33-761 Electrodynamics

Chapter 1 Electrostatics

1 Review of Maxwell's equations

Maxwell's Equations

$$\vec{\nabla}\cdot\vec{E}=\frac{1}{\varepsilon_0}\rho$$

$$\vec{\nabla}\cdot\vec{B}=0$$

$$\vec{\nabla}\times\vec{E}+\frac{\partial\vec{B}}{\partial t}=0$$

$$\frac{1}{\mu_0}\vec{\nabla}\times\vec{B}=\vec{J}+\varepsilon_0\frac{\partial\vec{E}}{\partial t}$$

Lorentz Force

$$\vec{F}=q(\vec{E}+\vec{v}\times\vec{B})$$

2 Coulomb's Law

Force between two charge particles,

$$\vec{F}_{12}=\frac{q_1q_2}{4\pi\varepsilon_0 r_{12}^2}\hat{r}_{12}$$

Electric field: (action at distance) for a charge q at the origin,

$$\vec{E}\left(\vec{r}\right)=\frac{q}{4\pi\varepsilon_0 r^2}\hat{r}$$

and the force on a charge q' is

$$\vec{F}=q'\vec{E}$$

Remarks:

1. $\vec{E}$ is in the radial direction, $\vec{\nabla}\times\vec{E}=0$.
2. There are two different types of charges–opposite charges attracts and same charges repluse.
3. The force is long range–no intrinsic length scale.

Principle of linear supposition: Electric field due to a collection of charges is just the sum of electric fileds due to each charge,

$$\vec{E}\left(\vec{r}\right)=\sum_i\vec{E}_i\left(\vec{r}\right)=\frac{q}{4\pi\varepsilon_0}\sum_i\frac{\left(\vec{r}-\vec{r}_i\right)}{\left|\vec{r}-\vec{r}_i\right|^3}$$

For continuous charge distribution,

$$\vec{E}\left(\vec{r}\right)=\frac{1}{4\pi\varepsilon_0}\int\frac{\rho\left(\vec{r}\right)\left(\vec{r}-\vec{r}'\right)}{\left|\vec{r}-\vec{r}'\right|^3}d^3r'$$

Identity

$$\frac{\vec{r}}{r^3}=-\vec{\nabla}\left(\frac{1}{r}\right)$$

The electric field can then be written as,

$$\vec{E}\left(\vec{r}\right) = -\vec{\nabla}\phi$$

where

$$\phi\left(\vec{r}\right) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho\left(\vec{r}\right)}{\left|\vec{r}-\vec{r}'\right|}d^3r' \qquad \text{Coulomb potential}$$

Example : Spherical Charge distribution $\rho\left(\vec{r}\right) = \rho\left(r\right)$, for $r \leq a$.

$$\phi\left(r\right) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho\left(\vec{r}\right)}{\left|\vec{r}-\vec{r}'\right|}d^3r' = \frac{1}{4\pi\varepsilon_0}\int r'^2dr'\rho\left(r'\right)\int \frac{\sin\theta' d\theta' d\phi}{\left|\vec{r}-\vec{r}'\right|}$$

Take $\vec{r}$ tobe the z-axis for $\vec{r}'$, then

$$\left|\vec{r}-\vec{r}'\right| = \left(r^2+r'^2-2rr'\cos\theta'\right)^{\frac{1}{2}}$$

and

$$\begin{aligned} I\left(r,r'\right) &= \int \frac{\sin\theta' d\theta' d\phi}{\left|\vec{r}-\vec{r}'\right|} = 2\pi\int \frac{dz}{\left(r^2+r'^2-2rr'z\right)^{\frac{1}{2}}} \qquad \text{where } z=\cos\theta' \\ &= 2\pi\frac{2}{\left(-2rr'\right)}\left[\left|r-r'\right|-\left(r+r'\right)\right] = 4\pi\begin{cases} \frac{1}{r} & r>r' \\ \frac{1}{r'} & r<r' \end{cases} \end{aligned}$$

