Information Objectivity and Accuracy in a Bank Run Model*

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(Preliminary Draft)

Abstract

In this paper, we develop and analyze the roles of two information properties in a simple bank run model. They are objectivity and accuracy of a given information source, originally discussed by an early, influential accounting work by Ijiri and Jaedicke (1966). We operationalize the two properties using a modeling technique from a recent work by Myatt and Wallace (2012) and embed the information source into an otherwise standard model of bank-runs in the spirit of Morris and Shin (2001). We show that the objectivity property plays a distinct and positive role for the economy, exhibiting a comparative advantage in mitigating inefficient, excess bank runs compared with the accuracy property. In fact, it is possible that improving objectivity discourages while improving accuracy encourages such runs.

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1 Introduction

In many situations of decision-making under uncertainty, economic agents utilize information available to assess the state-of-nature as well as what other agents know about the state-of-nature. The properties of available information sources may differ in assisting the decision-maker with respect to these two related but separate inference goals. We develop two information properties that speak to these two goals: objectivity and accuracy of an information source, originally discussed by an early, influential accounting work by Ijiri and Jaedicke (1966). We adapt the two properties using a modeling technique from a recent work by Myatt and Wallace (2012) and embed the information source into an otherwise standard model of bank-runs in the spirit of Morris and Shin (2001). We show that the objectivity property plays a distinct and positive role for the economy, exhibiting a comparative advantage in mitigating inefficient, excess bank runs compared to the accuracy property. In fact, it is possible in our model that improving objectivity discourages while improving accuracy encourages these bank runs.

To make the two properties more explicit, consider a decision-maker $i \in \{1, 2, ...\}$ is faced with a decision based on some signal x_i about some underlying payoff relevant state-variable $\tilde{r}$. The informational property of each private signal x_i is modeled as the state variable plus two independent mean-zero error terms:

$$x_i = \tilde{r} + \tilde{\eta} + \tilde{\varepsilon}_i$$

with the precision of $\tilde{r}$, $\tilde{\eta}$ and $\tilde{\varepsilon}_i$ denoted by α, γ and β respectively.

Key to this formulation is that the noise terms in the signal has two components: (1) a noise term $\tilde{\eta}$ common for all decision-makers and a noise term $\tilde{\varepsilon}_i$ independently applied for each decision-maker. Following the spirit of the seminal accounting work by Yuji Ijiri and Robert Jaedicke (1966), we call β, the precision of the agent-specific noise term, the objectivity of the signal and γ, the precision of the common noise term, the accuracy of the signal.[1] The higher the β, the realized measure is more objective (in the Ijiri-Jaedicke sense) as different decision-makers would more likely to "agree" with each other about the underlying state. Myatt and Wallace (2012) label this term information clarity. When β approaches positive infinity ($\beta = +\infty$), all agents receives the same signal and each decision-maker is able to forecast perfectly the signal others have received. The key is that β measures the *idiosyncratic* variability *among* different decision-makers. On the other hand, a higher γ indicates a more accurate signal as it places all decision-maker closer the state-variable ($\tilde{r}$). When γ approaches positive infinity ($\gamma = +\infty$), the consensus of all agents' signals would perfectly reveal the state-variable but each agent remains unsure about the signal others have received. The key is that γ measures the variability *common* to all decision-makers. Notice, either a higher β or a higher γ results in a more informative signal about the underlying state-variable for each individual decision-maker; but only a higher β makes one decision-maker's signal more informative about another decision-maker's signal. This is the subtlety of the distinction between objectivity and accuracy.

[1] In their influential work on reliability, Ijiri and Jaedicke proposed a simple, elegent model which decompose the so-called reliability property (of accounting measurement system) into two components: objectivity and bias. Suppose n measurers are asked to measure an object (e.g., the performance of a firm) using a given measurement system η. Let x^* be the true value of the object (e.g., the economic income of the firm) and x_i^η be the outcome (e.g., the net income of the firm) reported by measurer i by operating the measurement system η (e.g., a historical-cost based measurement system). Ijiri-Jaedicke proposed that Reliability of the measurement system η is given as

$$\begin{aligned} R_\eta &= \frac{1}{n}\sum_{i=1}^{n}(x_i^\eta - x^*)^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x}^\eta)^2 + (\bar{x}^\eta - x^*)^2 \\ &= V_\eta + B_\eta \end{aligned}$$

where $\bar{x}^\eta$ is the average outcome among n measurers.. Accordingly, $V \equiv \frac{1}{n}\sum_{i=1}^{n}(x_i^\eta - \bar{x}^\eta)^2$ is defined as Objectivity of the measurement system η and $B \equiv (\bar{x}^\eta - x^*)^2$ is defined as the Bias of the measurement system η.

Two key observations can be made. First, Objectivity, according to this conception, can be practically measured. As practicing accountants to use a measurement device and computed the variability among the reported measures would suffice. Bias, on the other hand, can be easy if the measurement object is tomorrow's rainfall but much harder if the measurement object is economic income of the firm next year.

Second, in practice, there may be a trade-off between objectivity and bias. For example, historical cost measurement system is said to have high Objectivity but large bias; while current-cost (or Fair value in today's language) measurement system may have lower bias but lower Objectivity. Obviously, a known bias (e.g., the value of bias is common knowledge) will not pose any economic problem so implicitly in the Ijiri-Jaedicke's idea is that problem posed by bias of any measurement system is that when the system is be deployed, the users are not sure exactly the precise bias even though they believe the bias is large or small in general.

A current measurement example is illustrative. Many choices among accounting method are confronted with the accuracy and objectivity trade-off. Consider the FASB's recent exposure draft on the contemplated change in the accounting for loan losses. Under the current "incurred loss" model, only expected losses over a specific time horizon that pass a "probable" threshold are recognized, resulting in a prohibition of recognizing loan losses on the same day of the loan origination. The newly proposed "current expected credit loss" (CECL) model removes the "probable" threshold and expands the time horizon of the expected losses. As a result, it is technically feasible that a loan loss may be recorded on day-one. According to the exposure draft, the expected credit loss is defined as: "An estimate of all contractual cash flows not expected to be collected from a recognized financial asset (or group of financial assets) or commitment to extend credit." Arguably, the "Incurred-loss" model can be more objective in Ijiri-Jaedicke sense than the CECL model because one may find less disagreements about the incurred-loss among reasonably trained professional accountants. However, it also natural to believe that CECL model may be more accurate because the proposed CECL model requires the estimated loan losses be based on relevant information about not only past events and current conditions (as required in the current incurred loss model) but also "reasonable and supportable forecasts that affect the expected collectability of the financial assets' remaining contractual cash flows," which are more forward-looking information than is permitted under the current U.S. GAAP.[2] (See 2013 FASB exposal draft on Credit Losses for more details.)

