Optimal Information Disclosure and Market Outcomes*

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Abstract

This paper addresses two central questions in markets with adverse selection: How does information impact the welfare of market participants (sellers and buyers)? Also, relatedly, what is the optimal information disclosure policy and how is it affected by the planner's relative welfare weight on sellers' surplus versus consumers' surplus? We find that as a result of improved information, prices become more strongly associated with the true quality of sellers and thus more dispersed. This will result in higher total surplus. Furthermore, we find that better information has opposing welfare effects on consumers and producers that could lead to limited disclosure depending on the social objective and market characteristics.

Keywords: Adverse Selection, Information Design, Consumer Surplus, Producer Surplus.

JEL codes: D21, D47, D60, D82, L11

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1 Introduction

Reputation mechanisms and ratings are widely used in markets with adverse selection. While relevant for any market with asymmetric information (e.g., hygiene ratings for restaurants or doctors' performance ratings), information design is a key consideration for the overall performance of the ever more popular online trading platforms, where transactions are decentralized and rarely repeated. Despite the importance of these mechanisms, little is known about their optimal design and how it might depend on the characteristics of the market, such as supply and demand. This paper sheds light on this question by considering the design of an optimal information disclosure mechanism and how it relates to market characteristics.

In particular, the paper addresses two central questions. First, how does information impact the welfare of market participants (i.e., sellers and buyers)? Second, what is the optimal information disclosure policy and how is it affected by the planner's relative welfare weight on sellers' versus consumers' surplus? Also, how do the answers to these questions depend on the properties of supply and demand in the market?

Our model considers a competitive market with a large set of buyers and sellers. Firms are endowed with different levels of quality, which is the only source of product differentiation.[1] The buyers do not observe the sellers' level of quality, but only observe the signals sent by the market designer, which is the source of asymmetric information. The model exhibits two features that are common to adverse selection settings. First, low-quality sellers benefit from being pooled with high-quality ones, while adversely affecting them. Second, high-quality sales are crowded out by low-quality ones.[2] Information disclosure, and in particular a rating system, helps reallocate sales from lower- to higher-quality producers, thus mitigating the problem of adverse selection.

We first consider the impact of information on consumer and producer surplus. As a result of improved information, prices become more strongly associated with the true quality of sellers and thus more dispersed. Demand is reallocated from lower- to higher-quality firms, and this reallo-

[1] While moral hazard might be a critical consideration in some markets, in others adverse selection might play a more critical role, as suggested by an empirical study on eBay (see Hui et al. (2018)). Optimal rating design with moral hazard and adverse selection is considered in Saeedi and Shourideh (2022) in a simplified market environment.

[2] While these features motivate our use of the term adverse selection, we note that in our model sellers' supply decisions do not directly depend on their quality and only through the price they receive in the equilibrium.

cation has a positive effect on the average quality of goods consumed and total surplus. However, the effect of improved information on total market size and consumer surplus is ambiguous and depends on the properties of the supply function. When supply is concave, the higher spread in prices results in a decrease in total output and lowers consumer surplus. The opposite occurs when the supply is convex. The impact of the changes in total output on consumer surplus also depends on demand elasticity: when demand is more elastic, consumer surplus will be less sensitive to the information structure.

We next consider the problem of optimal information provision by an informed social planner that maximizes a weighted sum of producer and consumer surplus. The information structure shared by all buyers follows the setting described in Ganuza and Penalva (2010) and Gentzkow and Kamenica (2016). A common prior over firm qualities and the information provided by the planner determine the distribution of expected posterior firm qualities. This provides a natural ordering of the quality of information, where better information is associated to a mean-preserving spread of the distribution of firms' expected values. At one extreme, there is no information, so all firms have an expected quality equal to the common prior mean; at the other extreme, there is full disclosure of information, so the buyers' posterior equals that of the planner. In the interior region, the planner can choose any garbling of this posterior.

Changes in the information structure have two effects on the planner's objective: a direct effect on expected profits and a general equilibrium effect, operating through the change in market size and equilibrium price. The direct effect will be an increase in profits of a magnitude that depends on the curvature of the profit function. The general equilibrium effect will affect firms and consumers in opposite directions and equal magnitudes. Moreover, the magnitude of the general equilibrium effect will vary directly with the curvature of the supply function. In particular, if the supply function is convex at a point, a mean-preserving spread at this point results in an increase in total output, and the general equilibrium effect will imply a transfer of utility from firms to consumers. A transfer in the opposite direction would occur if the supply function were concave at this point.

When the planner weighs equally producer and consumer surplus, this aforementioned surplus transfer does not affect total welfare, so full disclosure is optimal. When weights are not equal, limited disclosure may be optimal. For example, in regions where the supply function is concave, pooling can mitigate the reduction in output from improved information and its negative impact on consumer surplus. Where the supply function is convex, pooling decreases total output and increases prices, which might have a positive impact on producers. For those cases where full information is not optimal, we find that the region of pooling increases with the strength of the bias in the planner's preference for one or the other group.

Related Literature Our paper is related to the literature considering the impact of information disclosure on consumer and producer surplus.[3]

Most papers belonging to the first strand of literature consider the case where there is a single seller, or auctioneer, and multiple buyers, as opposed to multiple agents on both sides. Similar to our results, Schlee (1996) shows that information can hurt consumers when the cost function is sufficiently convex. Bergemann et al. (2015) consider the impact of information in third-degree price discrimination. They show that any distribution of surplus that is between the ones achieved by optimal pricing with no information and that with full information can be attained with some information structure. Bergemann and Pesendorfer (2007) show that in a private value setting, bidders can be worse off with better information even though total surplus increases. Board (2009) shows that this result depends on the number of bidders. Hoppe et al. (2011) consider a matching problem where for some distribution of types, consumers can be worse off with better information. Romanyuk and Smolin (2019) also considers a matching framework where full information leads to cream-skimming on the seller side and even market failure and shows that hiding some information can restore market efficiency. In our paper we show that better information always increases total surplus, but it might decrease consumer or producer surplus depending on the properties of the supply function. These considerations are absent in the matching framework, where supply is inelastic.

There is a large literature on certification and quality disclosure. Dranove and Jin (2010) provide an excellent survey of the earlier papers. Most of the literature has focused either on the incentives for firms to reveal their information or the incentives of certifiers to do so. The main question in

[3] Our paper focuses on a setting where uncertainty is about seller quality and information is provided to consumers. There is a growing literature that focuses on the reverse channel, where an intermediary transmits information about buyers to sellers. For a survey, see Bergemann and Bonatti (2019).

this literature is how much information will be revealed in equilibrium and how this might depend on the nature of competition in the product or certification markets. As an example, Lizzeri (1999) finds that while a monopoly certifier chooses to provide coarse information with a single and low threshold, competition among certifiers can lead to full information. Ostrovsky and Schwarz (2010) consider equilibrium information structures where colleges strategically choose how much information to reveal about their students' ability. DeMarzo, Kremer, and Skrzypacz (2019) consider a Bayesian game where agents choose the informativeness of testing but can withhold bad results.

Our paper differs from this certification literature in several dimensions. First, in our setting information is freely provided by a single informed certifier, and in particular it is exogenous to the firm, as occurs in the examples mentioned above. Secondly, information affects the payoffs of firms through two channels. The first is a standard one, where certification provides a signal of expected quality to consumers, directly affecting the price faced by the firm. The second one is that certification affects total equilibrium output and thus the equilibrium prices received by all firms, thus impacting both producer and consumer surplus. This effect is absent in most papers on certification in markets that usually assume inelastic supply. Another implication of elastic supply is that certification reallocates sales across firms, to a degree that is affected by supply elasticity. This plays an important role in the value and design of an optimal certification mechanism.

Information disclosure is the focus of the literature on Bayesian persuasion, where an informed sender chooses an information structure to influence the behavior of a receiver. Kamenica (2018) and Bergemann and Morris (2019) provide a great survey of this literature. Kolotilin (2018) and Dworczak and Martini, (2018) provide conditions on payoffs so that interval partitions are the optimal information structure. Onuchic and Ray (2021) study the problem of monotonic categorization when sender and receiver have different priors. In contrast to most of this literature, where a single receiver takes an action, in our setting the outcome is the result of the equilibrium choices of multiple agents.[4] We develop an approach that integrates results from this literature, considering

[4]While other papers have studied settings with multiple receivers, the analysis has often been suitable for games where a low-dimensional source of aggregate information is observed by a sender. For example, Bergemann and Morris (2013, 2016) characterize the outcome of all Bayesian persuasion games with multiple receivers. In principle, our problem could be potentially mapped into this framework, with an omniscient sender that observes the quality of a continuum of firms, but it would be impractical to solve it this way. Even for a simple two-player game, Bhaskar et al. (2016) show that computing the optimal public signal is NP-hard.

the constraints imposed by market equilibrium.

