



The Model

- Cost of effort for DM: $c(a,\theta)$, $\theta \sim F(\theta)$
- Payoff of DM

$$\int_Y \int_S \mathbb{E}\left[v|s\right] d\pi\left(s|y\right) d\sigma\left(y|a\right) - c\left(a,\theta\right) \qquad (\star)$$

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  - (a,θ): private to the DM
  - y observed by Int.
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- **Information:**
  - (a,θ): private to the DM
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- **Equilibrium**: Perfect Bayesian Nash Equilibrium
  - Given π and market beliefs, $a(\theta)$ maximizes $(\star)$
  - Market beliefs are consistent with π, $a(\theta)$, and prior according to Bayes' rule

Some Examples

- DM: Seller of a good on a platform: Airbnb, eBay
- Grading of a student's (DM) effort; Difficulty of exams
- Rating agency determining how to rate a corporate bond
- Manipulation:
  - Two actions:
    - ex-ante productive effort
    - ex-post costly manipulation of feedback
  - Intermediary observes manipulated feedback

First Step a la Revelation Principle

- Mechanism design without transfers
- First question: What allocations of effort $a(\theta)$ are affordable for an arbitrary information structure $(S, \pi(\cdot|y))$?
- Sufficient statistic for DM's decision

$$\int_Y \underbrace{\int_S \mathbb{E}[v|s]\, d\pi(s|y)}_{p(y)} d\sigma(y|a) - c(a, \theta)$$

- $p(y)$: Interim price or second-order expectation

First Step a la Revelation Principle

- **Incentive compatibility:**

$$\int p(y)\, d\sigma(y|a(\theta)) - c(a(\theta), \theta) \geq \int p(y)\, d\sigma(y|a) - c(a, \theta), \forall a \in A$$

- Interpretation: $p(\cdot)$ are monetary transfers; need to figure out feasibility imposed by

$$p(y) = \int \mathbb{E}[v|s]\, d\pi(s|y)$$

- Useful to define market values as, i.e., when $\pi(\{y\}|y) = 1$

$$\overline{v}(y) = \mathbb{E}[v|y]$$

Lemma

Lemma

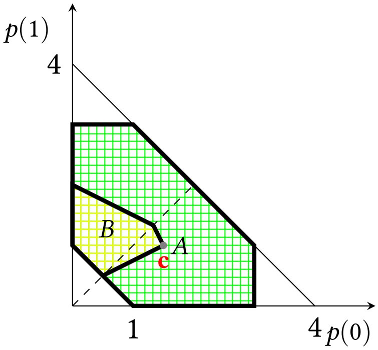
For any information structure (S, π) *and* $p(y)$ *defined above,* $p(\cdot)$ *second order stochastically dominates* $\overline{v}(\cdot)$*, i.e., for all concave and increasing function* $u : \mathbb{R} \to \mathbb{R}$*,*

$$\sum_{Y} \mu_Y(y)\, u(\overline{v}(y)) \leq \sum_{Y} \mu_Y(y)\, u(p(y))$$

$$\sum_{Y} \mu_Y(y)\, \overline{v}(y) = \sum_{Y} \mu_Y(y)\, p(y)$$

Example

- Is that also sufficient?
  Not necessarily
- Suppose $A = Y = \{0, 1, 3\}$,
- $v(a, y) = \overline{v}(a) = a$,
- $\sigma\left(Y'|a\right) = \mathbf{1}\left[a \in Y'\right]$,
- $\mu(\{a\}) = 1/3$.
- Set of mean-preserving contractions of Y: $A \cup B$,
- Set of interim prices B



First Step a la Revelation Principle

- Main result:

Theorem. Let $\overline{v}(y) = \mathbb{E}[v|y]$. Then,

1. If $p(\cdot)$ is derived from (S, π), then $p \succcurlyeq_{\text{SOSD}} \overline{v}$.
2. If $p \succcurlyeq_{\text{SOSD}} \overline{v}$ and $p(\cdot)$ and $\overline{v}(\cdot)$ are co-monotone, i.e., $p(y) > p(y') \Rightarrow \overline{v}(y) > \overline{v}(y')$, then there exists (S, π) that induces $p(\cdot)$.

