

A Caveat on the Use of the Quadratic Assignment Procedure

DAVID KRACKHARDT

*The Heinz School of Public Policy and Management
Carnegie Mellon University*

ABSTRACT: The Quadratic Assignment Procedure (QAP) has been suggested as a test of fit for structural data. An argument is made that this test is inappropriate on logical grounds and because its application leads to results that are difficult to interpret. By generating random samples under two simple models, this test is shown to be biased; moreover, the bias is not consistent but rather can be liberal or conservative, small or large, depending on parameters in the population from which the data are sampled: Quadratic Assignment Procedure, Goodness-of-fit, Network Analysis.

KEY WORDS:

Structural data, such as social network data or spatial data, pose a serious problem to the social scientist who wishes to test hypotheses. This problem stems from the fact that the observations are not mutually independent. In 1976, Hubert and Schultz published an influential paper in which they proposed a general method for testing hypotheses with data of this form. Their technique, called the Quadratic Assignment Procedure (QAP), was based on the pioneering work of Mantel (1967).

Since then, many papers have emerged that have generalized this work to varied problem areas (Sokal 1979; Baker and Hubert 1981; Hubert, Golledge and Costanzo 1981; Douglas and Endler 1982; Hubert and Golledge 1981; Nakao and Romney 1984; Faust and Romney 1985; Dow 1985; Dow and Cheverud 1985; Krackhardt 1987, 1988; Krackhardt and Kilduff 1990; Krackhardt and Porter 1986; see Hubert 1987, for a thorough review). The concern is that the enthusiasm generated around this procedure may have clouded our understanding of the bounds of its application. This paper was motivated by the observation that a growing number of articles are emerging that apply the procedure when it may not be appropriate.

QAP AS A GOODNESS-OF-FIT TEST

In an important article, Hubert and Golledge (1981) propose that the QAP can be used to perform four different kinds of statistical tests. One of these four is a test of "reconstruction" or "goodness-of-fit."

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When a given data set has been reconstructed by some model, a natural question arises as to how well the reconstruction exhausts the manifest data structure. . . . We propose to focus on the relation between an original and a residual matrix as a means of assessing whether the original data are reconstructed adequately . . . (If \mathbf{A} is a data set and \mathbf{C} represents some reconstruction, then $r_{A,A-C}$ is the correlation of the original data and a residual matrix. The closer $r_{A,A-C}$ is to zero, the better the correspondence between \mathbf{A} and \mathbf{C} . . . (A)nd a test of $r_{A,A-C}$ against zero would be a test of goodness-of-fit for the reconstruction. [Hubert and Golledge, 1981, p. 221]

Hubert and Golledge provide an example to demonstrate their proposed test of fit. An empirical matrix \mathbf{A} of proximities among 14 objects (colors) is converted to a distance matrix and then normalized so that $\sum_{i,j} \mathbf{A}_{i,j} = 0$ and $\sigma^2(\mathbf{A}_{i,j}) = 1$ (for all $i \neq j$). I will designate this normalized distance matrix as \mathbf{A}^* . Three multidimensional scaling solutions are calculated from the proximity matrix \mathbf{A} , one for one dimension, one for two dimensions, and one for three dimensions. For each solution, a corresponding interpoint distance matrix ($\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3$, respectively) was calculated. Each of these distance matrices was also normalized to mean = 0 and variance = 1, creating $\mathbf{B}_1^*, \mathbf{B}_2^*$, and \mathbf{B}_3^* . The following element-wise correlations were calculated from the normalized matrices, and the associated p-values were determined through the Hubert-Golledge QAP test of fit:

1. $r_{A^*, A^* - B_1^*} = 0.379 \quad p \leq 0.01$
2. $r_{A^*, A^* - B_2^*} = 0.125 \quad p \leq 0.09$
3. $r_{A^*, A^* - B_3^*} = 0.109 \quad p \leq 0.13$

The QAP test of fit was performed in the traditional manner of any QAP test. The probabilities are based on a sample of 99 of the 14! permutations of the \mathbf{A}^* matrix. For example, to test the first correlation above, the rows and columns of \mathbf{A}^* were randomly permuted to give $\mathbf{A}^{*'}$, and then $r_{A^{*'}, A^{*'} - B_1^*}$ was calculated to give one sample value for the reference distribution for the null hypothesis. This procedure was repeated 98 more times so that a total of 99 sample r 's are provided for the reference distribution. The p -value of 0.01 for the first correlation is strictly interpreted as meaning that none of the r 's obtained from the 99 permutations was greater than the observed $r_{A^*, A^* - B_1^*}$.