The most relevant empirical papers related to our theory are Saeedi (2019), Elfenbein et al. (2015), Fan et al. (2013), and Jin and Leslie (2003). Saeedi (2019) studies the value of reputation mechanisms and establishes a positive signaling value for the certification done by eBay. Elfenbein et al. (2015) study the value of certification badges across different markets. They find that certification provides more value when the number of certified sellers is low and when markets are more competitive. Fan et al. (2013) analyze the effect of badges on Taobao.com. They find sellers offer price discounts to move up to the next reputation level. Jin and Leslie (2003) use data on restaurant hygiene ratings to examine the effect of an increase in product quality information to consumers on firms' choices of product quality. Our paper also relates to the literature that analyzes the effects of changes in marketplace feedback mechanisms on price and quality (e.g., Hui et al. (2016), Filippas et al. (2018), and Nosko and Tadelis (2015)).

Section 2 describes the model. Section 3 considers the optimal information disclosure problem. Section 4 concludes the paper. Proofs are relegated to the appendix unless otherwise specified.

2 The Model

There is a unit mass of firms with qualities z distributed according to a continuous cumulative distribution function (cdf) $F(z)$ on a compact set Z. Production technology is the same for all firms and is given by a differentiable, strictly increasing, and strictly convex cost function of quantity, $c(q)$, and, correspondingly, a strictly increasing twice continuously differentiable supply function $S(p)$. On the demand side, there is mass M of consumers who face a discrete choice problem, with preferences

$$U(z, \theta, p) = z + \theta - p,$$

where z is the quality of the good purchased, θ is a taste parameter measuring the preference for goods offered in this market vis-Ã -vis an outside option, and p is the price of the good. The taste parameter $\theta \geq 0$ is distributed according to a continuous and strictly increasing cdf $\Psi(\theta)$, while the outside good's utility (no purchase) is normalized to zero.[5] Goods are differentiated only by their quality level, which is equally valued by all consumers. Given the linearity of the utility function in z, we can replace a good of quality z with a good of expected quality z and the utility of the consumer stays the same. Throughout the paper, we use z interchangeably as the quality or expected quality of a good.

We assume the following timing: (a) information about firm qualities is provided by the planner, (b) based on this information, consumers form posteriors about each firm's expected quality; (c) given these posteriors, perfectly competitive equilibrium prices are determined as a function of expected quality, considering the supply response of each firm to the corresponding price.[6] We will say that a firm has expected quality z if conditional on all signals received, that is the quality expected by all consumers. Denote by $G(z)$ the common posterior distribution over firms' expected qualities shared by consumers.[7]

Given expected quality z, equilibrium prices take the form $p(z) = p(0) + z$, where $p(0)$ corresponds to the demand price of a hypothetical good of quality zero. This expression for prices guarantees that consumers are indifferent between goods with different signal realizations with any positive sales, which is a necessary condition for an equilibrium. Given a baseline price $p(0)$, the marginal consumer's θ is found by setting $U(0, \theta, p(0)) = 0$, or simply $\theta(p(0)) = p(0)$. All consumers with $\theta \geq p(0)$ will make their unit purchase, so aggregate demand is $Q = M(1 - \Psi(p(0)))$. Inverting this function, we can define an inverse baseline demand function

$$P(Q) = \Psi^{-1}(1 - Q/M) = p(0). \tag{1}$$

On the supply side, each firm with expected quality z chooses its output, $q = S(p(z))$, so aggregate supply $Q = \int S(p(z))\, dG(z)$.[8]

[5]Alternatively, one can consider $-\theta$ to be the value of the outside good to consumers. Also note that we do not need to make any particular assumption on the distribution of θ ; additionally, linearity in z can be relaxed by modifying the distribution of the taste shock.

[6]Similar results can be obtained under Cournot competition in a model as the one considered by Salant and Shaffer (1999).

[7]This representation of the information structure is consistent with the approach followed in Ganuza and Penalva (2010) and Gentzkow and Kamenica (2016). Given a common prior $F(z_0)$ over firm qualities and a signal structure π, we can let $G(z)$ be the distribution of the expected posterior of firm quality.

[8]The reason that we mention expected quality of the firm instead of its quality is that the price the firm gets depends

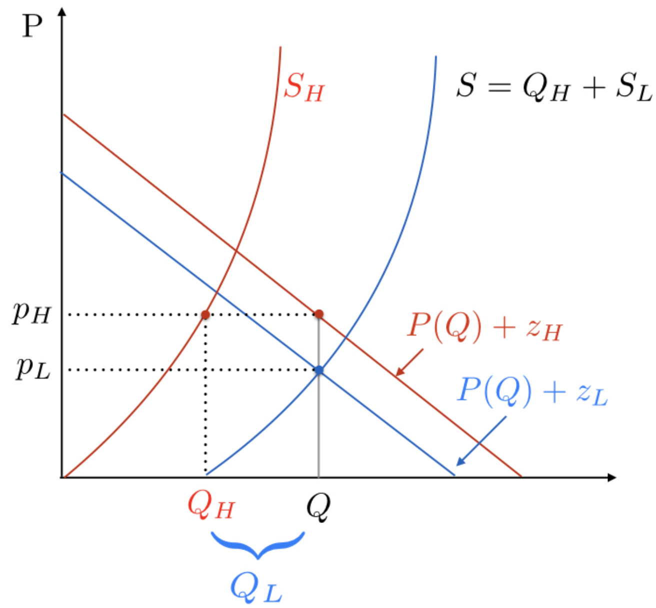

Figure 1: Equilibrium

Definition. An (interior) *equilibrium*, given the distribution of expected qualities $G(z)$, is given by prices $p(z) = P(Q) + z$, where total quantity $Q = \int S(p(z))\, dG(z)$.

Figure 1 shows graphically the derivation of an interior equilibrium for the case of a two-tier partition, where L represents the group of firms with quality below a threshold z^*, and H those above.[9] Denote by z_L (resp., z_H) the average quality of sellers in the L group (resp., H group). The two curves depict the demand curve for the goods in the L and H segment, respectively. Since all consumers value quality identically, the price difference between the goods in the two segments is the same as the difference between the two respective average qualities $p_H - p_L = z_H - z_L$. The first upward sloping curve is the supply function of the firms in the H group, $S_H = (1 - F(z^*))\, S(p_H)$. The second one is the supply function of the firms in the L segment, $S_L = F(z^*)\, S(p_L)$, displaced to the right by the equilibrium quantity of the H group, Q_H.[10] The marginal consumer is the one

on the information and the possibility of being pooled with other firms.

[9] Alternatively, this can be interpreted as a case of having two types of sellers with two levels of qualities.

[10] Note that given that we draw the supply and demand curves in the normal way, with price on the y-axis, the case depicted in the graph corresponds to concave supply.

that is indifferent between consuming either of these goods or none at the equilibrium prices; Q is also the total market supply of both goods.

To prove the existence of an interior equilibrium, we make the following assumptions.

Assumption 1. *There exists* $\widetilde{\theta}$ *in the support of* Ψ *such that*

$$M > \int S\left(\widetilde{\theta} + z\right) dG\left(z\right)$$

for all distributions G *such that* F *is a mean-preserving spread of* G*. In addition,* $\int S\left(p\left(0\right) + z\right) dG\left(z\right) > 0$ *for the same class of distributions.*

The first assumption rules out the possibility that all consumers make purchases in this market; in other words, we assume that the consumers are on the long side of the market.[11] The second assumption rules out no output as an equilibrium. While a corner equilibrium, if it exists, is also unique, we rule this out as a matter of convenience.