1. $r > a$

   $$\phi\left(r\right) = \frac{Q}{4\pi\varepsilon_0 r}, \qquad \text{where} \quad Q = 4\pi\int \rho\left(r\right)r^2dr \qquad \text{total charge}$$

   and

   $$\vec{E}\left(\vec{r}\right) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{r}$$

2. $r < a$

   $$\phi\left(r\right) = \frac{1}{\varepsilon_0}\left[\frac{1}{r}\int_0^r \rho\left(r'\right)r'^2dr' + \int_r^a \frac{\rho\left(r'\right)r'^2dr'}{r}\right]$$

   and

   $$\vec{E}\left(\vec{r}\right) = \frac{1}{4\pi\varepsilon_0}\frac{Q\left(r\right)}{r^2}\hat{r}$$

   where

   $$Q\left(r\right) = 4\pi\int_0^r \rho\left(r'\right)r'^2dr'$$

   is the charge inside the spherical surface of radius r.

3 Dirac Delta Function $\delta\left(x\right)$

Formal definition

$$\delta\left(x\right) = 0, \quad \text{for} \quad x \neq 0$$

and

$$\int_{-\infty}^{\infty}\delta\left(x\right)f\left(x\right)dx = f\left(0\right)$$

$\delta\left(x\right)$ as a limit of a series of functions

1.

$$\delta_n\left(x\right) = \begin{cases} 0 & x < -\frac{1}{2n} \\ n & -\frac{1}{2n} \leq x \leq \frac{1}{2n} \\ 0 & \frac{1}{2n} \leq x \end{cases}$$

2.
$$\delta_n(x) = \frac{n}{\sqrt{\pi}} \exp\left(-n^2x^2\right)$$

3.
$$\delta_n(x) = \frac{n}{\pi} \frac{1}{(1+n^2x^2)}$$

$$\delta_n(x) = \frac{\sin n\pi x}{\pi x} = \frac{1}{2\pi}\int_{-n}^{n} \exp(ixt)\, dt$$

Remark: $\lim_{n\to\infty} \delta_n(x)$ does not exist. But we do have

$$\int_{-\infty}^{\infty} \delta_n(x)\, dx = 1, \qquad \text{and} \qquad \lim_{n\to\infty}\int_{-\infty}^{\infty} \delta_n(x) f(x)\, dx = f(o)$$

$\delta(x)$ is called generalized function or distribution.
$\delta'(x)$ is defined by

$$\int_{-\infty}^{\infty} \delta^{'}(x) f(x)\, dx = -f'(o)$$

Useful properties of δ-functions

1.
$$\int_{-\infty}^{\infty} \delta(x-a) f(x)\, dx = f(a), \qquad \int_{-\infty}^{\infty} \delta'(x-a) f(x)\, dx = -f'(a)$$

2.
$$x\delta(x) = 0$$

3.
$$\int_{-\infty}^{\infty} \delta(ax) f(x)\, dx = \frac{1}{a} f(o)$$

4.
$$\delta(x) = \delta(-x), \qquad \delta'(-x) = -\delta'(x)$$

5.
$$\delta(g(x)) = \frac{1}{\left|\frac{dg}{dx}\right|}\delta(x-x_0), \qquad \text{where} \quad g(x_0) = 0$$

6.
$$\delta(x) = \frac{d}{dx}\theta(x)$$

7.
$$\delta(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \exp(-ikx)$$

$$x\delta'(x) = -\delta(x)$$

$\delta-$function and completeness

$\{f_i\}$ complete set of orthonormal functions. For arbitrary function $f(x)$,

$$f(x) = \sum_n c_n f_n(x), \qquad \text{with} \qquad c_n = \int_a^b f_n^*(y) f(y)\, dy$$