In our analytic model, we embed the objectivity-accuracy information structure above into a simple bank-run model in the spirit of Morris and Shin (2001). In this model, each depositor makes the withdraw decision (or run the bank) based on a signal it receives. Critically, each depositor rationally use the signal ($\tilde{x}_i$) to infer both (1) the underlying fundamental health of the bank $\tilde{r}$ (e.g.,

[2] For example, in addition to evaluating the borrowers' current creditworthiness, the CECL model also requires an evaluation of the forecasted direction of the economic cycle. Because the expansion of the time horizon, the estimate makes use of the time value concept such that expected losses are discounted at the asset's effective interest rate. Further, estimates should reflect both the possibility that a credit loss results and the possibility that no credit loss results.

quality of its loan portfolio) and (2) the likely signal other depositors have received (the $\tilde{x}_j$'s) in order to gauge the likelihood they would run the bank. The key message from our analysis is that objectivity property has a distinct role (from the accuracy property) in this bank-run setting with strategic complementarity. In particular, objectivity has a comparative advantage in facilitating coordination between depositors. First, we show that both objectivity and accuracy can improve collective decision-making and discourage inefficient and excessive bank-runs. This happens when the prior information about the bank-asset fundamentals is sufficiently imprecise. Second, the marginal effect of improving objectivity is higher than that of improving accuracy; and finally it is possible that improving objectivity discourages inefficient and excessive bank-runs while improving accuracy encourages such runs.

The key intuition of the analytic result comes from the different roles created by objectivity and by accuracy. Each depositor, deciding on whether to run the bank, uses the own signal to assess two related but different objects: the underlying state-variable and the signals that other depositors have received. The first allows the depositor to assess the payoff of the bank's loan portfolio when no one runs the bank, we call this the *fundamental* value of signal; and the second allows the depositor to assess the likely actions of other depositors, we call this the *strategic* value of signal. Both are important for the depositor's own decision because the equilibrium individual payoff depends on both the fundamentals as well as how many other depositors run the bank in equilibrium. Improving either of accuracy or objectivity plays similar roles in enhancing the fundamental value of the depositors' signals, diminishing the size of noises and moving the signals closer to true fundamentals. Thus in a setting in which the bank has only one depositor, without the need for knowing other's action, the lone depositor's use of information only depends on the signal's *fundamental* value and as a result, objectivity plays no distinct role from accuracy.

It is until we consider a setting with multiple depositors whose uses of information also depend on the *strategic* value of the signal that the distinction between objectivity and accuracy becomes critical. In fact, improving objectivity has the opposite effect on the strategic value to improving accuracy. Improving objectivity reduces the idiosyncratic variability of the depositors' signals and moves the signals closer to each other, which improves their strategic value. Improving accuracy, however, only reduces the common variability of the signals while leaving the idiosyncratic variability unaffected. As a result, the idiosyncratic variability takes a larger weight in the depositors' signals relative to the common variability, which in turn moves the depositors' signals further away from each other and impairs their strategic value. In this light, our paper identifies a situation in which the roles for the objectivity and the accuracy property are distinct and separate: objectivity reduces the dispersion among the depositors and thus facilitates their coordination while accuracy disrupts the coordination and encourages inefficient and excessive bank-runs.

The rest of paper is organized as follows. The rest of the introduction section provides related accounting and economics literature as well as connections to practice. Section 2 provides the model setup and the first-best benchmark results. Section 3 presents the analytic results of the paper and Section 4 concludes.

1.1 Connections to Literature and Practice

With respect to the accounting literature, our paper continues a long line of research which strive to endow information-economic meanings to accounting concepts. Starting in the late 1960s and early 1970s, accounting researchers begin linking accounting concepts to information economics concepts (see AAA monographs by Feltham 1972 and Mock 1976). The agenda is to build on the traditional approach under a purely measurement perspective,[3] and to tie the accounting measure-

[3] The approach, mainly analytic, was to derive a measurement basis from some self-evident postulates (e.g., entity, continuity, periodicity). Thus, the disagreements arose mainly from different definitions of assets and income and different postulates about accounting's environment. Naturally, the disagreements produced different procedures to measure the underlying stocks and flows. For example, on asset valuation side, historical cost was the key concept and on the income statement side, realization principle and matching are the key. Conservatism was the dominant

ment concepts to economic trade-off in decision making under uncertainty. The shift in perspective was well articulated by the seminal work of Beaver and Demski (1979). They argued that income measurement loses its economic foundation in a world with imperfect and incomplete markets. They offer a reinterpretation of income reporting and accrual notions in terms of a 'cost-effective' communication procedure (Beaver and Demski 1979, 38). Therefore, under this new information economics approach, the logical function for accounting in such a world is to carry information. Accounting notions like assets, liability, and earnings are treated as informative signals that tell the users something new about the entity. The usual connotations attached to these accounting labels are of less significance. What is important are their informational properties such as relevance, reliability, timeliness, etc. In turn, different uses of accounting information and the existence of other information sources besides the accounting source become important in understanding accounting. One major lesson from studying the strategic use of information has been that the mere production of information about some behavior may change the behavior being measured.[4] We add to this broad literature by isolating a distinct and positive role for the accounting notion of objectivity.

With respect to the economic literature of games with incomplete information, the information structure is critical and has been considered as one of the most important area of research (Morris and Shin 2011 discussion section). In a seminal paper, Diamond and Dybvig (1983) argue that banks play important roles in providing liquidity to the economy which, at the same time, exposes banks themselves to the risk of bank-runs. This fragility of banks is characterized by relying on a multiple-equilibrium: depending on the coordination among depositors, either a bank-run equilibrium or a socially optimal equilibrium prevails. However, which of these two equilibria occurs is either indeterminate or depends on extraneous variables ("sunspots"). This indeterminacy, despite its

rule in practice. These key concepts combine to create a somewhat robust system which achieved wide acceptance. Some likened the stability of the systems of accounting concepts to that of the Newtonian physics at the beginning of 20th century. See Chatfield (1974).

[4] As a result, some likened the information role of accounting to the instability of Quantum physics (see Borio and Tsatsaronis 2006 and Fellingham and Schroder 2006).

intuitive appeal and intellectually interesting, can be unsatisfactory and debilitating from a practical and policy stand point. For instance, Morris and Shin (2001) argue that such indeterminacy "runs counter to our theoretical scruples against indeterminacy" and generates "the obvious difficulties of any comparative-statics analysis." A recent development in the global games literature (Carlsson and van Damme, 1993; Morris and Shin, 1998, 2001) provides a key insight that challenges the indeterminacy in multiple-equilibrium models. Key to the global games approach in the model of bank runs is to reconsider the stark information environment condition (i.e., the fundamentals of the project is known to all). Under the approach, even when depositors observe the final payoff with just a very small amount of idiosyncratic uncertainty, the indeterminacy will disappear and the equilibrium becomes unique. Identifying a unique equilibrium in turn allows the study of various comparative statics, which is of central importance in generating policy implications. Built on the global games approach, this paper focuses on developing the information structure in a bank-run situation. Our findings suggest that introducing informational issues to the model of bank runs not only results in the uniqueness of equilibrium, but the information properties also play important roles in determining the equilibrium outcomes.