Lemma 1. *Under Assumption 1, there exists a unique interior equilibrium for all expected quality distributions* G *such that* F *is a mean-preserving spread of* G*.*

Proof. Given that the cdf Ψ is strictly increasing and continuous, the function $P\left(Q\right)$ is strictly decreasing and continuous. Define function $f\left(Q\right) = \int S\left(P\left(Q\right) + z\right) dG\left(z\right)$; by continuity and monotonicity of the supply function, function $f(Q)$ is strictly decreasing and continuous as well. By Assumption 1, $f\left(0\right) > 0$ and $f\left(M\right) < M$ since $P\left(M\right) \leq \widetilde{\theta}$. Hence, there exists a unique fixed point Q^* for this mapping which results in a unique equilibrium. □

3 Information Disclosure

Our previous analysis takes the distribution of observed mean qualities, G, as a primitive. Given the linearity of payoffs, this is a sufficient representation of information, as two products with the same posterior mean qualities are equivalent to consumers. As in Ganuza and Penalva (2010)

[11] As explained below, the assumption spans the set of all possible information structures.

and Gentzkow and Kamenica (2016), improvements in information can be represented by mean-preserving spreads of the distribution of mean qualities.[12] This section considers two related questions: (1) the impact of improved information on producer and consumer surplus and (2) optimal information disclosure by an informed principal.

3.1 Improved Information

This section examines the impact of improved information on producer and consumer surplus. Given total quantity Q, equilibrium prices are given by $P(Q) + z$, with mean $P(Q) + \bar{z}$. A mean-preserving spread of G increases the spread of prices around the mean while possibly changing the mean, too, as the equilibrium quantity Q changes.

The increased dispersion of prices has a direct positive effect on average profits, as a result of the convexity of the profit function. In turn, an increase (resp., decrease) in market size as measured by the change in total quantity Q has a negative (resp., positive) effect on profits. In contrast, as we now show, consumer surplus is affected only by changes in total quantity Q, and in the opposite direction of profits.

Consider a consumer of type θ who buys a good of quality z, with utility $\theta + z - p(z)$. Given the equilibrium price $p(z) = P(Q) + z$, the consumer's net utility is $\theta - P(Q)$. It follows that total consumer surplus is

$$M \int_{P(Q)} (\theta - P(Q))\, d\Psi(\theta) = \int_0^Q (P(x) - P(Q))\, dx,$$

where the equality follows from the change of variables $x = M(1 - \Psi(\theta))$ and our definition of $P(Q)$: $P(Q) = \Psi^{-1}(1 - Q/M)$. This implies that consumer surplus will move in the same

[12] For example, in Ganuza and Penalva (2010), who order the quality of information by the dispersion of beliefs, the distribution of expected qualities $\tilde{G}$ is more informative than distribution G if it is a mean-preserving spread of G. We will refer to this ordering as better information. As the maximal signal structure corresponds to perfect information, the class of all information structures can be represented by all garblings of F, i.e., all distributions G such that F is a mean-preserving spread of G. This corresponds to the ordering of integral precision of signal structures defined in Ganuza and Penalva (2010) and the ordering in Gentzkow and Kamenica (2016). Starting from a prior F_0, signal structure $\tilde{t}$ is more *integral precise* than signal t if the induced distribution of expected qualities $G(z)$ generated by $\tilde{t}$ is a mean-preserving spread of the one generated by t. In general, integral precision ordering is weaker than the likelihood ratio and other related orderings considered in the literature (see Ganuza and Penalva (2010) for references). We use this notion of garbling and more precise information in the proofs of this section.

direction as market size, as given by total quantity Q. It is worth noting that this general equilibrium effect has opposite impacts on consumer and producer surplus: the consumers prefer larger markets, whereas the producers prefer smaller markets. This observation will become relevant in Section 3.2 when considering optimal information disclosure.

Improved information, a mean-preserving spread in G, leads to an increased dispersion in prices. The impact of improved information on market size Q depends on the properties of the supply function. If it is linear, the increase in price dispersion has no effect on total output, so there is no change in Q. In contrast, when the supply function is convex (resp., concave), total output increases (resp., decreases) with price dispersion. More generally, the direction and magnitude of output change will depend on the shape of the supply function (i.e., convex or concave) and the magnitude of the changes in spread.

The following proposition summarizes these results.

Proposition 1. *An improvement in information quality, as given by a mean-preserving spread of G, has the following effects:*

i. *It increases (resp., decreases) total output if the supply function is convex (resp., concave).*

ii. *Consumer surplus changes in the same direction as total output.*

iii. *Producer surplus increases if total output does not increase.*

iv. *Total surplus increases.*

In particular, in the case of concave supply functions, consumers are better off with no information. There are some related results in the literature, though in different settings. For example, Schlee (1996) considers a single product monopoly seller in a vertically differentiated market. The quality of the good offered is exogenous and privately observed by the monopolist, who must choose the informativeness of a signal to be provided to consumers before observing the quality realization. It is shown that if the cost function is sufficiently convex, consumers are worse off ex ante with a more informative signal. Hoppe et al. (2011) consider a matching problem and show that under some conditions on the distribution of types, one of the sides (e.g., consumers) can be worse off by having a more precise information structure regarding the type of the other side.

Regarding producer surplus, there is an additional direct contribution of price dispersion to profits. So, total profits can still increase when output increases. While examples can be constructed where producer surplus decreases, for this to occur, the degree of convexity of supply needs to be very strong relative to the convexity of the profit function.

While improved information has ambiguous effects on consumer and producer surplus, it always increases total surplus. The intuition is as follows. In our model, for any distribution of qualities G, the unique competitive equilibrium also maximizes the sum of producer and consumer surplus, subject to that information structure. Now consider an alternative information structure $\tilde{G}$ that is more informative than G. Equipped with this information, a social planner could always replicate the original allocation, by ignoring the added information, so the original level of total surplus can still be achieved. But since the equilibrium corresponding to $\tilde{G}$ is different than the one for G, it must be that this improves total surplus. This result also implies that better information must benefit either producers or consumers, or both. In particular, average profits must rise when consumer surplus does not increase, as in the case of concave supply.

3.2 Optimal Information Disclosure

This section considers optimal information disclosure by a market designer, which we refer to as the planner. To motivate the analysis, we start with an example that captures in a stylized way some realistic features and helps motivate the necessity of information-pooling regions, which is a central feature of our finding for the optimal information mechanism. Moreover, the example highlights the differences between the optimal disclosure mechanism for sellers and consumers.

Example

Qualities $z \in [\underline{z}, \bar{z}]$ are distributed according to cdf $F(z)$. All firms inelastically supply $\bar{q}$ units provided price is above marginal cost $c > 0$. Note that because output is inelastically supplied, the only role of information is to exclude some low-quality producers from the market. This scenario might represent a market where retailers can acquire the good at a wholesale price c, at a limited capacity. For any minimum participation threshold z, let $Q(z) = (1 - F(z))\,\bar{q}$ denote total output, and $M(z)$ the average quality above a threshold z. Assume $P(Q(\underline{z}))+M(\underline{z}) < c < P(0)+M(\bar{z})$ to guarantee interior solutions.

Let z^c, z^o, z^p denote the optimal thresholds for consumers, an equal weights planner, and producers, respectively. As explained earlier, consumer surplus increases with output, so consumers are interested in maximizing output, subject to the participation constraint for producers being above the threshold

$$P(Q(z^c)) + M(z^c) \geq c.$$

Since this expression is strictly increasing in z^c, and given our previous assumption, it will bind for an interior choice.

An equal weights planner will exclude all producers that contribute negative value, and will thus choose

$$P(Q(z^o)) + z^o = c.$$

Finally, producers choose the threshold to maximize total profits. Differentiating with respect to z^p, the corresponding necessary condition is

$$P(Q(z^p)) + P'(Q(z^p))\, Q(z^p) + z^p = c.$$

The optimal threshold for producers thus equates the *marginal revenue* of the marginal firm to marginal cost. Since $P(Q(z)) + z$ is strictly increasing in z, it follows that $z^c < z^o < z^p$.

In order to implement the threshold z^c, pooling above z^c is needed. Intuitively, pooling the marginal firm with better firms allows increasing total participation, by lowering the threshold for the marginal firm. In contrast, to implement producers' preferred threshold z^p, all firms that are close to and below the marginal firm must be pooled with lower quality ones, to prevent them from participating. This can be achieved by pooling all firms below the threshold. This example shows the opposing incentives between consumers and producers in choosing thresholds and in their respective optimal disclosure policies.

General Theory

Information Structures

The planner's information is summarized by a distribution $F(z)$ of expected qualities across sellers with mean $\bar{z}$. This represents the maximal information that the planner could provide to buyers. Buyers have symmetric priors about seller quality. We first consider the extreme case where buyers have no information about firms, sharing a degenerate prior with mass 1 at mean quality $\bar{z}$. Results are then extended to non-degenerate priors in Section 3.3. Any partial revelation of information can be represented by a distribution $G \epsilon \mathcal{G}$, where $\mathcal{G}$ is the set of garblings or mean-preserving contractions of F.