We can write

$$f(x) = \sum_n c_n f_n(x) = \int_a^b dy f(y) \sum_n f_n^*(y) f_n(x)$$

Compare with

$$f(x) = \int_a^b dy f(y)\, \delta(x-y)$$

we get

$$\sum_n f_n^*(y)\, f_n(x) = \delta(x-y) \qquad \text{(completeness relation)}$$

Charge density for a pointed particle

Using $\delta-$function we can write the charge density for a pointed particle as

$$\rho\left(\vec{r}\right) = q\delta^3\left(\vec{r} - \vec{s}\,(t)\right)$$

where $\vec{s}\,(t)$ is the trajectory of the particle. Then

$$Q = \int \rho\left(\vec{r}\right) d^3r = q.$$

Note that the current density can be written as,

$$\vec{J}\left(\vec{r}\right) = q\vec{v}\,\delta^3\left(\vec{r} - \vec{s}\,(t)\right)$$

where $\vec{v} = \dfrac{d\vec{s}\,(t)}{dt}$ is the velocity of the particle.

4 Gauss Law

Electric field from a point charge q at origin,

$$\vec{E}\left(\vec{r}\right) = \frac{q}{4\pi\varepsilon_0}\frac{\hat{r}}{r^2}$$

Flux of $\vec{E}$ through an infinitesmal surface with unit normal $\vec{n}$ and area da is,

$$\vec{E}\cdot\vec{n}da = \frac{q}{4\pi\varepsilon_0}\frac{\hat{r}\cdot\vec{n}}{r^2}da = \frac{q}{4\pi\varepsilon_0}\frac{\cos\theta}{r^2}da = \frac{q}{4\pi\varepsilon_0}d\Omega$$

where θ is the angle between $\vec{E}$ and $\vec{n}$ and $d\Omega$ is the solid angle substended by the infinitesmal surface with respect to the charge q. The flux through a closed surface S is then,

$$\oint_S \vec{E}\cdot\vec{n}da = \frac{q}{4\pi\varepsilon_0}\int d\Omega = \frac{q}{\varepsilon_0}$$

It is clear that this can be generalized to a continuous charge distribution $\rho\left(\vec{r}\right)$,

$$\oint_S \vec{E}\cdot d\vec{S} = \frac{1}{\varepsilon_0}\int_V \rho\left(\vec{r}\right) d^3r$$

where V is the volume enclosed by the closed surface S. This is usually called the **Gauss law**.

Gauss theorem : For arbitrary vector field $\vec{V}$ the following relation holds,

$$\oint_S \vec{V}\cdot d\vec{S} = \int_V \vec{\nabla}\cdot\vec{V}d^3r$$

Then

$$\int_V \vec{\nabla}\cdot\vec{E}d^3r = \frac{1}{\varepsilon_0}\int_V \rho\left(\vec{r}\right) d^3r$$

Since we can make V arbitrary small, we get a local relation,

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho\left(\vec{r}\right)}{\varepsilon_0}$$

This is the differential form of the Gauss law.

Example : Spherical charge distribution centered at origin ,$\rho\left(\vec{r}\right) = \rho(r)$, for $r \le a$

From symmetry, it is easy to see that $\vec{E}$ is in the radial direction, $\vec{E}\left(\vec{r}\right) = E\left(r\right)\hat{r}$. Take a spherical surface with radius around the origin and apply the Gauss law, we get for $r > a$,

$$E\left(r\right) 4\pi r^2 = \frac{Q}{\varepsilon_0}$$

or

$$\vec{E}\left(\vec{r}\right) = \frac{Q}{4\pi r^2 \varepsilon_0}\hat{r}$$

where Q is the total charge.

Note that Gauss's law is more general than Coulomb's law and includes Coulomb's law as a special case. To see this we use the vector identity,

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{A}\right) = \vec{\nabla}\left(\vec{\nabla} \cdot \vec{A}\right) - \nabla^2 \vec{A}$$

to write formally,

$$\vec{A} = \frac{1}{\nabla^2}\left[\vec{\nabla}\left(\vec{\nabla} \cdot \vec{A}\right) - \vec{\nabla} \times \left(\vec{\nabla} \times \vec{A}\right)\right]$$

This implies that one needs both $\vec{\nabla} \cdot \vec{A}$ and $\vec{\nabla} \times \vec{A}$ to reconstruct the vector field $\vec{A}$. For the case of Coulomb field, $\vec{\nabla} \times \vec{E} = 0$ because it is in the radial direction.

