15-122: Principles of Imperative Computation

Recitation 3: Function Family Reunion   Nivedita Chopra, Andrew Benson

Big-O definition

The definition of big-O has a lot of mathematical symbols in it, and so can be very confusing at first. Let's familiarize ourselves with the formal definition and get an intuition behind what it's saying.

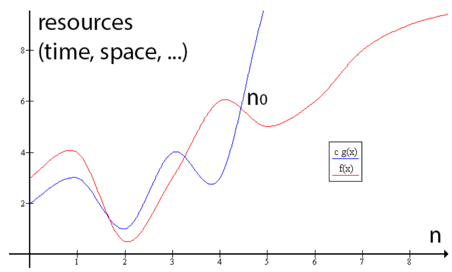
$O(g(n))$ is a set of functions, where $f(n) \in O(g(n))$ if and only if:

there is some ______________________________ and some _________________________________

such that for all ______________________________, ____.

Although it isn't technically correct set notation, it is also common to write $f(n) = O(g(n))$.

Big-O intuition



To the left of n_0, the functions can do anything.
To its right, $c * g(n)$ is always greater than or equal to $f(n)$.

Intuitively, $O(g(n))$ is the set of all functions that $g(n)$ can outpace in the long run (with the help of a constant multiplier). For example, n^2 eventually outpaces $3n\log(n) + 5n$, so $3n\log(n) + 5n \in O(n^2)$. Because we only care about long run behavior, we generally can discard constants and can consider only the most significant term in a function.

There are actually infinitely many functions that are in $O(g(n))$: If $f(n) \in O(g(n))$, then $\frac{1}{2}f(n) \in O(g(n))$ and $\frac{1}{4}f(n) \in O(g(n))$ and $2f(n) \in O(g(n))$. In general, for any constants k_1, k_2, $k_1 * f(n) + k_2 \in O(g(n))$.

Checkpoint 0

Rank these big-O sets from left to right such that every big-O is a subset of everything to the right of it. (For instance, $O(n)$ goes farther to the left than $O(n!)$ because $O(n) \subset O(n!)$.) If two sets are the same, put them on top of each other.

$O(n!)$ $O(n)$ $O(4)$ $O(n\log(n))$ $O(4n+3)$ $O(n^2+20000n+3)$ $O(1)$ $O(n^2)$ $O(2^n)$
$O(\log(n))$ $O(\log^2(n))$ $O(\log(\log(n)))$

Checkpoint 1

Using the formal definition of big-O, prove that $n^3 + 300n^2 \in O(n^3)$.

Simplest, tightest bounds

Something that will come up often with big-O is the idea of a tight bound on the runtime of a function.

It's technically correct to say that binary search, which takes around $\log(n)$ steps on an array of length n, is $O(n!)$, since $n! > \log(n)$ for all $n > 0$ but it's not very useful. If we ask for a tight bound, we want the closest bound you can give. For binary search, $O(\log(n))$ is a tight bound because no function that grows more slowly than $\log(n)$ provides a correct upper bound for binary search.

Unless we specify otherwise, we want the simplest, tightest bound!

Checkpoint 2

Simplify the following big-O bounds without changing the sets the represent:

$O(3n^{2.5} + 2n^2)$ can be written more simply as ______________________________

$O(\log_{10}(n) + \log_2(7n))$ can be written more simply as ______________________________

One interesting consequence of the second result in Checkpoint 2 is that $O(\log_i(n)) = O(\log_j(n))$ for all i and j (as long as they're both greater than 1), because of the change of base formula:

$$\log_i(n) = \frac{\log_j(n)}{\log_j(i)}$$

But $\frac{1}{\log_j(i)}$ is just a constant! So, it doesn't matter what base we use for logarithms in big-O notation.

When we ask for the simplest, tightest bound in big-O, we'll usually take points of if you write, for instance, $O(\log_2 n)$ instead of the simpler $O(\log n)$.

Checkpoint 3

Give the simplest, tightest bound for the following functions:

$f(n) = 16n^2 + 5n + 2 \in$ ______________________________

$g(n, m) = n^{1.5} \times 16m \in$ ______________________________

$h(x, y, z) = \max(x, y) + z^{16} \in$ ______________________________

Checkpoint 4

A water main break in GHC has unexpectedly removed `for` loops and all contracts except for `//@assert` from the C0 compiler! Since this means the function below won't compile anymore, rewrite the `for` loop and the contracts so that it will compile, but all the same operations (contract checks, loop guard checks, assignments...) still happen in the same order as when this code is compiled with the normal compiler with `-d`.

```
int search(int x, int[] A, int n)
//@requires n == \length(A);
//@ensures -1 <= \result && \result < n;
{
  for (int i = 0; i < n; i++)
  //@loop_invariant 0 <= i;
  {
    if (A[i] == x) return i;
  }
  return -1;
}
```