Lemma 2. *The set $\mathcal{G}$ is convex and compact.*[13]

Proof. We use the definition that F is a mean-preserving spread of G if $\int f dG \geq \int f dF$ for all concave functions f. Convexity follows immediately. Finally, compactness follows since $\mathcal{G}$ is a closed subset of the set of probability measures. By the Banach–Alaoglu theorem – see Dunford and Schwartz (1988), Theorem 3.15 – the set of signed measures of bounded norm over a compact set is compact in the weak-* topology. It then follows that $\mathcal{G}$ is compact. □

Optimal Program

For any $G \in \mathcal{G}$, let $\bar{Q}(G)$ denote total output in the unique equilibrium.[14] Letting $0 \leq \gamma \leq 1$ denote the weight given to consumers, the planner's problem is

$$\max_{G \in \mathcal{G}} (1-\gamma) \int \pi \left(P\left(\bar{Q}(G)\right) + x\right) dG(x) + \gamma \int_0^{\bar{Q}(G)} \left(P(q) - P\left(\bar{Q}(G)\right)\right) dq, \tag{2}$$

where $P\left(\bar{Q}(G)\right)$ is equal to the price of a hypothetic good of quality zero, where

$$\bar{Q}(G) = \int S\left(P\left(\bar{Q}(G)\right) + z\right) dG(z), \tag{3}$$

[13] We consider the weak* topology of measures. Under this topology, the set of probability measures is compact.
[14] Given Lemma 1, the equilibrium is unique for any given G.

and $\pi(.) = qp - C(q)$ is the profit function of sellers which is maximized at $q = S(p)$.

The first term in (2) corresponds to producer surplus, and the second term to consumer surplus, as explained before. Equation (3) implicitly defines the unique equilibrium output as a function of G. We say that output Q is *attainable* if there exists some $G \in \mathcal{G}$ such that Q is the equilibrium output corresponding to information structure G. Let $\mathcal{Q}$ denote the set of attainable output levels.

Lemma 3. *The function $\bar{Q}(G)$ defined implicitly by (3) is continuous and the set $\mathcal{Q}$ is given by an interval $[Q_L, Q_H]$.*

From the previous lemma it follows easily that the objective function defined by (2) is continuous on the compact set $\mathcal{G}$, so it has a maximum. We now derive the properties of the optimal solution.

Changes in the information structure, as represented by G, have two effects on the planner's objective: a direct effect on expected profits and a general equilibrium effect, operating through the change in Q and p. To provide some intuition, consider a small mean-preserving spread of G around quality x. The direct effect will be an increase in profits of a magnitude that depends on the curvature of the profit function around $p + x$, $(1-\gamma)\pi''(p+x)$. The equilibrium effect will be given by a marginal change in prices dp with welfare effect

$$\left[(1-\gamma)\int \frac{\partial \pi(p+x)}{\partial p} dG(x) - \gamma \bar{Q}(G)\right] dp = (1-2\gamma)\bar{Q}(G)\, dp,$$

which follows from the envelope condition $\frac{\partial \pi(p+x)}{\partial p} = S(p+x)$. Thus the general equilibrium effect will affect firms and consumers in opposite directions and equal magnitudes. Moreover, since $dp = P'(Q)\, dQ$, its magnitude will vary directly with the intensity (and sign) of the output change dQ. This in turn depends on the curvature of the supply function $S''(p+x)$ around the point of the mean-preserving spread x. In particular, if the supply function is convex around this point, total output will increase and the general equilibrium effect will imply a transfer of utility from firms to consumers. A transfer in the opposite direction would occur if the supply function were concave at this point.

In standard Bayesian persuasion models, the objective function has the form $\int u(x)\, dG(x)$, where $G(x)$ is the posterior distribution induced by the signal structure and is thus linear in G.

This is also the case in our problem when considering the direct effect only, holding fixed total output Q and the implied price $p = P(Q)$. But because of the general equilibrium effect discussed above, total output will most likely change with G, introducing a nonlinearity in the optimization problem. As a result, we cannot use directly the methods developed in this literature. In order to leverage those results, we introduce a method for solving our problem in two stages. First, we solve for an optimal information structure in the subset $\mathcal{G}(Q)$ that delivers a given equilibrium output Q. As we show below, this problem has both an objective function and a constraint that are linear in G. We then optimize this result with respect to Q, in the set $\mathcal{Q}$. Consider the constrained optimization problem:

$$\begin{aligned} U(Q) &= \max_{G \in \mathcal{G}} (1-\gamma) \int \pi(P(Q)+z)\, dG(z) + \gamma \int_0^Q (P(q) - P(Q))\, dq \\ & \quad \text{subject to } Q = \int S(P(Q)+z)\, dG(z), \end{aligned} \tag{4}$$

where both the objective function and the constraint are linear in G. Note that if we set $Q = Q(G)$, where G is the solution to the original problem 2, and we solve the above problem, we should find the same optimal level for G. However, the linearity of the above problem can help us in finding properties that our optimal solution should satisfy. The corresponding Lagrangian function is given by

$$\begin{aligned} \mathcal{L}(Q,\lambda) &= \max_{G \in \mathcal{G}} (1-\gamma) \int \pi(P(Q)+z)\, dG(z) + \gamma \int_0^Q (P(q) - P(Q))\, dq \\ & \quad -\lambda \left(Q - \int S(P(Q)+z)\, dG(z) \right). \end{aligned} \tag{5}$$

We prove the following result:

Proposition 2. *Let $\bar{Q}(G)$ denote the unique total equilibrium output corresponding to information structure G.*

i. For any $Q_L \le Q \le Q_H$, there exists a solution G for the optimal constrained program (4) and the function $U(Q)$ is continuous.

ii. (Lagrange) For any $Q_L < Q < Q_H$ and any optimal solution $\bar{G}(Q)$, there exists $\lambda(Q)$ such that

$\bar{G}(Q)$ solves the Lagrangian 5 for $\lambda = \lambda(Q)$ and Q is the unique equilibrium for that information structure.

iii. *For $Q \in \{Q_H, Q_L\}$, exactly one of the following two statements are correct:*

   (a) *There exists $\lambda(Q) \in \mathbb{R}$ such that any optimal information structure $\bar{G}(Q)$ solves the Lagrangian 5 for $\lambda = \lambda(Q)$ and Q is the unique equilibrium for that information structure.*

   (b) *$U(Q) < \max_{Q' \in [Q_L, Q_H]} U(Q')$, i.e., Q cannot be an optimal quantity.*

Once this constrained problem is solved, we can find the optimal level of output Q^* by maximizing the continuous function $U(Q)$ in the compact set $[Q_L, Q_H]$.

We now characterize the optimal information structure when the value $Q^* \in (Q_L, Q_H)$ by using conditions 2 and 3a of Proposition 2. The value $\lambda(Q^*)$ can be obtained by differentiating $\mathcal{L}(Q, \lambda(Q))$ with respect to Q at $Q = Q^*$ and equating to zero. This gives

$$\lambda(Q^*) = \frac{(1-2\gamma) Q^* P'(Q^*)}{1 - P'(Q^*) \int S'(P(Q^*) + z) \, dG(z)}, \tag{6}$$

which, dividing both numerator and denominator by $Q^* P'(Q)$ and multiplying by $P(Q)$, can be rewritten as

$$\lambda(Q^*) = \frac{-(1-2\gamma) P(Q)}{\varepsilon_D + \varepsilon_S}, \tag{7}$$

where ε_D and ε_S are demand elasticity, in absolute value, and supply elasticity, respectively. It follows that λ is positive if and only if $\gamma > 1/2$. This captures the intuitive idea, discussed above, that increases in total output represent, at the margin, a transfer from firms to consumers, or vice versa. In addition, the strength of this transfer effect is dampened by demand and supply elasticities. This in turn implies that larger supply and demand elasticities reduce the importance of the general equilibrium effect, which intuitively should lead to more information provision. In the extreme, if demand is perfectly elastic, consumer surplus is zero regardless of the level of output so $G = F$, so all information is revealed.

Letting

$$V(z) = (1-\gamma) \pi (P(Q^*) + z) + \lambda S (P(Q^*) + z), \tag{8}$$

which includes only components of 5 which change by changing information given a fixed level of Q, the optimal information structure solves

$$\max_{G\in\mathcal{G}}\int V\left(z\right)dG\left(z\right),$$

which is a linear optimization problem. As in Kolotilin (2018), it follows from Jensen's inequality that full revelation is optimal when $V\left(z\right)$ is convex, and no revelation is optimal when it is concave. Noting that

$$V'\left(z\right)=\left(1-\gamma\right)S\left(P\left(Q\right)+z\right)+\lambda S'\left(P\left(Q\right)+z\right),$$

we can easily derive the following sufficient conditions for full revelation.