With respect to practical and policy considerations, managing information flow is of critical importance in many settings large and small, from corporate communication (earning releases and conference calls) to policy-maker announcements (Federal Reserve meeting transcripts releases and Q&A at congress). When coordination impact of information releases become important, our result implies that corporations or policy makers should be careful to balance the objectivity and accuracy of disclosure. Sometimes the best course of action is to provide the an objective, abeit less accurate, disclosure. During the recent financial crisis when the big financial institutions are under stress, possibly due the coordination failure among their investors. Two alternative fundamental measures of banks well-being came to the limelight: Tier-I capital and Tangible Common Equity

(TCE). Arguably, Tier-I capital may be more accurate measure of bank equity than TCE. But TCE is more likely to induce less "disagreements" among investors given its simplicity (i.e., more objective in Ijiri-Jaedicke sense) while Tier-I capital may be subject to many interpretations given its many-layered construction. One phenomenon, which could be consistent with the spirit of our result, was that financial institutions began voluntarily emphasize TCE measures toward their shareholders. For example, Citigroup began formally disclose and comment its TCE measures in their 10K reports for fiscal year 2008.

2 Model

2.1 Setup

We examine a simple bank-run model that has three dates, a continuum of depositors, and a bank endowed with an investment project (an illiquid loan). At date 0, the bank finances the project by issuing deposits to the depositors. At date 1, depositors learn an intermediate signal $\tilde{x}_i$ and decide whether to withdraw their deposits from the bank. At date 2, the project yields a stochastic payoff that depends on both the fundamentals of the project and the amount of deposits withdrawn. The time line of the model is shown below.

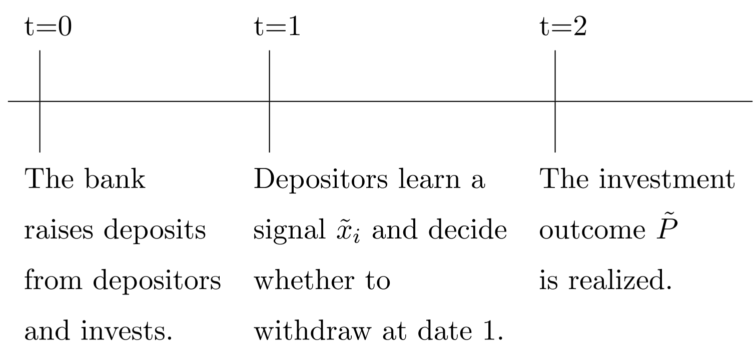

Figure 1.1: Time line.

We now describe and explain the decisions and events at each date in more detail.

Date 0

At date 0, the bank is endowed with an investment project that yields a stochastic gross rate of return, $\tilde{R} = e^{\tilde{r}}$ realized on date-2 where $\tilde{r}$ is normally distributed with a mean $\bar{r}$ and a variance $\frac{1}{\alpha}$. α measures the precision of depositors' common prior about $\tilde{r}$. We assume that $0 < \bar{r} \leqslant \frac{1}{2}$.[5] The bank finances the project by attracting deposits from a group of depositors, with unit mass and indexed by the unit interval $[0, 1]$, each of whom contributes 1 unit of the consumption good. All depositors have the log utility function such as:

$$u_i = \log(c_{i1} + c_{i2}), \tag{1}$$

where c_{i1} and c_{i2} denote depositor i's consumption at date 1 and at date 2 respectively. The bank invests all the deposits in its project.

Date 1

At date 1, each depositor i observes a private signal $\tilde{x}_i$ about the project's fundamentals, $\tilde{r}$:

$$\tilde{x}_i = \tilde{r} + \tilde{\eta} + \tilde{\varepsilon}_i, \tag{2}$$

where the various noise terms are all independently distributed with $\tilde{\eta} \sim N(0, \frac{1}{\gamma})$ and $\tilde{\varepsilon}_i \sim N(0, \frac{1}{\beta})$. Myatt and Wallace (2012) interpret equation (2) as that the bank discloses some information (e.g., a financial report) $\tilde{x}_p = \tilde{r} + \tilde{\eta}$ about its project with a noise $\tilde{\eta}$. The accuracy of the disclosure is denoted by the precision of the noise η, γ. A depositor i can only interpret the disclosure imperfectly

[5] As we will show later, assuming $r \leqslant \frac{1}{2}$, the bank's disclosure of information helps to reduce the risk of bank runs. However, when the common prior about the bank's project is sufficiently good (i.e., $r > \frac{1}{2}$), the release of information actually exacerbates the risk of runs, which prevents the bank from disclosing in the first place. This is because, in such cases, the bank prefers the depositors to rely more on the favorable prior rather than to respond to new information, which is likely to be worse than the prior.

as $\tilde{x}_i = \tilde{x}_p + \varepsilon_i$. The precision of the individual noise ε_i, β, thus measures the degree of consensus among the depositors who interpret the same disclosure by the bank. The higher the β, the less disperse depositors' interpretations of $\tilde{x}_p$ are. In the accounting literature, the consensus among a group of observers is often related to the concept of objectivity (Ijiri and Jaedicke, 1966). That is, the bank's disclosure is more objective if the depositors agree more on its interpretations. In this light, throughout this paper, we define the consensus β as a measure of the objectivity of the bank's disclosure. Notice that the total noise in a depositor's signal, $\tilde{\eta} + \tilde{\varepsilon}_i$, is determined by both the accuracy and the objectivity.

Based on the information $\tilde{x}_i$, depositor i updates her beliefs about the project's fundamentals and other depositors' beliefs and decides whether to withdraw her deposit or not. For simplicity, we assume that if depositor i withdraws, she is repaid at the face value, 1 unit of the consumption good. Following Diamond and Dybvig (1983) and Morris and Shin (2001), we assume that the bank's project is illiquid and the net rate of return obtainable at date 2 is decreasing in the proportion of the deposit withdrawn at date 1, as denoted by $l \in [0, 1]$. Specifically, we assume that at date 2, the net rate of return is:

$$\tilde{P} = \tilde{R}\, e^{-l} = e^{\tilde{r}-l}, \tag{3}$$

where the term $e^{-l} < 1$ captures the cost of liquidating the illiquid project to meet the depositors' withdrawals.