The conclusion Hubert and Golledge draw from these results is: "[W]e see that a two-dimensional solution is suggested by size of the associated . . . p -values" (p. 223). That is, since the first correlation based on a one-dimensional solution is significant, we are to conclude that the one-dimensional solution is not an adequate reconstruction of the data. Since the two-dimensional correlation is *not* significant (apparently Hubert and Golledge were using a traditional 0.05 level of significance as a decision

rule), we are to conclude that the reconstruction based on a two-dimensional solution is adequate, that the reconstruction fits the data.

This procedure has been used by several authors to test the goodness-of-fit against their data (e.g., Gale, Hubert, Tobler, and Golledge 1983; Nakao and Romney 1984; Dow 1985; Krackhardt and Brieger 1985). I claim that the QAP should not be used as a test of fit as Hubert and Golledge suggested. I argue that the null hypothesis is ambiguous with at least two possible interpretations. Under each interpretation of the null hypothesis, I argue that the probabilities attributed to the test results are not meaningful.

INTERPRETATION 1: $H_0: \rho_{A, A^* - C^*} = 0$

As stated explicitly by Hubert and Golledge, the test of fit of the reconstruction is . . . “a test of $r_{A, A-C}$ against zero . . .” What this phrase means is unclear; testing the sample statistic $r_{A, A-C}$ “against zero” is not a clear statement about what the null hypothesis is. But one possible interpretation is that it is a test against the null hypothesis that the true population correlation between A and $A^* - C^*$ is 0, where A^* and C^* are normalized scores with mean 0 and variance 1. In other words, one might interpret this test as a test of the null hypothesis that explicitly states that $H_0: \rho_{A, A^* - C^*} = 0$ (I use the Greek ρ to refer to the true population value and r to refer to the observed value).

It is easy to show that this proposed test statistic is determined by the size of $r_{A, C}$ in the following way (see Appendix for proof):

$$r_{A, A^* - C^*} = \frac{\sqrt{1 - r_{A, C}}}{2}$$

This formula (and its proof) equally applies to the population parameters, so that ρ may be substituted for r . Note that, substituting ρ for r in this formula, the only way $\rho_{A, A^* - C^*} = 0$ is if $\rho_{A, C} = 1$. It follows that the null hypothesis can be equivalently restated as follows:

$$H_0: \rho_{A, C} = 1$$

If we assume an underlying model where $\rho = 1$, then every sample generated by this model must also equal 1 (although the converse is not necessarily true). One can think of this problem as sampling from a straight line. Any set of points sampled from a straight line (where $\rho = 1$) must also lie on a straight line. The introduction of error at any point means that it is no longer true that $\rho = 1$. A value of $\rho = 1$ means that the reconstruction is perfect, that there is no error of any kind (measurement or otherwise). Therefore, if we observe any $\rho_{A, C} \neq 1$ (or, equiva-

lently, $r_{A^*-C^*} \neq 0$), we know with certainty this particular null hypothesis cannot be true. And, since we know the null hypothesis cannot be true, then performing a test of that null hypothesis is unreasonable. Hubert and Golledge seem to acknowledge that this is unreasonable on p. 221, where they interpret the statistic “. . . as long as $r_{A,C} \neq 1$.”

INTERPRETATION 2: MODEL IS CORRECT EXCEPT FOR ERROR

Another possible justification for this test of fit of a reconstruction is to claim that the procedure is testing a different null hypothesis, something other than $\rho_{A,A^*-C^*} = 0$. The remainder of this paper is devoted to showing how the test behaves if we were to choose this alternative interpretation.

A reasonable argument could proceed as follows. A traditional goodness-of-fit test assesses the probability that we could draw a sample similar to the one we observe from some hypothesized model. The probability assessment, then, has a precise interpretation: The resulting p -value from the goodness-of-fit test is the probability that we would find the observed test-statistic value (or some value more extreme) if we were to generate data repeatedly from this model. Note that this logic does not require us to generate data where $\rho_{A,A^*-C^*} = 0$; it only requires that the test provide us with an accurate assessment of the probability that the sample was generated from some specified model.