Proposition 3. *Full revelation is always optimal in the following cases:*

i. $\gamma=1/2$*;*

ii. $\gamma<1/2$ *and* S *is concave;*

iii. $\gamma>1/2$ *and* S *is convex; and*

iv. demand is infinitely elastic.

In addition, when $\gamma=1$*, full revelation is optimal only if* S *is convex.*

In all of these cases the implied function $V\left(z\right)$ is convex, after factoring in the corresponding sign of the multiplier λ. The first case confirms our previous result that full revelation is optimal when the planner maximizes total surplus. The second result follows intuitively from the fact that when S is concave, improved information decreases output, implying a transfer from consumers to firms, which is desirable as $\gamma<1/2$. Similarly, the last result follows from the fact that when S is convex, improved information increases output, implying a transfer from firms to consumers, which is desirable as $\gamma>1/2$. [15]

When demand is infinitely elastic, all consumer surplus is appropriated by firms regardless of the amount of information provided. Given the convexity of the profit function, total profits are

[15] This proposition holds also when $Q^*\in\{Q_L,Q_H\}$. Indeed, in part (2) of the proposition, full information implies that $Q^*=Q_L$, and in part (3) it implies that $Q^*=Q_H$.

maximized under full information. Note also that demand sensitivity seems to play an important role more generally. An examination of the Lagrange multiplier (6) suggests that it will be larger in absolute value when demand is steeper. Intuitively, the transfer between producers and consumers will be larger the steeper the demand curve is, as $P'(Q)\,Q$ measures the change in consumer surplus (and corresponding decrease in producer surplus) when total output increases.

Sufficient conditions for no revelation of information are harder to obtain. Because the first term in (8) is convex, the conditions needed for $V(z)$ to be concave seem to be stronger. In the extreme case when $\gamma = 1$, the sufficient conditions will hold when the supply function S is concave, which, as we found before, is the case where consumers are better off with no information. More generally, when $\gamma > 1/2$, the supply function has to be sufficiently concave relative to the profit function for no information to be optimal, while if $\gamma < 1/2$, the supply function has to be sufficiently convex.

When considering the question of information provision using certification criteria, an issue that often arises is how hard should the test be? As an example, eBay's increase in the requirements to qualify as eBay Top Rated Seller was an attempt to make the test harder to pass. An easy test allows creating differentiation at the lower end, while a harder one, at the upper end. So, where is information revelation more valuable? Our previous analysis suggests that more differentiation of firm qualities is of greater value in regions where the degree of convexity of $V(z)$ is stronger. In particular, when $V''(z)$ is increasing (resp., decreasing), we should expect full disclosure (resp., pooling) starting from a point z^* and pooling (resp., full disclosure) in the region below this point. These correspond to situations when $V(z)$ is concave-convex (resp., convex-concave). The next proposition provides conditions under which these properties hold.

Proposition 4. *Full disclosure up to some threshold z^* and complete pooling above is optimal in the following cases:*

i. $\gamma > 1/2$ *and* S''/S' *is decreasing,*

ii. $\gamma < 1/2$ *and* S''/S' *is increasing.*

Complete pooling up to some threshold z^* and full disclosure above is optimal in the following cases:

i. $\gamma > 1/2$ *and* S''/S' *is increasing,* *ii.* $\gamma < 1/2$ *and* S''/S' *is decreasing.*

The intuition for these results is as follows. A small increase in spread around z has a direct positive impact on expected profits that is proportional to $\pi''(P(Q)+z)$, the curvature of the profit function around this point. Likewise, it has an impact on total output and a transfer from producers to consumers that is proportional to $S''(P(Q)+z)$. This transfer is positive if S is convex at this point, and negative otherwise. The ratio $S''(z)/\pi''(z)$ measures the transfer relative to the profit increase resulting from this small increase in spread. The higher $S''(z)$ is relative to $\pi''(z)$ (lower in absolute value), the smaller the transfer (loss) is relative to the direct profit gain. In this case, it is optimal to provide information disclosure for higher values of this ratio. Our intuitive argument suggests that when $S''(z)/\pi''(z)$ is increasing (resp., decreasing), disclosure should occur in an upper interval (resp., lower interval). Note that this is precisely the case where $V(z)$ is concave-convex (resp., convex-concave).

As an example, consider the homogenous supply function $S(p) = p^{\alpha}$, where $S''(p)/S'(p) = (\alpha-1)p$. This is increasing for $\alpha > 1$ and decreasing otherwise. So for $\alpha > 1$, where the supply function is convex, there will be full disclosure (resp., pooling) followed by pooling (resp., full disclosure) when $\gamma < 1/2$ (resp., $\gamma > 1/2$). The reverse pattern occurs when $\alpha < 1$, where the supply function is concave. These two cases correspond, respectively, to elastic and inelastic supply.

In the realistic case where the supply function has a minimum strike price and an upper bound on output, there must be a convex region for lower values and one that is concave for higher ones. The example at the beginning of this section fits in this class of supply functions. This, in turn, implies that when $\gamma = 1$, $V(z)$ will be convex-concave, while in case $\gamma = 0$, it will be necessarily convex in the upper region, where S is concave. Under these conditions, consumers would prefer separation for lower quality levels (screening out bad quality products) and pooling at the top, the latter to maximize output. In contrast, producers would always want separation at the high end and possibly pooling of lower qualities.

We end this section considering the effect of asymmetries in the weights of consumers and producers. For those cases where full information is not optimal, we find that the region of pooling increases with the strength of the bias in the planner's preference for one group or the other.

Proposition 5. *Suppose the optimal disclosure policy is given by a threshold with pooling on one side of the threshold (above or below) and complete separation on the other side. Consider an increase in* γ*. If* $\gamma > 1/2$*, then the pooling region increases with* γ*, while if* $\gamma < 1/2$*, the pooling region decreases with* γ*.*

The intuition for this result is as follows. When $\gamma > 1/2$, it must be the case that S is concave in the pooling region; otherwise, there would be full disclosure, as stated in Proposition 3. Thus pooling takes place to mitigate the reduction in output from improved information and its negative impact on consumer surplus. The larger the weight of consumers, the larger this pooling region will be. When $\gamma < 1/2$, it must be that the supply function is convex in the pooling region, and pooling occurs precisely to mitigate the increase in output and its negative impact on producers. The lower the weight of producers (higher γ), the smaller this pooling region will be.

3.3 Buyer's Prior Information

In this section we extend our results to the case where buyers have non-degenerate priors, given as follows. There is a finite partition of sellers into N groups with respective shares $\alpha_j, j = 1, ..., N$. For all sellers in a group, buyers share symmetric information given by a Dirac prior on mean quality z_j^0. This could represent, for example, identical realizations for a finite set of ratings. For each of these groups, the planner's information can be represented by a distribution F_j of expected qualities across these sellers, with mean z_j^0. Any partial revelation of information can be represented by a distribution $G_j \epsilon \mathcal{G}_j$, where $\mathcal{G}_j$ is the set of garblings or mean-preserving contractions of F_j. This information structure implies a distribution $G = \sum_j \alpha_j G_j$ over expected qualities of sellers which refines the information of consumers up to the information held by the planner. Let $\mathcal{G}$ denote the set of distributions that can be obtained this way. The optimal problem is identical to (2), optimized over this set of distributions.

The constrained optimization problem we specified in (5) can be adapted to this case. We solve for the optimal disclosure policy G_j within each element of the information partition of buyers, holding fixed the vector of total output Q_j for each. Since the only connection between all of these planning problems is through aggregate output, holding it fixed makes the problem separable. Moreover, as total output is the sum of the output Q_j of all partitions, the multipliers λ_j are identical. In consequence, all properties derived above translate to each element of the partition. In particular, Propositions 3 and 4 as well as Proposition 5 hold.

4 Final Remarks

This paper considered the optimal design of quality ratings in markets with adverse selection. We first studied the problem of optimal rating design for a planner with a flexible objective function. We found that better information has opposing welfare effects on consumers and producers that could lead to limited disclosure. For example, in regions where the supply function is concave, pooling can mitigate the reduction in output from improved information and its negative impact on consumer surplus. Where the supply function is convex, pooling decreases total output and increases prices, which might have a positive impact on producers. For those cases where full information is not optimal, we found that the region of pooling increases with the asymmetry in the weights of the two groups in the objective function of the planner.