Main Result: Idea of Proof

- One direction is obvious: existence of $\pi \to$ stochastic dominance
- For the other direction: a geometric approach similar to Strassen's theorem
- Suppose Y is finite, $|Y| = m$.
- Let

$$S = \left\{ \hat{p} \in \mathbb{R}^m \,|\exists \left(S, \pi\right), \hat{p}\left(y\right) = \mathbb{E}\left[\overline{v}|y\right] \right\}$$

- Convex and closed set of probability measures

Main Result: Idea of Proof

- Separating Hyperplane Theorem:

$$p \in S \iff \forall \lambda \in \mathbb{R}^m, \exists \hat{p} \in S, \lambda \cdot p \leq \lambda \cdot \hat{p}$$

- If p and $\bar{v}$ are comonotone and $p \succcurlyeq_{\text{SOSD}} \bar{v}$, we can construct an information structure for each λ.
  - Depends on the comonotonicity of λ with p
  - In general, construct inductively by pooling two states appropriately

Remark on Theorem

- Our result is reminiscent of the result of Blackwell (1953), Rothschild and Stiglitz (1970) and Strassen (1965)
- What's the difference
  - It is stated for the second order conditional expectation
  - The key intricacy is that the same signal structure that generates the random variable $\mathbb{E}\left[\overline{v}|s\right]$ must be used to generate $\mathbb{E}\left[\mathbb{E}\left[\overline{v}|s\right]|y\right]$.
  - The equivalent of Blackwell's result does not hold in general and can only be shown when $\overline{v}$ and p are co-monotone.

Implication of the Theorem

- When the comonotonicity of $p(\cdot)$ and $\overline{v}(\cdot)$ is without loss of generality, we can solve the mechanism design problem by solving for $p(y)$ and $a(\theta)$ that satisfy:
    1. Incentive compatibility:
       $a(\theta) \in \arg\max_{a \in A} \int p(y)\, d\sigma(y|a) - c(a, \theta)$
    2. Stochastic dominance: $p(y) \succcurlyeq_{\text{SOSD}} \overline{v}(y)$
- We'll show two applications of this

Majorization

- Instead of using the conditions for second order stochastic dominance we will be using majorization conditions
- Helps to use a Lagrangian method to solve for the optimal rating systems
- When $Y = \mathbb{R}$, we can write

$$p \succcurlyeq_{SOSD} \overline{v} \iff \int_{-\infty}^{y} p\left(\hat{y}\right) d\mu_Y\left(\hat{y}\right) \geq \int_{-\infty}^{y} \overline{v}\left(\hat{y}\right) d\mu_Y\left(\hat{y}\right), \forall y \in \mathbb{R}.$$

- With equality at the top.

Application 1: Rating Design Under Productive Effort

- Market values $v(a, y) = y,\ y \in [0, 1]$
- $\Theta = \{\theta_1, \cdots, \theta_n\}$
- Objective: pareto optimality

$$\sum_{\theta \in \Theta} f(\theta)\, \lambda(\theta) \left[\int p(y)\, dG(y | a(\theta)) - c(a(\theta), \theta) \right]$$

- Monopolist intermediary is a special case. Full weight on lowest participating type

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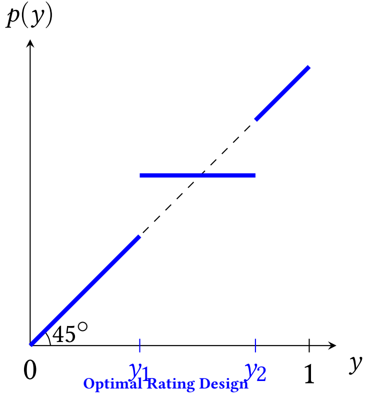
Proposition. Under productive effort, pareto optimal rating systems are monotone partitions.

A Two-Type Case

- Suppose $\Theta = \{\theta_1 < \theta_2\}$.
- Objective: Maximize revenue of a monopolist intermediary
- Two key forces:
    - Market size effect: pooling states lead to reshuffling profits to θ_1 and allows the intermediary to charge a higher fee
    - Incentive effect: pooling leads to reduced incentive for both types

A Two-Type Case

Proposition. Suppose that Assumption 3 holds. If at the optimum $a_2 \geq a_1$, then there exists two thresholds $y_1 < y_2$ where optimal monopoly rating system is fully revealing for values of y below y_1 and above y_2 while it is pooling for values of $y \in (y_1, y_2)$.