Date 2

The net rate of return $\tilde{P}$ is realized and the proceeds from the project is distributed to the depositors.

2.2 The First-Best Benchmark

We first solve for the first-best in our model as a benchmark. Consider a situation in which the bank's project is financed by only one depositor. Thus the investment is no longer plagued by the coordination problem among multiple depositors. For expositional purpose, denote the "de-meaned" value of the signal $\tilde{x}_i$ as $\tilde{y}_i \equiv \tilde{x}_i - \bar{r}$. It is straightforward to verify that, without loss of generality, only one kind of strategy, a switching strategy, needs to be considered, where the depositor chooses to withdraw if and only if she observes a $\tilde{y}_i$ below some threshold y^{**}:

$$s(\tilde{y}_i) = \begin{cases} \text{Withdraw} & \text{if } \tilde{y}_i \leqslant y^{**}, \\ \text{Not to Withdraw} & \text{if } \tilde{y}_i > y^{**}. \end{cases} \tag{4}$$

Consider a marginal depositor whose signal $\tilde{y}_i$ is exactly equal to y^{**}. If the depositor withdraws, her expected utility is $\log(1) = 0$. If she chooses not to withdraw, she earns the rate of return, $\tilde{R}$, and her expected utility conditional upon the signal $\tilde{y}_i = y^{**}$ is:

$$E[\log(\tilde{R})|\tilde{y}_i = y^{**}] = E[\tilde{r}|\tilde{y}_i = y^{**}] = \bar{r} + \frac{\frac{1}{\alpha}}{\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}}\, y^{**}. \tag{5}$$

Denote $k_1 \equiv \frac{\frac{1}{\alpha}}{\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}}$ and k_1 measures the importance of the marginal depositor's signal y^{**} in estimating the fundamentals. In equilibrium, the marginal depositor is indifferent between staying and withdrawing:

$$\bar{r} + k_1\, y^{**} = 0, \tag{6}$$

which gives,

$$y^{**} = -\frac{\bar{r}}{k_1}, \tag{7}$$

a negative first-best withdraw threshold. We summarize the equilibrium in the first-best benchmark in the lemma below.

Lemma 1 *In the benchmark that has only one depositor to invest in the bank, the depositor withdraws if and only if* $\tilde{y}_i \leqslant y^{**} = -\frac{\bar{r}}{k_1}$.

Notice that in this benchmark, a depositor makes her decision only based on her belief about the fundamentals $\tilde{r}$, not on what the other depositors' belief about $\tilde{r}$. Therefore, the information owned by the depositors is only valuable in estimating the fundamentals, as characterized by k_1:

$$k_1 = \frac{\frac{1}{\alpha}}{\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}}. \tag{8}$$

As is intuitive, the symmetry of β and γ implies nothing special about the objectivity of a signal, separate and distinct from its accuracy. At an extreme, when $\beta = \gamma \equiv c$, a marginal improvement in the objectivity (higher β) generate the same benefit to the decision-maker as a marginal improvement in the accuracy (higher γ): $\frac{\partial k_1}{\partial \beta} = \frac{\partial k_1}{\partial \gamma} = \frac{\alpha}{(c+2\,\alpha)^2}$. Alternatively, we can appreciate mootness of objectivity as the fact that the decision-maker would be indifferent to swapping the accuracy and the objectivity. Let $c_1 < c_2$, k_1 is unchanged between measurement system with $\langle \beta = c_1, \gamma = c_2 \rangle$ and another measurement system $\langle \beta = c_2, \gamma = c_1 \rangle$.

2.3 The Equilibrium

We now solve for the equilibrium in our model. As shown in Morris and Shin (2001), it suffices to consider only the switching strategy where the depositor chooses to withdraw if and only if she observes a $\tilde{y}_i$ below some threshold y^*:

$$s(\tilde{y}_i) = \begin{cases} \text{Withdraw} & \text{if } \tilde{y}_i \leqslant y^*, \\ \\ \text{Not to Withdraw} & \text{if } \tilde{y}_i > y^*. \end{cases} \tag{9}$$

Consider a marginal depositor whose signal $\tilde{y}_i$ is exactly equal to y^*. If she withdraws, her expected utility is 0. If she chooses not to withdraw, her expected utility is equal to:

$$E[\log(\tilde{P})|\tilde{y}_i = y^*] = E[\tilde{r} - l|\tilde{y}_i = y^*]. \tag{10}$$

Recall that the depositor's updated belief of $\tilde{r}$ is:

$$E[\tilde{r}|\tilde{y}_i = y^*] = \bar{r} + k_1\, y^*. \tag{11}$$

We now compute, from the marginal depositor's perspective, the portion of depositors who choose to withdraw $E[l|\tilde{y}_i = y^*]$. Since the noises are all independently distributed, the expected portion of depositors who withdraw is equal to the probability that a particular depositor j withdraws. Since depositor j also follows the same switching strategy, she withdraws if and only if her signal $\tilde{y}_j \leqslant y^*$. Thus we have,

$$E[l|\tilde{y}_i = y^*] = \Pr(\tilde{y}_j \leqslant y^*|\tilde{y}_i = y^*). \tag{12}$$

Given the marginal depositor's signal $\tilde{y}_i = y^*$, she thinks that depositor j's signal is normally distributed with a mean $\rho\, y^*$ and a variance $(1-\rho^2)\left(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}\right)$, where $\rho = \frac{\frac{1}{\alpha}+\frac{1}{\gamma}}{\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}}$ denotes the correlation between the signals of the marginal depositor and depositor j.[6] Therefore,

$$
\begin{aligned}
\Pr(\tilde{y}_j \leqslant y^* | \tilde{y}_i = y^*) &= \Pr\left(\frac{\tilde{y}_j - \rho\, y^*}{\sqrt{(1-\rho^2)\left(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}\right)}} \leqslant \frac{y^* - \rho\, y^*}{\sqrt{(1-\rho^2)\left(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}\right)}} | \tilde{y}_i = y^* \right) \quad (13) \\
&= \Phi\left(\sqrt{\frac{1-\rho}{1+\rho}\frac{1}{\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}}}\, y^* \right).
\end{aligned}
$$

Denote $k_2 \equiv \sqrt{\frac{1-\rho}{1+\rho}\frac{1}{\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}}}$ and the correlation ρ in k_2 measures the importance of the marginal depositor's signal y^* in estimating the risk of bank runs (the expected proportion of depositors who withdraw). We have

$$
E[l|y^*] = \Phi\left(k_2\, y^*\right). \tag{14}
$$

In equilibrium, the depositor who observes a $\tilde{y}_i$ equal to y^* is indifferent between staying and withdrawing:

$$
E[\tilde{r} - l|y^*] = 0, \tag{15}
$$

which can be reduced into:

$$
\bar{r} + k_1\, y^* = \Phi\left(k_2\, y^*\right). \tag{16}
$$

In the following proposition, we show that for α sufficiently small, there exists a unique equilibrium in our model.