In a follow-up paper, we consider the more practical question of optimal ratings when the number of signals that the planner can provide is limited, Hopenhayn and Saeedi (2022). In that paper we characterize the constrained optimal ratings which have the form of interval partitions or rankings of quality. The paper also shows that an optimal unrestricted information structure can be well approximated with a small number of rankings.

Our analysis has been limited to case of homogenous consumer preference for quality. A natural extension is considering the case where these preferences are heterogenous as in the standard vertical differentiation model. In this setting, information not only helps reallocate market share to higher quality sellers but also improves the matching between goods' quality and consumers' preferences. This additional force will result in larger increases in total surplus as a result of improved information. But the impact on the distribution of the gains will depend on the response of equilibrium prices, as in our simplified setting. Other extensions worth considering are a more detailed modeling of entry, following results obtained in the empirical literature by Hui et al. (forthcoming). Finally, we have abstracted from moral hazard considerations, which can be important in some settings; exploring their impact on the design of optimal ratings is left to future research.

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5 Appendix. Proofs

Proof of Proposition 1

Proof. Assuming that G_1 and G_2 are distribution of expected qualities of sellers, if G_1 has improved information over G_2, then By definition of integral precision, it follows that G_1 second order stochastically dominates G_2 and it is mean preserving spread over G_2. Assume the corresponding levels of total market quantity are Q_1 and Q_2. Suppose the supply function $S(p)$ is convex and, by way of contradiction, $Q_1 < Q_2$. Let $p_1(z) = P(Q_1) + z$ denote the equilibrium price for a good of expected quality z, and define similarly $p_2(z)$. It follows immediately that $p_1(z) > p_2(z)$, since P is strictly decreasing. Therefore

$$\begin{aligned} Q_1 &= \int S(p_1(z))\, dG_1(z) \geq \int S(p_2(z))\, dG_1(z) \\ &\geq \int S(p_2(z))\, dG_2(z) = Q_2, \end{aligned}$$

where the first inequality follows from the monotonicity of the supply function and the second inequality follows from convexity of $S(p)$. The above contradicts the original hypothesis, proving that $Q_1 \geq Q_2$. The proof is similar for concave $S(p)$.

To show that total surplus increases with better information, we show that there exists a correspondence between competitive equilibria and allocations that maximize total surplus. Given a distribution of mean qualities $G(z)$, the problem of maximizing total surplus solves

$$S = \max_{q(z)} \int_0^Q P(x)\, dx + \int [zq(z) - C(q(z))]\, dG(z)$$

subject to

$$Q = \int q(z)\, dG(z).$$

The first order conditions for the choice of $q(z)$ are

$$z - C'(q(z)) + \lambda = 0 \tag{9}$$

and this holds for all points in the support of G, where the Lagrange multiplier of the constraint $\lambda = P(Q)$. Substituting in (9) and letting $p(z) = P(Q) + z$ implies $p(z) = C'(q(z))$, which is the condition defining the profit maximizing output $q(z)$ in the competitive equilibrium. Hence the allocation $q(z)$ and the prices $p(z)$ are the ones that correspond to the unique competitive equilibrium.

Consider now a distribution of expected qualities $\tilde{G}$ corresponding to a better information system than G so it is a mean-preserving spread of G. Following the characterization in Rothschild and Stiglitz (1970), there exists a garbling of signals that generates G from $\tilde{G}$. This means that a social planner could ignore the additional information contained in $\tilde{G}$ and reproduce the quantity-weighted distribution of average qualities corresponding to the optimal allocation under G and thus the same value. While this allocation is feasible under $\tilde{G}$, it is not optimal. This follows from the easily verified property that the unique competitive equilibrium (which as argued is also the optimal allocation) differs across these two information structures. □

Proof of Lemma 3

Proof. The set of attainable output levels is bounded, as z is bounded above and output for any G is bounded by the equilibrium value corresponding to a distribution with point mass at the highest z. We next show that the mapping $\bar{Q}(G)$ defined implicitly by (3) is continuous. Let $G_n \in \mathcal{G}$ converge to G and let Q_n be the corresponding sequence of total output. Assume that this sequence converges to $Q \geq 0$ (otherwise consider a subsequence). It suffices to show that $\int S(P(Q_n) + z)\, dG_n \to \int S(P(Q) + z)\, dG$. By the triangle inequality,

$$\begin{aligned} \left| \int S(P(Q_n) + z)\, dG_n - \int S(P(Q) + z)\, dG \right| \quad \leq \\ \int |S(P(Q_n) + z) - S(P(Q) + z)|\, dG_n \\ + \left| \int S(P(Q) + z)\, dG_n - \int S(P(Q) + z)\, dG \right| \end{aligned}$$

The first term converges to zero, since Q_n is bounded and thus $S(P(Q_n)+z) - S(P(Q)+z)$ is uniformly continuous. The second term converges to zero, since the function defined by $S(P(Q)+z)$ is continuous in z. Having proved that the function $\bar{Q}(G)$ is continuous, it follows from Lemma 2 that $\mathcal{Q}$ is compact, so there are minimal and maximal attainable output levels Q_L and Q_H, respectively. Let G_L and G_H be the corresponding quality distributions and define $G(\alpha) = \alpha G_H + (1-\alpha) G_L \in \mathcal{G}$ by Lemma 2, and let $\phi(\alpha) = \bar{Q}(G(\alpha))$. This is a continuous function on $[0,1]$, and $\phi(0) = Q_L$ and $\phi(1) = Q_H$. By the intermediate value theorem, $\phi([0,1]) = [Q_L, Q_H]$. □

Proof of Proposition 2

To prove this proposition we first state and prove the following two Lemmas. We will be using the following notation throughout of the proof. Define the following two correspondences:

$$\mathcal{G}^{+}(Q) = \left\{ G \in \mathcal{G} | Q \geq \int S(P(Q)+z)\, dG \right\}$$
$$\mathcal{G}^{-}(Q) = \left\{ G \in \mathcal{G} | Q \leq \int S(P(Q)+z)\, dG \right\}$$

Lemma 4. *Assume the supply function S is uniformly continuous. The correspondences $\mathcal{G}^{+}(Q)$ and $\mathcal{G}^{-}(Q)$ are continuous at Q_L and $Q_{H,}$ respectively.*

Proof. We proof continuity of $\mathcal{G}^{+}(Q)$. The proof for the other correspondence is analogous. Consider first upper hemi-continuity. Take a sequence $Q_n \to Q_L$ and $G_n \in \mathcal{G}^{+}(Q_n)$, where $G_n \to G$.

$$\begin{aligned}
&\left| \int S(P(Q_n)+z)\, dG_n(z) - \int S(P(Q_L)+z)\, dG(z) \right| \\
&\leq \left| \int S(P(Q_n)+z)\, dG_n(z) - \int S(P(Q_L)+z)\, dG_n(z) \right| \\
&+ \left| \int S(P(Q_L)+z)\, dG_n(z) - \int S(P(Q_L)+z)\, dG(z) \right|
\end{aligned}$$

The second term converges by the definition of convergence of measures since S is continuous. Regarding the first term, uniform continuity implies that for any ε there exists N such that for all

$n \geq N$:

$$\left| \int S\left(P\left(Q_n\right)+z\right) dG_n\left(z\right) - \int S\left(P\left(Q_L\right)+z\right) dG_n\left(z\right) \right|$$
$$< \varepsilon \int dG_n\left(z\right) = \varepsilon$$

so the first term also converges to zero.

Lower hemi-continuity at $G \in \mathcal{G}^+\left(Q_L\right)$ follows immediately since

$$Q \geq Q_L \geq \int S\left(P\left(Q_L\right)+z\right) dG \geq \int S\left(P\left(Q\right)+z\right) dG$$

so $G \in \mathcal{G}^+\left(Q\right)$. □

Lemma 5. *Assume the profit function $\pi\left(p\right)$ and the supply function $S\left(p\right)$ are uniformly continuous. Then the function*

$$\begin{aligned} \zeta\left(Q,\lambda,G\right) &= (1-\gamma)\int \pi\left(P\left(Q\right)+z\right) dG + \gamma\int_0^Q \left(P\left(q\right)-P\left(Q\right)\right) dq \\ &\quad -\lambda\left(Q - \int S\left(P\left(Q\right)+z\right) dG\right) \end{aligned}$$

is jointly continuous.