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- Roughly speaking assumption 3 says that likelihood ratio function g_a/g is concave and increasing enough
- Holds for:
  - Power distributions: $G(y|a) = y^{\alpha \cdot a}$, $G(y|a) = 1 - (1-y)^{\alpha/a}$
  - Exponential distribution: $G(y|a) = \frac{e^{\lambda(a)y}-1}{e^{\lambda(a)}-1}$

Separable Distributions

Proposition. Suppose that $g(\cdot|a)$ satisfies the following separability assumption

$$g(y|a) = \alpha(a) + \beta(a)\, m(y)$$

Then optimal monopoly rating system is full disclosure.

Application 2: Rating Design Under Manipulation

- Market valuation: $y \sim G(y|a)$, $y \in [0,1]$; Only one type of DM
- After realization of y, DM reports x to intermediary at cost

$$c_m(x,y) = k\frac{(x-y)^2}{2} + \tau|x-y|, k \geq 0, \tau \in [0,1]$$

- Objective: maximize payoff of DM

App 2: Ratings and Manipulation

- How does our theorem apply here?
- Equilibrium:
  - Manipulation strategy $\hat{x}(y)$
  - productive effort: a
- Market is smart and has correct beliefs about $\hat{x}(y)$
- Interim price

$$p(y) = \mathbb{E}\left[\mathbb{E}\left[\hat{x}^{-1}(x)\,|s\right]|\hat{x}(y)\right]$$

- Incentive compatibility of manipulation strategy plus single-crossing for $c_m(\cdot,\cdot)$:
  - $p(y)$ and $\hat{x}(y)$ have to be increasing in y.
- Our Theorem says: Existence of π is equivalent to $p(y) \succcurlyeq_{\text{SOSD}} y$

App 2: Ratings and Manipulation

- Orduous manipulation

Proposition. There exists $\overline{\tau}$ such if $1 \geq \tau > \overline{\tau}$, then for optimal rating:

1. There is no manipulation in equilibrium: $\hat{x}(y) = y$,
2. Optimal rating satisfies

$$\pi(\{s\}|y) = \begin{cases} \tau & s = y \\ 1 - \tau & s = N \end{cases}$$

App 2: Ratings and Manipulation

- When manipulation is costly no point in trying to let people manipulate
  - Note: $p(y) = y$ is the solution absent manipulation
- An interpretation of optimal rating:
  - Involves rating uncertainty
  - It is as if the intermediary hides features of the rating system from the DM
  - Some evidence for value of this in Nosko and Tadelis (2015) based on an experiment in eBay

App 2: Ratings and Manipulation

- Let's make manipulation effortless: $\tau = 0$;
- Trade-off between manipulation and ex-ante incentives
  - Marginal cost of manipulation is 0 at $\hat{x} = y$
  - Need variation in $p(y)$ for ex-ante incentives, i.e., a or productive effort

Assumption. The distribution function satisfies

1. $\frac{G_a}{g}$ is convex in y for all values of a.

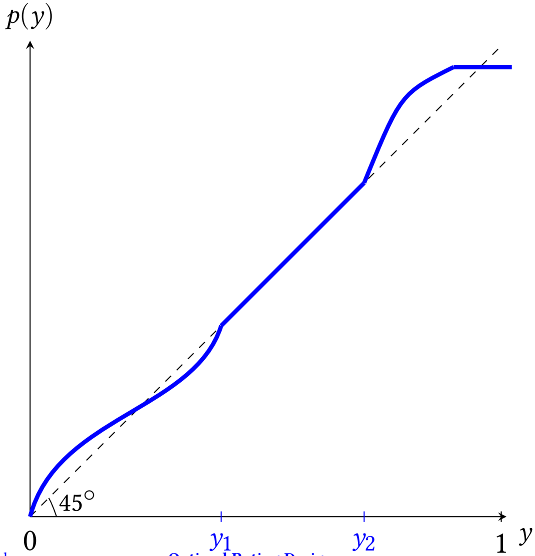
App 2: Ratings and Manipulation

Theorem. When $\tau = 0$, optimal rating satisfies

1. If $k \geq \hat{k}_1$, then optimal rating involves randomization and is non-separating.
2. If $k \in \left[\hat{k}_2, \hat{k}_1\right]$, then optimal rating involves three regions:
   2.1 For high and low values of y optimal rating involves randomization and non-separation
   2.2 For mid-values of y, the optimal rating is fully revealing.

App 2: Ratings and Manipulation

- Interim prices for $k \in \left[\hat{k}_1, \hat{k}_2\right]$



Conclusion

- Studied optimal rating design in presence of incentives
- Characterization of feasible outcomes
- Optimal rating design under productive and unproductive effort, i.e., manipulation