Proposition 1 *Define α_H as the unique positive solution to*

$$
k_1 = k_2 \sqrt{\frac{1}{2\,\pi}}. \tag{17}
$$

Given that $\alpha \leqslant \alpha_H$, there exists a unique equilibrium such that every depositor withdraws if and

[6] The derivations regarding the conditional distribution are included in the Appendix I.

only if $\tilde{y}_i < y^*$, where y^* is the unique solution to

$$\bar{r} + k_1 \, y^* = \Phi \left(k_2 \, y^* \right). \tag{18}$$

Proposition 1 shows that when the common prior among depositors is sufficiently diffuse, the equilibrium in a bank run game becomes unique, which often appears in the higher-order beliefs and global game literature (Morris and Shin, 1998, 2001, 2002; Plantin, Sapra and Shin, 2008). This result shows that the occurrence of bank runs depends critically on the information disclosed by the bank: a depositor withdraws upon receiving a bad signal about the bank's project. In addition, we find that the depositor tends to withdraw more often in equilibrium than in the first-best benchmark. That is, $y^* \geqslant y^{**}$, as summarized in the corollary below.

Corollary 1 *Given that $\alpha \leqslant \alpha_H$, in equilibrium, every depositor tends to withdraw more often than in the first-best benchmark,*

$$y^* \geqslant 0 > y^{**}. \tag{19}$$

Corollary 1 depicts the coordination failure in the bank run situation. Since a depositor is concerned with the risk of runs by other depositors, she will withdraw when anticipating others will withdraw, even if the bank's project yields an expected payoff higher than its liquidation value. As a result, in equilibrium, the project is liquidated more often than what is optimal in the first-best ($y^* > y^{**}$). The results indicate that in our model, the runs are always "inefficient" in terms of investment efficiency and hence reducing the withdrawal threshold always enhances the social welfare. Our findings also suggest that since we consider a situation where all the depositors hold a common pessimistic prior $\bar{r} \leqslant \frac{1}{2}$, a depositor will choose not to withdraw only when she receives an updated signal that is sufficiently more favorable than her prior (that is, $\tilde{y}_i = \tilde{x}_i - \bar{r} > y^* \geqslant 0$). This point turns out to be of critical importance in understanding the roles of improving the accuracy and the clarity in affecting the risk of bank runs.

3 Equilibrium Analysis

Identifying the unique equilibrium in the bank-run game allows us to analyze the properties of the equilibrium in which we focus on studying the role of two important properties of information, the accuracy γ and the objectivity β, in affecting the threshold for withdrawals y^*. We think such analyses can shed light on the optimal design of the information system in order to reduce the occurrences of inefficient bank runs.[7] Since in most analyses, we are interested in comparing the effects of improving the accuracy and the objectivity, we will focus on the case $\beta = \gamma$ to "level the playing field."

We first show that improving both of the accuracy and the objectivity of the information system can help to reduce the threshold for withdrawals in the following proposition.

Proposition 2 *Given that $\alpha \leqslant \alpha_H$, the following holds:*

1. *Improving the objectivity always decreases the threshold for withdrawals,*

$$\frac{\partial y^*}{\partial \beta} < 0; \tag{20}$$

2. *There exists a threshold $\alpha_L \in [0, \alpha_H]$, such that for $\alpha < \alpha_L$, improving the accuracy decreases the threshold for withdrawals,*

$$\frac{\partial y^*}{\partial \gamma} < 0. \tag{21}$$

Proposition 2 characterizes the roles of the accuracy and the objectivity of the information disclosure in affecting the risk of bank runs. We find that improving the objectivity always reduces

[7] Note that since $y^* \geqslant y^{**}$, in equilibrium, withdrawals are always inefficient and, therefore, reducing y^* improves the investment efficiency.

the withdrawal threshold, which increases the social welfare. Improving the accuracy, on the contrary, has a non-monotonic effect. The second part of Proposition 2 identifies a sufficient condition for the accuracy to be welfare-enhancing: the common prior α needs to be sufficiently imprecise. In other regions, the effect of the accuracy is ambiguous. In fact, as we will discuss later in Proposition 4, when the common prior is sufficiently precise, improving the accuracy can actually increase the risk of runs. This result implies that although improving the accuracy and the objectivity play similar roles in reducing the noises associated with the information disclosure, the two have distinct effects on the risk of bank runs; improving the objectivity seems to be more effective than improving the accuracy in terms of reducing runs. Indeed, we verify this conjecture in the proposition below.

Proposition 3 *Given that $\alpha \leqslant \alpha_H$, improving the objectivity dominates improving the accuracy in terms of decreasing the threshold for withdrawals,*

$$\frac{\partial y^*}{\partial \beta} < \frac{\partial y^*}{\partial \gamma}. \tag{22}$$

Proposition 3 indicates the superiority of improving the objectivity to improving the accuracy in mitigating the risk of bank runs. This proposition also sheds some light on the trade-off between accuracy and objectivity in terms of policy implications for designing the optimal information system. Our results suggest that in scenarios that have conflicts between objectivity and accuracy, it could be welfare-enhancing to sacrifice accuracy to improve objectivity, especially when reducing the risk of bank runs is the central concern.

To understand the intuition behind the results in Proposition 2 and 3, it is illuminating to return to the equation that determines the threshold for withdrawal:

$$\bar{r} + k_1 \, y^* = \Phi \left(k_2 \, y^* \right). \tag{23}$$

Equation (23) illustrates well that, from the marginal depositor's standpoint, her information $\tilde{y}_i = y^*$ serves two purposes: on the left-hand side of the equation, y^* is used to estimate the fundamentals of project, where the importance of this usage is characterized by k_1; on the right-hand side, y^* is used to forecast the probability that other depositors will withdraw (the risk of bank runs), where the importance of this usage is characterized by ρ. We call the first value of the information a *fundamental* value and the second a *strategic* value.