Proof. For the first and last term, the proof follows the same lines as the proof of upper hemi-continuity in the previous Lemma. The second term is obviously continuous in Q as the inverse demand function $P\left(Q\right)$ is so. □

Now using the above two lemmas we continue the Proof of Proposition 2.

1. For any $Q \in \left[Q_L, Q_H\right]$, the set of measures G satisfying the constraint in (4) is nonempty and closed, as defined by a linear functional on $\mathcal{G}$, and thus compact. The optimized functional is linear and thus continuous, so there exists a maximum. Finally, since the objective is continuous and the constraint correspondence is a compact set of probability measures and continuous in Q, Berge's Maximum Theorem implies that the maximized value $U\left(Q\right)$ is continuous in Q.

2. We will show that the solution of the constrained problem (4) coincides with the solution of a relaxed problem when we use either $\mathcal{G}^+(Q)$ or $\mathcal{G}^-(Q)$. To simplify notation, suppress the dependence of $\mathcal{G}^+$ and $\mathcal{G}^-$ on Q in what follows. It is immediate to verify that these relaxed problems satisfy the requirements of Theorem 1 in section 8.3 of Luenberger (1997). As a result Lagrange multipliers must exist associated with the inequality constraint.

To show this, we first show that since $Q \in (Q_L, Q_H)$ the inequalities defining $\mathcal{G}^+$ and $\mathcal{G}^-$ hold strictly for some distributions in the corresponding sets. To see this, let G_L be the probability measure that satisfies

$$Q_L = \int S\left(P\left(Q_L\right) + z\right) dG_L$$

and let G_H be defined accordingly for Q_H. Since $Q > Q_L$ and $P(\cdot)$ is decreasing, we must have

$$Q > Q_L = \int S\left(P\left(Q_L\right) + z\right) dG_L \geq \int S\left(P\left(Q\right) + z\right) dG_L.$$

Hence $G_L \in \mathcal{G}$ and continuity of the function $\Gamma(G) = \int S(P(Q) + z)\, dG$ with respect to G implies that $G_L \in \text{int}(\mathcal{G}^+)$. Similarly, since $Q < Q_H$, we have

$$Q < Q_H = \int S\left(P\left(Q_H\right) + z\right) dG_H \leq \int S\left(P\left(Q\right) + z\right) dG_H.$$

As a result, $G_H \in \text{int}(\mathcal{G}^-)$.

Next, letting $\Pi(G, Q) = (1-\gamma)\int \pi(P(Q)+z)\, dG + \gamma \int_0^Q [P(q) - P(Q)]\, dq$, we show that

$$\begin{aligned}\mu_0 &= \max_{G \in \mathcal{G}^+ \cap \mathcal{G}^-} \Pi(G, Q) \\ &= \min\left\{\max_{G \in \mathcal{G}^+} \Pi(G, Q), \max_{G \in \mathcal{G}^-} \Pi(G, Q)\right\}\end{aligned}$$

Given the definition of μ_0, obviously it must be that $\mu_0 \leq \max_{G \in \mathcal{G}^+} \Pi(G, Q), \mu_0 \leq \max_{G \in \mathcal{G}^-} \Pi(G, Q)$. Now, suppose to the contrary that $\mu_0 < \max_{G \in \mathcal{G}^+} \Pi(G, Q), \max_{G \in \mathcal{G}^-} \Pi(G, Q)$. If $\Pi(G^+) = \max_{G \in \mathcal{G}^+} \Pi(G, Q)$ and $\Pi(G^-) = \max_{G \in \mathcal{G}^-} \Pi(G, Q)$, there must exist $\lambda \in [0, 1]$ such that $\lambda G^+ +$

$(1-\lambda)\,G^{-} \in \mathcal{G}^{+} \cap \mathcal{G}^{-}$. This implies that

$$\begin{aligned}\mu_0 &\geq \Pi\left(\lambda G^{+} + (1-\lambda)\,G^{-}\right) \\ &= \lambda \Pi\left(G^{+}\right) + (1-\lambda)\,\Pi\left(G^{-}\right) \\ &\geq \min\left\{\max_{G\in\mathcal{G}^{+}} \Pi\,(G,Q)\,,\max_{G\in\mathcal{G}^{-}} \Pi\,(G,Q)\right\}\end{aligned}$$

which is a contradiction.

Now, suppose that $\mu_0 = \max_{G\in\mathcal{G}^{+}} \Pi\,(G,Q)$. Applying Luenberger's theorem to the latter optimization, we have that $\lambda \leq 0$ exists such that

$$\mu_0 = \max_{G\in\mathcal{G}} \Pi\,(G,Q) - \lambda\left\{Q - \int S\,(P\,(Q)+z)\,dG\right\}$$

The case with $\mu_0 = \max_{G\in\mathcal{G}^{-}} \Pi\,(G,Q)$ is similar.

3. We prove the claim for $Q = Q_L$. The proof for the case $Q = Q_H$ follows almost identically.

Note that at $Q = Q_L$, the inequality defining $\mathcal{G}^{-}$ holds strictly for some G. This is because at G_H, we have

$$Q_L < Q_H = \int S\,(P\,(Q_H)+z)\,dG_H < \int S\,(P\,(Q_L)+z)\,dG_H.$$

Therefore, if $\mu_0 \equiv U\,(Q_L) = \max_{G\in\mathcal{G}^{-}} \Pi\,(G,Q_L)$, then we can apply Luenberger's theorem and show that $\lambda\,(Q_L)$ exists. Otherwise, we must have that

$$\mu_0 = \max_{G\in\mathcal{G}^{+}} \Pi\,(G,Q_L) < \max_{G\in\mathcal{G}^{-}} \Pi\,(G,Q_L)$$

The above also implies that for values of Q close enough to Q_L, the same property holds. Hence, for such values of $Q > Q_L$, $\lambda\,(Q) \leq 0$.

Next, suppose that along some subsequence $Q_i \to Q_L$ the corresponding Lagrange multipliers $\lambda_i \to \lambda_L$, where $0 > \lambda_L > -\infty$. For any $\tilde{G} \in \mathcal{G}$,

$$\Pi\left(\tilde{G},Q_i\right) - \lambda\,(Q_i)\left(Q_i - \int S\,(P\,(Q_i)+z)\,d\tilde{G}\,(z)\right) \leq \mathcal{L}\,(Q_i,\lambda\,(Q_i)) \qquad (10)$$

By Lemma 5 and applying Berge's theorem of the maximum, Taking limits on both sides as $Q_i \to Q_L$ and $\lambda(Q_i) \to \lambda_{L,}$ applying Berge's theorem of the maximum, $\mathcal{L}(Q_i, \lambda(Q_i)) \to \mathcal{L}(Q_L, \lambda_L)$. Taking limits on the left hand side of (10), it follows that $\Pi\left(\tilde{G}, Q_L\right) - \lambda_L\left(Q_L - \int S(P(Q_L) + z)\, d\tilde{G}\right) \leq \mathcal{L}(Q_L, \lambda_L)$. Without loss of generality we can choose the sequence $\{Q_i\}$ so that $Q_L < Q_i < Q_H$, by part (1) of this Proposition it follows that $U(Q_i) = \mathcal{L}(Q_i, \lambda_i)$. Finally, using Lemma 4 and Lemma 5, and applying again Berge's maximum theorem, $U(Q_i) \to U(Q_L)$, so

$$\Pi\left(\tilde{G}, Q_L\right) - \lambda_L\left(Q_L - \int S(P(Q_L) + z)\, d\tilde{G}\right) \leq U(Q_L)$$

with equality when $\tilde{G} \in G(Q_L)$.

Suppose to the contrary that the sequence $\lambda(Q_i)$ converge to $-\infty$ as $\lambda(Q_i) \leq 0$. Using Envelope theorem we have the following for all $Q > Q_L$:

$$U'(Q) = (1-\gamma)\int \pi'(P(Q) + z)\, P'(Q)\, dG - \gamma Q P(Q) - \lambda\left\{1 - \int S'(P(Q) + z)\, P'(Q)\, dG\right\}$$

The first two terms of the above statements are bounded. For the third term, the expression in the bracket is always positive and $-\lambda \to \infty$. This results in $U'(Q) > 0$ as $Q \to Q_L$ and thus Q_L cannot be optimal. The same argument holds for $Q = Q_H$.