Scalar potential

Any vector fields which satisfies

$$\vec{\nabla} \times \vec{E} = 0$$

can be written as,

$$\vec{E} = -\vec{\nabla}\phi$$

This is the case for the static electric field (from Faraday's law). In this cas, Gauss's law becomes,

$$\nabla^2 \phi = -\frac{\rho}{\varepsilon_0} \qquad \text{Poisson equation}$$

The general solution is of the form,

$$\phi\left(\vec{r}\right) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho\left(\vec{r}\right)}{\left|\vec{r} - \vec{r}'\right|} d^3r' + u\left(\vec{r}\right)$$

where $u\left(\vec{r}\right)$ is an arbitrary solution to the Laplace equation,

$$\nabla^2 u\left(\vec{r}\right) = 0.$$

This can be verified by using the relation,

$$\nabla^2\left(\frac{1}{\left|\vec{r} - \vec{r}'\right|}\right) = -4\pi\delta^3\left(\vec{r} - \vec{r}'\right).$$

Thus, up to a solution to the Laplace equation, the Coulomb potential is the most general solution to Gauss's law for the case of the static electric field.

Example: Potential from a uniformly charged disk at distance z from the center

$$\phi = \frac{1}{4\pi\varepsilon_0}\int \frac{\sigma r' dr' d\theta'}{\left|\vec{x} - \vec{x}'\right|}$$

where σ is the surface charge density. From

$$\left|\vec{x} - \vec{x}'\right| = \left(r'^2 + z^2\right)^{\frac{1}{2}}$$

we get

$$\phi = \frac{\sigma}{4\pi\varepsilon_0}\int_0^{2\pi} d\theta \int_0^a \frac{r'dr'}{\left(r'^2+z^2\right)^{\frac{1}{2}}} = \frac{\sigma}{2\varepsilon_0}\left[\sqrt{(z^2+a^2)}-z\right]$$

The $z-$component of electric field is

$$E_z = -\frac{\partial \phi}{\partial z} = \frac{\sigma}{2\varepsilon_0}\left[\frac{-z}{\sqrt{(z^2+a^2)}}+1\right]$$

5 Potential from Surface charge distribution

Let $\sigma\left(\vec{x}\right)$ be the surface charge distribution (charge per unit area). The potential is then

$$\phi\left(\vec{x}\right) = \frac{1}{4\pi\varepsilon_0}\int \frac{\sigma\left(\vec{x}'\right)ds'}{\left|\vec{x}-\vec{x}'\right|}$$

Example : Uniform charge distribution on a plane $(z=0)$, $\sigma\left(\vec{x}'\right) = \sigma_0$

$$\left|\vec{x}-\vec{x}'\right| = \left[(x-x')^2+(y-y')^2+z^2\right]^{1/2}$$

Then

$$\phi\left(\vec{x}\right) = \frac{\sigma_0}{4\pi\varepsilon_0}\int \frac{dx'dy'}{\left[(x-x')^2+(y-y')^2+z^2\right]^{1/2}}$$

Introduce the polar coordinates,

$$x-x' = \rho\cos\theta, \qquad y-y' = \rho\sin\theta$$

then

$$\phi\left(z\right) = \frac{\sigma_0}{2\varepsilon_0}\int_0^\infty \frac{\rho d\rho}{\left(\rho^2+z^2\right)^{1/2}}$$

This is a divergent integral. However the electric field is finite,

$$E_z = -\frac{\partial\phi}{\partial z} = \frac{\sigma_0}{\varepsilon_0}\int_0^\infty \frac{z\rho d\rho}{\left(\rho^2+z^2\right)^{3/2}} = \frac{\sigma_0}{\varepsilon_0}\frac{z}{|z|} = \begin{cases} \frac{\sigma_0}{\varepsilon_0}, & z>0 \\ -\frac{\sigma_0}{\varepsilon_0}, & z<0 \end{cases}$$

Boundary conditions

In this example, we see that the surface charge density gives rise to discontinuity in the electric field. More generally we have from Gauss's law