The effects of improving the two informational properties on the risk of bank runs depends critically on their effects on the two usages of the information. Recall that, as we discussed in the first-best benchmark, improving the accuracy and the objectivity are similar in affecting the fundamental value of the information (that is, $\frac{\partial k_1}{\partial \beta} = \frac{\partial k_1}{\partial \gamma}$ for $\beta = \gamma$). Improving either of the two increases the precision of the marginal depositor's signal and as a result, the marginal depositor places a larger weight (a higher k_1) on the new signal, relative to her prior. Recall that in our model, the marginal depositor holds a pessimistic prior and will choose not to withdraw only upon receiving a new signal that is sufficiently more favorable than the prior (that is, $y^* = x^* - \bar{r} \geqslant 0$). Therefore, as the marginal depositor places a larger weight on the more favorable new signal y^*, she forms a more optimistic expectation about the project's return. As a result, at the previous withdrawal threshold y^*, she now prefers to leave her money within the bank, which means the solvency threshold needs to lower. In other words, a marginal depositor needs to receive a worse signal than the previous one in order to be indifferent between withdrawing and staying. These analyses show that through magnifying the fundamental value of the marginal depositor's favorable information, both improving the objectivity and the accuracy reduce the withdrawal threshold.

Improving the accuracy and the objectivity have the opposite impacts on the strategic value of the information. The disclosure of more objective information facilitates the depositor's ability to forecast others' actions while more accurate information impairs her forecasting ability. This is because the accuracy and the objectivity affect differently the correlation between depositors' signals $\rho = \frac{\frac{1}{\alpha}+\frac{1}{\gamma}}{\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}}$, which is the key determinant of the strategic value of the information. On one hand, an improvement in the objectivity reduces the degree of disagreement among depositors' interpretation of the bank's disclosure and hence shrinks the magnitude of the individual noise ε_i. Drawing an analogy between our model and the theory of CAPM, enhancing the objectivity decreases the relative portion of the "idiosyncratic" component $(\frac{\frac{1}{\beta}}{\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}})$ in the information while increasing the portion of the "systematic" one $(\frac{\frac{1}{\alpha}+\frac{1}{\gamma}}{\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}})$. As a result, the depositor's signal is of greater importance in predicting others' actions as the depositors' signals become more correlated (a higher ρ). On the other hand, however, making the disclosure more accurate only diminishes the size of the systematic noise while leaving the idiosyncratic noise unaffected. As a result, in each depositor's information, the idiosyncratic component takes a larger weight at the expense of reducing the weight on the systematic one, which in turn diminishes the strategic value of the information (a lower ρ).

Now consider how these changes in the strategic value of the information affect the withdrawal threshold. For a marginal depositor with a more favorable signal than her prior, when improving the objectivity enhances the value of the information in forecasting others' actions, her signal indicates that a larger portion of the other depositors also receive favorable signals and hence will choose to stay. As a result, from the marginal depositor's perspective, the risk of bank runs is lower and hence she prefers to stay at the previous threshold before the improvement in the objectivity. That is, improving the objectivity, through amplifying the strategic value of the marginal depositor's favorable information, reduces the withdrawal threshold. Similarly, improving the accuracy decreases the strategic use of the marginal depositor's favorable signal and induces her to believe that the others will be more likely to withdraw. In response, she will prefer to withdraw at the previous threshold before the change, which means the withdrawal threshold needs to be higher.

Our analyses combined suggest that improving the objectivity increases both the fundamental and the strategic value of the information, which collectively reduce the withdrawal threshold; however, the effect of improving the accuracy is non-monotonic, depending on the trade-off between the increase in the fundamental value of the information and the decrease in its strategic value. This non-monotonicity implies that there may exist a region in which improving the objectivity yields the opposite effect on the risk of bank runs to improving the accuracy, which we show in the proposition below.

Proposition 4 *Consider a case where $\bar{r}$ is sufficiently close to $\frac{1}{2}$ and given that $\alpha \leqslant \alpha_H$, the following holds:*

1. *For $\alpha < \alpha_L$, improving the objectivity and the accuracy both decrease the threshold for withdrawals,*

$$\frac{\partial y^*}{\partial \beta} < \frac{\partial y^*}{\partial \gamma} < 0; \tag{24}$$

2. *For $\alpha_L < \alpha \leqslant \alpha_H$, improving the objectivity decreases the threshold for withdrawals, while improving the accuracy increases the threshold for withdrawals,*

$$\frac{\partial y^*}{\partial \beta} < 0 < \frac{\partial y^*}{\partial \gamma}. \tag{25}$$

Proposition 4 summarizes the main policy implications of our findings. The proposition shows that in a case where the depositor's prior is sufficiently close to $\frac{1}{2}$, when the depositors' prior information about the bank is poor ($\alpha < \alpha_L$), improvements in the objectivity and the accuracy both decrease the risk of bank runs. However, when the depositors have good prior information ($\alpha > \alpha_L$), there is a striking difference between the effects of the objectivity and the accuracy on bank runs: it is welfare-enhancing to make the information disclosed more objective while more accurate disclosure actually impairs social welfare. The intuition behind this result is as follows. When the depositors' prior is sufficiently diffuse, the systematic component ($\frac{\frac{1}{\alpha}+\frac{1}{\gamma}}{\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}}$) already takes a significant portion in the depositors' information. A variation in the accuracy hence will not substantially change the systematic component and its impact on the strategic value of the information is small. As a result, with the fundamental value of the information as the dominant force, the roles of the objectivity and the accuracy are similar. It is until the depositors' prior becomes sufficiently precise that changing the accuracy has a considerable impact on the strategic value of the information. Only with this additional effect do the roles of the objectivity and the accuracy differ from each other.

The two cases characterized in Proposition 4 can be interpreted as the descriptions of two types of bank runs. The case of poor prior information ($\alpha < \alpha_L$) can be viewed as depicting "old-fashioned", ordinary depositors' runs on traditional commercial banks, such as the ones that occurred repeatedly in the 19th century (Allen and Gale, 1998). Our results imply that in mitigating these runs, it makes no qualitative difference between improving objectivity and accuracy of banks' disclosure; the trade-off between objectivity and accuracy hence may seem rather muted. The other case in which the depositors hold much better prior information ($\alpha > \alpha_L$) can be related to the "modern-day" runs on shadow banks (Shin, 2009). The group of "depositors" in these shadow banks are primarily contained with sophisticated institutional investors, such as mutual funds, investment banks and hedge funds. Different from ordinary depositors, these depositors are trained professional investors well equipped with prior knowledge about shadow banks' operations. In the recent financial turmoil, shadow banks had experienced catastrophic runs by their investors, which led to severe liquidity dry-ups and economic downturns. Our results suggest that in dealing with these modern-day runs, it is of vital importance to understand the trade-off between objectivity and accuracy. Our model predicts that the disclosure of highly objective information helps to stabilize investors' runs. Our results also serve at least as a message of caution regarding regulatory initiatives that aim solely at improving the accuracy of disclosure, since such initiatives can have the "unintended" consequence of triggering runs. An illuminating example related to this point is the popular disclosure of the tangible common equity (TCE) by banks during the recent crisis.[8] Although the TCE is an extremely simple measure and hence may not reflect banks' operating status fairly accurately, it is the same simplicity that limits the room for alternative interpretations and hence makes the TCE highly objective. Our findings suggest that banks may choose to disclose this somewhat inaccurate but highly objective information in the hope of stabilizing runs by investors.

