Basics -- 2

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Carnegie Mellon

Review: Logarithms, Powers and Exponentials

Let b be any real greater than 1 and let x be any real greater than 0. The logarithm to the base b of x, denoted $\log_b x$ is defined to be the number y such that

$$b^y = x$$

$\log_b 1 = 0$
$\log_b x > 0$ if $x > 1$
$\log_b b = 1$
$\log_b x < 0$ if $0 < x < 1$

Any logarithmic function is a monotone increasing function of its argument, that is

$$\log_b x_1 > \log_b x_2 \text{ provided that } x_1 > x_2$$

Doubling the argument increases the base 2 logarithm by 1. That is,

$$\log_2 2x = (\log_2 x) + 1$$

Why?

$$2^{(\log_2 x) + 1} = 2^{\log_2 x} \cdot 2^1 = 2x$$

$$2^{\log_2 2x} = 2x$$

Review: Logarithms

$$\log_b (x_1 \cdot x_2) = \log_b x_1 + \log_b (x_2)$$

$$\log_b (x_1 / x_2) = \log_b x_1 - \log_b x_2$$

$$\log_b x^c = c \cdot \log_b x$$

Suppose *a* and *b* are both greater than 1, what is the relation of $\log_a x$ to $\log_b x$?

This is a constant. So, any two logarithmic functions differ only by a constant factor.

Since $x = a^{\log_a x}$

$$\log_b x = \log_b(a^{\log_a x})$$

$$= \log_a x \cdot \log_b a$$

For example, suppose we know that an algorithm executes lg x instructions, where x is the size of the input.

$$\lg x = \lg(10^{\log_{10} x})$$

$$= \log_{10} x \cdot \lg 10$$

$$= \log_{10} x * \sim 3.32$$

The number of bits in the usual binary notation for the positive integer N is $\lfloor \mathrm{Lg} N \rfloor + 1$.

For example, how many bits are required to represent

41	**0**	**1**	**0**	**1**	**0**	**0**	**1**	**6 bits**	$\lfloor \mathrm{Lg} 41 \rfloor + 1 = 5+1 = 6$
	64	32	16	8	4	2	1		
3					**0**	**1**	**1**	**2 bits**	$\lfloor \mathrm{Lg} 3 \rfloor + 1 = 1+1 = 2$
					4	2	1		

The number of digits in the usual base 10 notation for the positive integer N is $\lfloor \text{Log}_{10} N \rfloor + 1$.

For example, how many digits are required to represent 31?

31   3 1   2 digits $\lfloor \text{Log}_{10} 31 \rfloor + 1 =$ $1+1 = 2$

10 1

Any function from reals to reals of the form $g(x) = x^{\alpha}$ for some constant $\alpha > 0$ is called a simple power. Any simple power is an increasing function of its argument.

Examples:

x^2, x^3 and $x^{1/3}$ are simple powers

An exponential function is one of the form $h(x) = c^x$ for some constant $c > 1$.

Examples:

2^x and 100^x are exponential functions of x

Dominates

- Let f and g be functions from reals to reals. f dominates g if the ratio $f(n) / g(n)$ increases without bound as n increases without bound. In other words, if for any $c > 0$ there is an $n_0 > 0$ such that $f(n) > c \cdot g(n)$ for all $n > n_0$.

- Examples: $f(n) = n^2$ dominates $g(n) = 2n$ since for any c $n^2 > c \cdot 2n$ whenever $n > 2c$.

- $f(n) = 10n$ does not dominate $g(n) = 2n$ since the ratio of $f(n) / g(n)$ is never larger than 5.

Theorem:

Any exponential function dominates any simple power, any simple power dominates any logarithmic function.

Let N be the set of nonnegative integers {0,1,…}. Let R be the set of real numbers and let R^* be the set of nonnegative real numbers.

Let g be a function from N to R^*. Then O(g) is the set of all functions f from N to R^* such that, for some constants $c > 0$ and $N_0 \geq 0$.

$$f(N) \leq c \cdot g(N) \text{ for all } N \geq N_0 .$$

In other words, $f \in O(g)$ if the value of f is bounded from above by a fixed multiple of the value of g for all sufficiently large values of the argument.

Examples:

For any f it is the case that $f \in O(f)$.
Any constant multiple of f is in $O(f)$.
$F(n) = 13n + 7$ is in $O(n)$.

Why?

$$13n + 7 \leq 14n \text{ for } n \geq 7$$

So the definition is satisfied with $c = 14$, $n_0 = 7$.

$$1000n \in O(.0001n^2)$$

Why?