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$$

Take a pill box with height infinitesmally small across the boundary of charge distribution. Using Gauss's theorem, we get

$$\int\left(\vec{\nabla}\cdot\vec{E}\right)d^3x = \oint_S \vec{E}\cdot\vec{dS} \simeq \left(E_\perp^{(1)}-E_\perp^{(2)}\right)A = \frac{\sigma}{\varepsilon_0}A$$

or

$$\boxed{\mathrm{E}_\perp^{(1)}\text{-}\mathrm{E}_\perp^{(2)}=\frac{\sigma}{\varepsilon_0}}$$

Since this follows directly from the Gauss's law, this holds whether electric field is static or not. But for the static electric field we have in addition,

$$\vec{\nabla}\times\vec{E} = 0$$

Take a rectangular surface across the surface and apply Stokes's theorem,

$$\int_A\left(\vec{\nabla}\times\vec{E}\right)\cdot\vec{dS} = \oint\vec{E}\cdot\vec{dl} \simeq \vec{E}_\parallel^{(1)}-\vec{E}_\parallel^{(2)} = 0$$

Thus the paralle components of the electric field are continuous across the surface.

Dipole amd dipole layers

Electrostatic potential from a pair of charges q and $-q$, separately by a distance a, is of the form

$$\phi(\vec{x}) = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{\left|\vec{x}-\vec{x}'\right|} - \frac{1}{\left|\vec{x}-\vec{x}'+\vec{a}\right|}\right]$$

When $\left|\vec{x}-\vec{x}'\right| \gg \left|\vec{a}\right|$,

$$\frac{1}{\left|\vec{x}-\vec{x}'+\vec{a}\right|} \simeq \frac{1}{\left|\vec{x}-\vec{x}'\right|} + \left(\vec{a}\cdot\vec{\nabla}\right)\left(\frac{1}{\left|\vec{x}-\vec{x}'\right|}\right) + O\left(a^2\right)$$

Then

$$\phi(\vec{x}) = -\frac{1}{4\pi\varepsilon_0}\left(\vec{p}\cdot\vec{\nabla}\right)\left(\frac{1}{\left|\vec{x}-\vec{x}'\right|}\right)$$

where $\vec{p} = q\vec{a}$ is the dipole moment of the charge pair. We can generalize this to a layer of dipoles to get

$$\begin{aligned}\phi(\vec{x}) &= -\frac{1}{4\pi\varepsilon_0}\int\left(D(\vec{x}')\,\vec{n}\cdot\vec{\nabla}'\right)\left(\frac{1}{\left|\vec{x}-\vec{x}'\right|}\right)da \\ &= \frac{1}{4\pi\varepsilon_0}\int\frac{\vec{n}\cdot\left(\vec{x}-\vec{x}'\right)}{\left|\vec{x}-\vec{x}'\right|^3}D(\vec{x}')\,da\end{aligned}$$

where $D(\vec{x})$ is the dipole per unit area and $\vec{n}$ is the unit normal to the surface.

Example: uniform layer of dipole on $xy-$ plane

Again

$$\left|\vec{x}-\vec{x}'\right| = \left[(x-x')^2+(y-y')^2+z^2\right]^{1/2}$$

and

$$\begin{aligned}\phi\left(\vec{x}\right) &= \frac{D_0}{4\pi\varepsilon_0}\int\frac{z\,dx'dy'}{\left[(x-x')^2+(y-y')^2+z^2\right]^{3/2}} \\ &= \frac{D_0 z}{2\varepsilon_0}\int_0^\infty\frac{z\rho\,d\rho}{\left(\rho^2+z^2\right)^{3/2}} = \frac{D_0}{2\varepsilon_0}\frac{z}{|z|}\begin{cases}\dfrac{D_0}{2\varepsilon_0}, & z>0 \\ -\dfrac{D_0}{2\varepsilon_0}, & z<0\end{cases}\end{aligned}$$

where D_0 is the dipole density. Thus layer of dipoles will give rise to a discontinuity in the static potential. More generally we have,

$$\phi_2\left(\vec{x}\right) - \phi_1\left(\vec{x}\right) = \frac{D\left(\vec{x}\right)}{\varepsilon_0}$$