4 Conclusion

In this paper, two information properties, objectivity and accuracy, are analyzed in a simple bank run model. We show that the objectivity property plays a distinct and positive role for the economy, exhibiting a comparative advantage in mitigating inefficient, excess bank runs compared with the accuracy property. In fact, it is possible that improving objectivity discourages bank runs while improving accuracy encourages such runs. We believe this is only the beginning of a new line of accounting reserach that explores the role of accounting information in settings featuring coordination as the key economic tension. We hope future work would use our results as the building blocks to develop a theory of corporate disclosure based on the need to encourage or discourage coordination. Given public information plays a special role in coordination-based games, we believe

[8] See an example of reporting the TCE by Citibank at http://online.wsj.com/news/articles/SB123577012189796905.

there are much to learn about the role accounting disclosure in these settings including what make accounting special compared to other information sources such as share price. This new line of research would complement current disclosure theories build on moral harzard and adverse selection tensions.

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Appendix I: Derivations of the Conditional Distribution of $\tilde{y}_j$ Given $\tilde{y}_i = y^*$

In this appendix, we derive the conditional distribution of a depositor j's signal $\tilde{y}_j$, given the marginal depositor's signal $\tilde{y}_i = y^*$. The two signals are:

$$\begin{aligned} \tilde{y}_i &= \tilde{x}_i - \bar{r} = \tilde{r} - \bar{r} + \eta + \varepsilon_i, \\ \tilde{y}_j &= \tilde{x}_j - \bar{r} = \tilde{r} - \bar{r} + \eta + \varepsilon_j. \end{aligned} \tag{26}$$

Since the random variables $\tilde{r} - \bar{r}$, η, ε_i, and ε_j are independently normally distributed, their linear combinations $\tilde{y}_i$ and $\tilde{y}_j$ are jointly normally distributed such as,

$$\begin{bmatrix} \tilde{y}_i \\ \tilde{y}_j \end{bmatrix} \tilde{\ } N \left[\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} & \rho \, (\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}) \\ \rho \, (\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}) & \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} \end{bmatrix} \right], \tag{27}$$

where the correlation between the two signals $\rho = \frac{\frac{1}{\alpha} + \frac{1}{\gamma}}{\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}}$. As a result, the conditional distribution of $\tilde{y}_j$ given $\tilde{y}_i = y^*$ is also normally distributed with the conditional expectation

$$E[\tilde{y}_j | \tilde{y}_i = y^*] = 0 + \frac{\rho \, (\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma})}{\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}} (y^* - 0) = \rho \, y^*, \tag{28}$$

and the conditional variance

$$Var[\tilde{y}_j | \tilde{y}_i = y^*] = (1 - \rho^2) \, (\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}). \tag{29}$$

Appendix II: Proofs

Proof of Proposition 1

Proof. As shown in the main text, the equilibrium threshold y^* is given by the following equation,

$$\bar{r} + k_1\, y^* = \Phi\left(k_2\, y^*\right). \tag{30}$$

It is straightforward to verify that $\frac{k_1}{k_2}$ is strictly decreasing in α. Thus when $\alpha \leqslant \alpha_H$, $\frac{k_1}{k_2} \geqslant \frac{k_1}{k_2}|_{\alpha=\alpha_H} = \sqrt{\frac{1}{2\pi}}$. Therefore,

$$k_1 \geqslant k_2 \sqrt{\frac{1}{2\pi}} \geqslant k_2\, \phi(k_2\, y^*), \tag{31}$$

that is, the slope of the LHS of equation (30), $\bar{r} + k_1\, y^*$, is always greater than the slope the RHS of equation (30), $\Phi\left(k_2\, y^*\right)$, which guarantees a unique solution to the equation. As a result, for $\alpha \leqslant \alpha_H$, there exists a unique equilibrium such that every depositor withdraws if and only if $\tilde{y}_i < y^*$. ■

Proof of Corollary 1

Proof. In equilibrium, the threshold y^* solves,

$$\bar{r} + k_1\, y^* = \Phi\left(k_2\, y^*\right). \tag{32}$$

In the first-best benchmark, the threshold y^{**} solves,

$$\bar{r} + k_1\, y^{**} = 0. \tag{33}$$

Therefore,

$$\bar{r} + k_1\, y^* = \Phi\left(k_2\, y^*\right) \geqslant 0 = \bar{r} + k_1\, y^{**}, \tag{34}$$

which gives,

$$y^* \geqslant y^{**}. \tag{35}$$

Since $y^{**} = -\frac{\bar{r}}{k_1}$ and k_1 is positive, $y^{**} < 0$. In addition, it can be verified that given $\bar{r} \leqslant \frac{1}{2}$, $y^* \geqslant 0$. To see this, consider the function $f(y)$:

$$f(y) = \bar{r} + k_1\, y - \Phi\left(k_2\, y\right), \tag{36}$$

$f(y)$ is continuous in y, $f(0) = \bar{r} - \frac{1}{2} \leqslant 0$, and $f(\frac{1-\bar{r}}{k_1}) > 0$. Therefore, by the intermediate value theorem, the unique root of the equation $f(y) = 0$, y^*, must be between 0 and $\frac{1-\bar{r}}{k_1}$. That is, $y^* \geqslant 0$.

■

Proof of Proposition 2

Proof. Since y^* solves,

$$\bar{r} + k_1\, y^* = \Phi\left(k_2\, y^*\right), \tag{37}$$

using the implicit function theorem and taking the derivative with respect to β and γ on the both sides of the equation, we have,

$$k_1 \frac{\partial y^*}{\partial m} + y^* \frac{\partial k_1}{\partial m} = \phi(k_2\, y^*) \left(k_2 \frac{\partial y^*}{\partial m} + y^* \frac{\partial k_2}{\partial m} \right),\ m \in \{\beta, \gamma\}, \tag{38}$$

which gives,

$$\frac{\partial y^*}{\partial m} = \frac{y^* \left(\phi(k_2\, y^*)\, \frac{\partial k_2}{\partial m} - \frac{\partial k_1}{\partial m} \right)}{k_1 - k_2\, \phi(k_2\, y^*)},\ m \in \{\beta, \gamma\}. \tag{39}$$

Since $\alpha \leqslant \alpha_H$,

$$k_1 \geqslant k_2 \sqrt{\frac{1}{2\pi}} \geqslant k_2\, \phi(k_2\, y^*), \tag{40}$$

and we have shown that $y^* \geqslant 0$ in Corollary 1, thus the denominator of $\frac{\partial y^*}{\partial m}$ is always positive.