Let $c = 10^7$ and $n_0 = 0$ in the definition of O().

$$f(n) = n$$

$$f(n) \in O(n^2)$$

Little o

For any function g, o(g) is the set of all functions that are dominated by g. That is, the set of all functions f such that for each constant $c > 0$ there is an $n_0 > 0$ such that

$$f(n) < c \cdot g(n) \text{ for all } n > n_0.$$

Examples:

Let $f(n) = n$ and $g(n) = n^2$ then
$f(n) \in o(g(n))$

Let $f(n) = n^2$ and $g(n) = 2^n$ then
$f(n) \in o(g(n))$

Theorem: Growth Rates

1. The power n^{α} is in $O(n^{\beta})$ if and only if $\alpha \le \beta$ ($\alpha, \beta > 0$); and n^{α} is in $o(n^{\beta})$ if and only if $\alpha < \beta$.

Examples:	Intuitively
n $\in O(n^3)$	$n <= n^3$
n $\in o(n^3)$	$n < n^3$

2. $\log_b n \in o(n^{\alpha})$ for any b and α.

Examples:

$$\log_{10} n \in o(n)$$

$$\log_2 n \in o(n^{1/2})$$

3. $n^{\alpha} \in o(c^{n})$ **for any** $\alpha > 0$ **and** $c > 1$**.**

Examples:

$n^{2} \in o(4^{n})$

$n^{100} \in o(2^{n})$

4. $\log_a n \in O(\log_b n)$ **for any** a **and** b**.**

$\log_2 n \in O(\log_{10} n)$

$\log_{10} n \in O(\log_2 n)$

5. $c^n \in O(d^n)$ if and only if $c \leq d$, and $c^n \in o(d^n)$ if and only if $c < d$.

Examples:

$3^n \in O(4^n)$

$3^n \in o(4^n)$

6. Any constant function *f(n)* = *c* is in *O*(1).

For example:

A 32-bit add instruction O(1).

Big-O only provides an upper bound.

For example:

$$\mathbf{17}n^{2} \in \mathbf{O}(n^{2}) \text{ but}$$
$$\mathbf{17}n^{2} \in \mathbf{O}(n^{37})$$
$$\mathbf{17}n^{2} \in \mathbf{O}(2^{n})$$

Big Omega (Big- Ω):

Big-Ω notation is exactly the converse of Big-O notation; $f \in \Omega(g)$ if and only if $g \in O(f)$.

$f \in O(g)$ implies that f grows at most as quickly as g.
$f \in \Omega(g)$ implies that f grows at least as quickly as g.

Examples:

let $f(n) = n$

$f(n) \in O(n^2)$

$n^2 \in \Omega(n)$

Big theta (Big θ):

$\theta(f) = O(f) \cap \Omega(f)$

Example:

Let $f(n) = 4n$ **then**

$f(n) \in O(n)$

$f(n) \in \Omega(n)$

so

$f(n) \in \theta(n)$

The set of functions $\theta(f)$ **is the order of** f**.**

A Quiz

Suppose we use a phone book to look up a number in the standard way. Let T(n) be the number of operations (comparisons) required.

In the worst case T(n) $\in$ ***O(Log N).***

True or False:

Also, in the worst case,

T(n) $\in$ O(n) ______

T(n) $\in$ θ(n) ______

T(n) $\in$ Ω(n) ______

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Also, in the worst case,

T(n) $\in$ ***O(n)*** ***True***

T(n) $\in$ $\boldsymbol{\theta}$***(n)*** ***False***

T(n) $\in$ $\boldsymbol{\Omega}$***(n)*** ***False***

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T(n) $\in$ $\boldsymbol{\Omega}$***(n)*** _____

A Quiz

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True or False:

Also, in the best case,

T(n) $\in$ O(n)   True

T(n) $\in$ θ(n)   False

T(n) $\in$ Ω(n)   False

A Quiz

Suppose we want to delete the last item on a singly linked list. Let T(n) be the number of operations (comparisons) required.

There are no cases to consider: T(n) $\in$ ***O(n).***

True or False:

$$T(n) \in O(Lg\ n)$$
$$T(n) \in \theta(n)$$
$$T(n) \in \Omega(Lg\ n)$$

A Quiz

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There are no cases to consider: T(n) $\in$ ***O(n).***

True or False:

$T(n) \in O(Lg\ n)$ ***False***
$T(n) \in \theta(n)$ ***True***
$T(n) \in \Omega(Lg\ n)$ ***True***

Remember

When working with Big O, Big θ and Big Ω, be sure to always consider only large n.

In addition, pin the case down first and then consider Big O, Big θ, and Big Ω.

Lastly, remember that sometimes "case" does not apply.

Algorithms and Problems

In this class, we will mostly be analyzing algorithms (counting operations) in terms of Big O, Big θ and Big Ω.

Problems may also be analyzed. The lower bound for a particular problem is the worst case running time of the fastest algorithm that solves that problem.

Later, we will look at an argument that comparison based sorting is $\Omega(n \text{ Log } n)$. What does that mean?