Note that

$$\left(\vec{n}\cdot\vec{\nabla}'\right)\left(\frac{1}{\left|\vec{x}-\vec{x}'\right|}\right)da = \frac{\vec{n}\cdot\left(\vec{x}-\vec{x}'\right)}{\left|\vec{x}-\vec{x}'\right|^3}da = -d\Omega$$

where $d\Omega$ is the solid angle substended by the surface da with repected to $\vec{x}$. Then we have

$$\phi(\vec{x}) = -\frac{1}{4\pi\varepsilon_0}\int D\left(\vec{x}'\right)d\Omega$$

6 Electrostatic Energy

Electric force on a charge particle is

$$\vec{F} = q\vec{E}$$

For the case where $\vec{E}$ is derived from a potential,

$$\vec{E} = -\vec{\nabla}\phi$$

the equation of motion is then

$$m\frac{d^2\vec{x}}{dt^2} = -q\vec{\nabla}\phi$$

Take scalar product with $\frac{d\vec{x}}{dt}$,

$$\frac{d}{dt}\left[\frac{m}{2}\left(\frac{d\vec{x}}{dt}\right)^2\right] = -q\frac{d\vec{x}}{dt}\cdot\vec{\nabla}\phi$$

or

$$\frac{d}{dt}\left[\frac{m}{2}\left(\frac{d\vec{x}}{dt}\right)^2 + q\phi\right] = 0$$

We recognize this is the energy conservation and $q\phi$ is just the potential energy associated with the static electic field. Another way to get the same result is to look at the work done in moving charge q in the external electric field from A to B,

$$W = -\int_A^B \vec{F}\cdot d\vec{l} = q\int_A^B \vec{\nabla}\phi\cdot d\vec{l} = q\int_A^B d\phi = q\left(\phi_B - \phi_A\right)$$

So $q\phi$ is seen to be the potential energy. Note that this result is independent of the path it takes from A to B. Thus for a closed path $A = B$, and

$$0 = \oint_C \vec{E}\cdot d\vec{l} = \int_S \left(\vec{\nabla}\times\vec{E}\right)\cdot d\vec{S}$$

Since S is arbitrary, we get

$$\vec{\nabla}\times\vec{E} = 0$$

Suppose we bring charges $q_1, \ldots q_n$, in succession from infinity to a localized region. The work done on charge q_i is

$$W_i = q_i\phi_i\left(\vec{x}_i\right)$$

where $\phi_i\left(\vec{x}_i\right)$ is the potential due to charges $q_1, \ldots q_{i-1}$,

$$\phi_i\left(\vec{x}\right) = \frac{1}{4\pi\varepsilon_0}\sum_{j<i}\frac{q_j}{\left|\vec{x} - \vec{x}_j\right|}$$

The total potential energy is then,

$$W = \sum_i W_i = \frac{1}{2}\sum_{i\neq j}\frac{q_i q_j}{4\pi\varepsilon_0\left|\vec{x}_i - \vec{x}_j\right|}$$

This generalize to the case of continuous charge distribution,

$$W = \frac{1}{4\pi\varepsilon_0}\frac{1}{2}\int d^3x d^3y \frac{\rho\left(\vec{x}\right)\rho\left(\vec{y}\right)}{\left|\vec{x} - \vec{y}\right|}$$

Or

$$W = \frac{1}{2}\int d^3x\rho\left(\vec{x}\right)\phi\left(\vec{x}\right)$$

From Gauss's law, $\rho\left(\vec{x}\right) = -\varepsilon_0\nabla^2\phi$, we can write

$$W = -\frac{\varepsilon_0}{2}\int d^3x\phi\nabla^2\phi = \frac{\varepsilon_0}{2}\int d^3x\vec{E}^2$$

We can identify the energy density as

$$u\left(\vec{x}\right) = \frac{\varepsilon_0}{2}\vec{E}^2$$