It remains to check the sign of several derivatives, $\{\frac{\partial k_1}{\partial \beta}, \frac{\partial k_2}{\partial \beta}, \frac{\partial k_1}{\partial \gamma}, \frac{\partial k_2}{\partial \gamma}\}$. For $\beta = \gamma$, it is straightforward to verify that $\frac{\partial k_1}{\partial \beta} > 0$, $\frac{\partial k_2}{\partial \beta} < 0$, $\frac{\partial k_1}{\partial \gamma} > 0$ and $\frac{\partial k_2}{\partial \gamma} > 0$. Therefore,

$$\frac{\partial y^*}{\partial \beta} = \frac{y^* \left(\phi(k_2\, y^*)\, \frac{\partial k_2}{\partial \beta} - \frac{\partial k_1}{\partial \beta} \right)}{k_1 - k_2\, \phi(k_2\, y^*)} < 0. \tag{41}$$

For $\frac{\partial y^*}{\partial \gamma} = \frac{y^* \left(\phi(k_2\, y^*)\, \frac{\partial k_2}{\partial \gamma} - \frac{\partial k_1}{\partial \gamma} \right)}{k_1 - k_2\, \phi(k_2\, y^*)}$, its sign is ambiguous and depends on the comparison between the two terms in the numerator, $\phi(k_2\, y^*)\, \frac{\partial k_2}{\partial \gamma}$ and $\frac{\partial k_1}{\partial \gamma}$. However, we find that for $\beta = \gamma$, there exists a $\alpha_L \in [0, \alpha_H]$, that solves the equation:

$$g(\alpha) = \sqrt{\frac{1}{2\,\pi}}\, \frac{\partial k_2}{\partial \gamma} - \frac{\partial k_1}{\partial \gamma} = 0, \tag{42}$$

such that for $\alpha < \alpha_L$,

$$\frac{\partial y^*}{\partial \gamma} < 0. \tag{43}$$

To see this, it can be verified that $g(\alpha)$ is continuous and strictly increasing in α, $g(0) \leqslant 0$, and

$g(\alpha_H) \geqslant 0$. Therefore, by the intermediate value theorem, there exists an $\alpha_L \in [0, \alpha_H]$ such that

$$g(\alpha_L) = 0. \tag{44}$$

Moreover, for $\alpha < \alpha_L$,

$$\sqrt{\frac{1}{2\,\pi}}\,\frac{\partial k_2}{\partial \gamma} - \frac{\partial k_1}{\partial \gamma} = g(\alpha) < g(\alpha_L) = 0, \tag{45}$$

and

$$\phi(k_2\, y^*)\,\frac{\partial k_2}{\partial \gamma} - \frac{\partial k_1}{\partial \gamma} < \sqrt{\frac{1}{2\,\pi}}\,\frac{\partial k_2}{\partial \gamma} - \frac{\partial k_1}{\partial \gamma} < 0, \tag{46}$$

thus we have

$$\frac{\partial y^*}{\partial \gamma} = \frac{y^*\,\left(\phi(k_2\, y^*)\,\frac{\partial k_2}{\partial \gamma} - \frac{\partial k_1}{\partial \gamma}\right)}{k_1 - k_2\,\phi(k_2\, y^*)} < 0. \tag{47}$$

■

Proof of Proposition 3

Proof. From the proof of Proposition 2, we have $\frac{\partial k_2}{\partial \beta} < 0$ and $\frac{\partial k_2}{\partial \gamma} > 0$, thus

$$\phi(k_2\, y^*)\,\frac{\partial k_2}{\partial \gamma} > \phi(k_2\, y^*)\,\frac{\partial k_2}{\partial \beta}, \tag{48}$$

and for $\beta = \gamma$, we verify that $\frac{\partial k_1}{\partial \gamma} = \frac{\partial k_1}{\partial \beta}$. Therefore,

$$\phi(k_2\, y^*)\,\frac{\partial k_2}{\partial \gamma} - \frac{\partial k_1}{\partial \gamma} > \phi(k_2\, y^*)\,\frac{\partial k_2}{\partial \beta} - \frac{\partial k_1}{\partial \beta}, \tag{49}$$

and

$$\frac{y^*\,\left(\phi(k_2\, y^*)\,\frac{\partial k_2}{\partial \gamma} - \frac{\partial k_1}{\partial \gamma}\right)}{k_1 - k_2\,\phi(k_2\, y^*)} > \frac{y^*\,\left(\phi(k_2\, y^*)\,\frac{\partial k_2}{\partial \beta} - \frac{\partial k_1}{\partial \beta}\right)}{k_1 - k_2\,\phi(k_2\, y^*)}, \tag{50}$$

that is,

$$\frac{\partial y^*}{\partial \beta} < \frac{\partial y^*}{\partial \gamma}. \tag{51}$$

■

Proof of Proposition 4

Proof. In the proof of Proposition 2, we have shown $\frac{\partial y^*}{\partial \beta} < 0$ and for $\alpha < \alpha_L$, $\frac{\partial y^*}{\partial \gamma} < 0$. It thus remains to show that for $\alpha_L < \alpha \leqslant \alpha_H$ and $\bar{r}$ sufficiently close to $\frac{1}{2}$, $\frac{\partial y^*}{\partial \gamma} > 0$. Notice first that when $\bar{r}$ is sufficiently close to $\frac{1}{2}$, y^* approaches 0 and $\phi(k_2\, y^*)$ approaches $\sqrt{\frac{1}{2\pi}}$. Also as shown in the proof of Proposition 2, for $\alpha \in (\alpha_L, \alpha_H)$,

$$\sqrt{\frac{1}{2\,\pi}}\,\frac{\partial k_2}{\partial \gamma} - \frac{\partial k_1}{\partial \gamma} > 0, \tag{52}$$

therefore, when $\bar{r}$ is sufficiently close to $\frac{1}{2}$, $\phi(k_2\,y^*)\,\frac{\partial k_2}{\partial \gamma} - \frac{\partial k_1}{\partial \gamma}$ approaches $\sqrt{\frac{1}{2\,\pi}}\,\frac{\partial k_2}{\partial \gamma} - \frac{\partial k_1}{\partial \gamma}$ and is positive. Hence we have,

$$\frac{\partial y^*}{\partial \gamma} = \frac{y^*\,\left(\phi(k_2\,y^*)\,\frac{\partial k_2}{\partial \gamma} - \frac{\partial k_1}{\partial \gamma}\right)}{k_1 - k_2\,\phi(k_2\,y^*)} > 0. \tag{53}$$

■