To pursue this logic, it might be argued that the QAP test of reconstruction could be used if it provided us with a reasonable answer to this probabilistic question. That is, we could assert that the QAP test of reconstruction performs the same function as a traditional goodness-of-fit test. Suppose we were to repeatedly generate sample data from some known model. Suppose further we were to test each sample using the procedure suggested by Hubert and Golledge. And finally, suppose that we found that the probability value ascribed to the QAP test roughly corresponded to the probability of actually finding such a sample generated from that known model. Then, one could argue, the QAP test is relatively unbiased and may be a useful test of fit.

Our present task is to explore the extent of bias in the QAP goodness-of-fit test by comparing the outcomes of the statistical decisions based on the QAP test to the probability of observing the statistical values when the data are generated in accordance with a known structural model.

A STEP-BY-STEP EXAMPLE OF THE QAP TEST OF FIT

Again, I am assuming that Hubert and Golledge's test of “adequate recon-

struction" is comparable in logic to a test of fit. Goodness-of-fit tests are a two-step procedure. First, model parameters (which Hubert and Golledge refer to as the "reconstruction") are estimated from the data. Second, we assume those estimates are correct and ask the question how likely it is that these data could have been generated by those parameter (or reconstruction) value.

To illustrate the Hubert and Golledge procedure, I will use a row-dominated model. The row-dominated model is a model of row parameters that generate the matrices of observations. Dow (1985) used such a model in his test of rhesus monkeys' migration patterns. In this case, the null model is that the entries in the cell are largely determined (except for minor errors) by the attributes of the social group, represented by the row of the matrix. Formally, this model is specified below:

$$\mathbf{A} = \mathbf{M} + K\varepsilon$$

Where:

\mathbf{M} is a matrix of order $N \times N$ (less the diagonals) of parameters whose values are identical within rows (i.e., $\mathbf{M}_{ij} = \mathbf{M}_{ik}$). Each row parameter (\mathbf{M}_i) is drawn from a $N(0, 1)$ distribution.

ε is a same order matrix of error terms, $\sim N(0, 1)$.

K is a constant weighting factor for the error terms.

That is, a model (\mathbf{M}) was built by fixing a matrix of N row parameter values. For each sample, a matrix of observed data (\mathbf{A}) was created by adding a weighted normal error term of each cell of \mathbf{M} .

The next step was to create \mathbf{C} , the "reconstruction" matrix. In this case, we are testing whether the data are born from a row-dominated model. To estimate the row parameters, \mathbf{C} will consist of row-average values of the observed matrix \mathbf{A} :

$$\mathbf{C}_{ij} = \frac{\sum_{j=1}^N \mathbf{A}_{ij}}{N-1}, \text{ for } i \neq j$$

Both \mathbf{A} and \mathbf{C} are normalized to mean = 0 and variance = 1, and these normalized matrices will be referred to as \mathbf{A}^* and \mathbf{C}^* , respectively. Finally, the test of $r_{\mathbf{A}^*, \mathbf{A}^* - \mathbf{C}^*}$ is calculated as described earlier.

To be clear about this procedure, an example is provided below (refer to Table 1). \mathbf{M} is the matrix of parameters from which the observations \mathbf{A} are derived. \mathbf{C} refers to the reconstructed matrix against which \mathbf{A} is compared.

In this example, $N = 5$, and $K = 0.1$. The first matrix, \mathbf{M} , is composed of a set of 20 parameters, wherein the row parameters are all equal (recall that diagonals are ignored here). In other words, the values in the first row are all -0.33454 , the values in the second row are all -0.7757 , and so on.