Proof of Proposition 4

Proof. Following the first condition given in Proposition 3 part (i) in Kolotilin (2018), full disclosure up to some threshold z^* and complete pooling above is optimal when $V''(z)$ changes sign from positive to negative. Note that $V''(z)$ has the same sign as

$$\frac{V''(z)}{\pi''(z)} = (1-\gamma) + \lambda\frac{S''(z)}{\pi''(z)}.$$

So when either of the two first conditions given in the proposition holds, then $V''(z)$ can be always positive, always negative, or switch sign from positive to negative. If it is always positive, full disclosure is optimal, and if it is always negative, no disclosure is optimal. In these two cases z^* is at an extreme. When $V''(z)$ changes sign, z^* is an interior point. In this case, there is an interval with full disclosure followed by no disclosure.

The proof of the second part is analogous to the first one, using instead the second condition in Proposition 3 part (i) in Kolotilin (2018). $\square$

Proof of Proposition 5

To prove this proposition, we first need to state the following two lemmas.

Lemma 6. *1) Let $Q(z)$ be the equilibrium output for an upper interval disclosure policy with threshold z. Let $m_L(z)$ denote the conditional mean below z (the pooling interval), $p_L = P(Q(z) + m_L)$ and $p = P(Q(z) + z)$. Then $Q'(z)$ has the same sign as*

$$S'(p_L)(p - p_L) - (S(p) - S(p_L)).$$

2) Let $Q(z)$ be the equilibrium output for a lower interval disclosure policy with threshold z. Let $m_H(z)$ denote the conditional mean above z (the pooling interval), $p_H = P(Q(z) + m_H)$ and $p = P(Q(z) + z)$. Then $Q'(z)$ has the same sign as

$$S'(p_H)(p_H - p) - (S(p_H) - S(p)).$$

Proof. Consider the upper interval disclosure with threshold z. Equilibrium output $Q(z)$ is the solution to

$$Q(z) = F(z) S(P(Q(z)) + m_L(z)) + \int_z S(P(Q(z)) + s) dF(s),$$

where $m_L(z)$ is the conditional mean below the threshold z (the pooling interval). Differentiating with respect to z,

$$Q'(z) = f(z) \frac{S'(P(Q(z)) + m_L(z))(z - m_L(z)) - (S(P(Q(z)) + z) - S(P(Q(z)) + m_L(z)))}{1 - P'(Q(z)) \left(F(z) S'(P(Q(z)) + m_L(z)) + \int_z S'(P(Q(z)) + s) dF(s)\right)}.$$

The denominator is positive, so the sign of $Q'(z)$ is equal to the sign of the numerator:

$$S'(P(Q(z)) + m_L(z))(z - m_L(z)) - (S(P(Q(z)) + z) - S(P(Q(z)) + m_L(z))).$$

The proof of the second part follows similar calculations. □

The following lemma gives the conditions that determine the sign of the above expressions.

Lemma 7. *Consider an optimal disclosure policy that is given by a threshold with pooling on one side of the threshold (above or below) and complete separation on the other side. Then a marginal increase in the pooling region will increase total output (resp. decrease total output) when* $\gamma > 1/2$ *(resp. when* $\gamma < 1/2$*).*

Proof. Consider first the case where z corresponds to the threshold of a lower disclosure interval. Let $m_H(z)$ denote the mean above z (the pooling interval), $p_H = P(Q(z) + m_H)$ and $p = P(Q(z) + z)$. Following Kolotilin (2018), $V(m_H(z)) - V(z) - (m_H(z) - z)V'(m_H(z)) = 0$. Since $V(s) = \pi(P(Q(z)) + s) + \lambda S(P(Q(z)) + s)$ and the first term is convex, for this equality to hold it is necessary that

$$\lambda[S(P(Q(z)) + m_H(z)) - S(P(Q(z)) + z)] > \lambda[(m_H(z) - z)S'(P(Q(z)) + m_H(z))].$$

For $\gamma > 1/2$, $\lambda > 0$, so this implies that $S(p_H) - S(p) > (p_H - p)S'(p_H)$, and by Lemma 6, it follows that $Q'(z) < 0$. An increase in the pooling region corresponds to a decrease in z, so total output increases. The reverse is obviously true when $\gamma < 1/2$.

Consider now the case where z corresponds to the threshold of an upper disclosure interval. Let $m_L(z)$ denote the mean above z (the pooling interval), $p_L = P(Q(z) + m_L)$ and $p = P(Q(z) + z)$. Following Kolotilin (2018), $V(z) - V(m_L(z)) - (z - m_L(z))V'(m_L(z)) = 0$. Since $V(s) = \pi(P(Q(z)) + s) + \lambda S(P(Q(z)) + s)$ and the first term is convex, for the equality to hold it is necessary that

$$\lambda[S(P(Q(z)) + m_H(z)) - S(P(Q(z)) + z)] < \lambda[(m_H(z) - z)S'(P(Q(z)) + m_H(z))].$$

For $\gamma > 1/2$, $\lambda > 0$, so this implies that $S(p_H) - S(p) < (p_H - p) S'(p_H)$, and by Lemma 6, it follows that $Q'(z) > 0$. An increase in the pooling region corresponds to an increase in z, so total output increases. The reverse is obviously true when $\gamma < 1/2$. □

Now using the above two lemmas, we continue the proof of Proposition 5.

Proof. Consider first the setting where the optimal policy is a left disclosure interval, as in Proposition 4.

$$\begin{aligned} W(z,\gamma) &= (1-\gamma)\left[\int_0^z \pi(P(Q(z))+s)\,dF(s) + (1-F(z))\,\pi(P(Q(z))+m_H)\right] \\ &\quad + \gamma\int_0^{Q(z)} (p - P(Q(z)))\,dz, \end{aligned}$$

where $Q(z)$ is the equilibrium output under this policy.

$$\frac{\partial W}{\partial z} = (1-\gamma) f(z)(\pi(z) - \pi(m_H)) + (1-\gamma) f(z)(m_H - z)\pi'(m_H) + (1-2\gamma) Q(z) P'(Q(z)) Q'(z)$$

Taking derivative with respect to γ,

$$\begin{aligned} \frac{\partial^2 W}{\partial z \partial \gamma} &= -f(z)\left[(\pi(z) - \pi(m_H)) + (m_H - z)\pi'(m_H)\right] - 2Q(z) P'(Q(z)) Q'(z) \\ &= -\frac{(1-\gamma)}{1-\gamma} f(z)\left[(\pi(z) - \pi(m_H)) + (m_H - z)\pi'(m_H)\right] - \frac{2(1-\gamma)}{1-\gamma} Q(z) P'(Q(z)) Q'(z) \\ &= 0 - \frac{1}{1-\gamma} Q(z) P'(Q(z)) Q'(z). \end{aligned}$$

When $\gamma > 1/2$, Lemma 7 implies that $Q'(z) < 0$, so the cross partial is negative and the pooling region increases with γ. When $\gamma < 1/2$, $Q'(z) > 0$, so the cross partial is positive and the pooling region decreases with γ.

Consider next the case where the optimal policy is a right disclosure interval, as in Proposition

4:

$$
\begin{aligned}
W(z,\gamma) &= (1-\gamma)\left[F(z)\,\pi\left(P\left(Q(z)\right)+m_L\right)+\int_z \pi\left(P\left(Q(z)\right)+s\right)dF(s)\right] \\
&\quad +\gamma\int_0^{Q(z)}\left(p-P\left(Q(z)\right)\right)dz,
\end{aligned}
$$

where $Q(z)$ is the equilibrium output under this policy.

$$
\frac{\partial W}{\partial z} = (1-\gamma)\,f(z)\left(\pi(m_L)-\pi(z)\right)+(1-\gamma)\,f(z)\,(z-m_L)\,\pi'(m_L)+(1-2\gamma)\,Q(z)\,P'\left(Q(z)\right)Q'(z)
$$

Taking derivative with respect to γ,

$$
\begin{aligned}
\frac{\partial^2 W}{\partial z\partial\gamma} &= -f(z)\left[\left(\pi(m_L)-\pi(z)\right)-(z-m_L)\,\pi'(m_L)\right]+-2Q(z)\,P'\left(Q(z)\right)Q'(z) \\
&= -\frac{1}{1-\gamma}Q(z)\,P'\left(Q(z)\right)Q'(z)\,.
\end{aligned}
$$

When $\gamma > 1/2$, 7 implies that $Q'(z) > 0$, so the cross partial is positive and the pooling region increases with γ. When $\gamma < 1/2$, $Q'(z) < 0$, so the cross partial is negative and the pooling region decreases with γ. □