TABLE I
 Example of calculations for simulation of row dominated model ($N = 5$, $K = 0.1$; values rounded to the nearest hundredths)

	0.00	-0.33	-0.33	-0.33	-0.33	
M =	-0.78	0.00	-0.78	-0.78	-0.78	Model of
	-1.32	-1.32	0.00	-1.32	-1.32	"true"
	0.08	0.08	0.08	0.00	0.08	parameters
	-0.67	-0.67	-0.67	-0.67	0.00	
	0.00	-0.42	-0.48	-0.27	-0.37	
A =	-0.70	0.00	-0.66	-0.81	-0.92	"Observed" data
	-1.24	-1.31	0.00	-1.36	-1.24	(= model + error)
	0.20	-0.01	-0.14	0.00	-0.06	
	-0.67	-0.69	-0.68	-0.65	0.00	
	0.00	-0.38	-0.38	-0.38	-0.38	
C =	-0.77	0.00	-0.77	-0.77	-0.77	Reconstruction as
	-1.29	-1.29	0.00	-1.29	-1.29	estimated from
	-0.01	-0.01	-0.01	0.00	-0.01	"observed" data
	-0.67	-0.67	-0.67	-0.67	0.00	
	0.00	0.48	0.34	0.82	0.59	
A* =	-0.18	0.00	-0.07	-0.43	-0.69	Normalized
	-1.42	-1.58	0.00	-1.69	-1.44	"observed" data
	1.89	1.41	1.12	0.00	1.29	
	-0.10	-0.15	-0.12	-0.62	0.00	
	0.00	0.57	0.57	0.57	0.57	
C* =	-0.35	0.00	-0.35	-0.35	-0.35	Normalized
	-1.56	-1.56	0.00	-1.56	-1.56	reconstruction
	1.45	1.45	1.45	0.00	1.45	
	-0.11	-0.11	-0.11	-0.11	0.00	
	0.00	-0.08	-0.23	0.25	0.02	
A* - C* =	0.17	0.00	0.28	-0.08	-0.34	Difference in
	0.14	-0.02	0.00	-0.13	0.12	normalized
	0.44	-0.05	-0.34	0.00	-0.16	matrices
	0.01	-0.04	-0.01	0.05	0.00	
observed $r_{A^*, A^* - C^*} = 0.096$						
First permutation						
	0.00	0.48	0.34	0.59	0.82	
A*' =	-0.18	0.00	-0.07	-0.69	-0.43	A sample
	-1.42	-1.58	0.00	-1.44	-1.69	permutation
	-0.10	-0.15	-0.12	0.00	-0.06	of A*
	1.89	1.41	1.12	1.29	0.00	
$r_{A^*, A^* - C^*} = 0.009$						

The “observed” data are assumed to be born from these parameter values, and they differ from the parameters by a small error term, $K\varepsilon_{ij}$. Then the model, C , is estimated or “reconstructed” by averaging the row elements of A . The elements of A and C are then normalized to mean = 0 and variance = 1 in A^* and C^* , respectively. The difference between A^* and C^* is then calculated. The observed correlation of this difference matrix with the A^* matrix (or the A matrix, it makes no difference) is the test statistic Hubert and Golledge define, here equal to 0.096279.

To calculate the reference distribution against which this observed statistic is compared, one must permute either the difference matrix, $A^* - C^*$, or the A^* . I will permute the latter. The first random permutation, given as $A^{*'}$, is provided at the bottom of Table 1. Rows and Columns 4 and 5 are switched in this permutation. After the permutation, the test statistic, $R_{A^{*'}, A^* - C^*}$, is recalculated (= 0.009372).

In the particular case of $N = 5$, it is easy to enumerate all 120 permutations of the matrix. With each permutation, the test statistic is recalculated, and the reference distribution is thereby generated. The reference distribution for this example is given in Table 2. As one can see from this table, the observed correlation (= 0.096279) in this example was ranked 110 out of 120, or 11th from the top. Thus, one would conclude that this observed statistic was significant at the $p = 0.092$ (= 11/120).

STATISTICAL BIAS AND THE USE OF RESTRICTED PERMUTATIONS

We should note at this point that Hubert and Golledge anticipated a liberal bias in their article. Specifically, they discuss bias in a section that proposes a different application of the QAP test (not the goodness-of-fit test): . . .

[T]he comparison scheme we have just outlined is so general that great care must be used when interpreting the results obtained from specific applications. . . . [C]ertain comparisons may be biased by the way in which they are constructed. . . . Whenever a positive bias may exist in the comparison under study, a failure to reject could be the most interpretable outcome. Stated somewhat differently, a significant value for $r_{A, B - C}$ can be viewed as a necessary but not sufficient condition for arguing, say, for the superiority of one reconstruction [B , in this case] over the second [C]. [p. 220].

And in subsequent articles, Hubert and others suggested there was reason to suspect a sizeable liberal bias in the goodness-of-fit test because of the constraint that the statistic $r_{A, A^* - C^*}$ must be positive. To correct for this, Gale et al. (1983), recommended comparing the observed r value to

TABLE II
Rank order of test statistics based on 120 permutations (values rounded to the nearest ten thousandths)

1	-0.1578	31	-0.0445	61	0.0050	91	0.0478
2	-0.1396	32	-0.0441	62	0.0050	92	0.0491
3	-0.1328	33	-0.0441	63	0.0053	93	0.0491
4	-0.1328	34	-0.0402	64	0.0053	94	0.0527
5	-0.1145	35	-0.0402	65	0.0068	95	0.0527
6	-0.1090	36	-0.0347	66	0.0068	96	0.0551
7	-0.1090	37	-0.0347	67	0.0090	97	0.0554
8	-0.1014	38	-0.0286	68	0.0090	98	0.0554
9	-0.0945	39	-0.0286	69	0.0094	99	0.0575
10	-0.0945	40	-0.0281	70	0.0131	100	0.0575
11	-0.0874	41	-0.0281	71	0.0131	101	0.0661
12	-0.0830	42	-0.0280	72	0.0171	102	0.0721
13	-0.0830	43	-0.0280	73	0.0171	103	0.0757
14	-0.0810	44	-0.0279	74	0.0180	104	0.0757
15	-0.0810	45	-0.0279	75	0.0181	105	0.0767
16	-0.0794	46	-0.0258	76	0.0181	106	0.0767
17	-0.0774	47	-0.0258	77	0.0222	107	0.0793
18	-0.0774	48	-0.0186	78	0.0258	108	0.0793
19	-0.0703	49	-0.0178	79	0.0272	109	0.0802
20	-0.0703	50	-0.0064	80	0.0272	110	0.0963
21	-0.0648	51	-0.0064	81	0.0281	111	0.1044
22	-0.0648	52	-0.0063	82	0.0281	112	0.1148
23	-0.0647	53	-0.0063	83	0.0302	113	0.1148
24	-0.0647	54	-0.0034	84	0.0302	114	0.1175
25	-0.0640	55	-0.0034	85	0.0305	115	0.1185
26	-0.0640	56	-0.0026	86	0.0347	116	0.1185
27	-0.0538	57	-0.0026	87	0.0347	117	0.1212
28	-0.0451	58	-0.0005	88	0.0437	118	0.1345
29	-0.0451	59	0.0043	89	0.0437	119	0.1459
30	-0.0445	60	0.0043	90	0.0478	120	0.1459

a restricted reference distribution composed of only permutations that generate positive values (Dow, 1985, also uses this procedure).

Referring back to the row-dominated model described in Tables 1 and 2, Gale et al. (1983) would adjust our prior conclusion about the significance level of the observed statistic. Instead of 120 permutations, we would only count the permutations that result in correlations greater than 0 in Table 2 as part of our reference distribution. One can see that there are 62 such permutations. Thus, our new significance level would be $p = 0.177$ ($= 11/62$) instead of the $p = 0.092$ calculated earlier. If we were using a prior alpha level of 0.10, then we would have concluded that the data were significantly different from the reconstructed model using the

unrestricted permutation test, but we would have concluded that the data were not significantly different from the model using the restricted permutation test, I note here that the unrestricted permutation test will necessarily result in a smaller p -value, and consequently the unrestricted test will always increase the chances that a sample data set will be found to be significantly different from the reconstruction, relative to the restricted test.

To see how these tests behave, I conducted Monte Carlo simulations of the exact procedure recommended by Hubert and Golledge. First, a "true" model of parameters is assumed. Second, "observed" data are generated by adding a small amount of error to the fixed parameter values. Third, the model parameters are estimated, or reconstructed, from the observed data. Fourth, the data are tested using the Hubert-Golledge test to determine whether the observations are significantly different from the reconstructed estimates. If the Hubert-Golledge test has a reasonable probabilistic interpretation, then the probability of the test showing significant results at the alpha level should be approximately alpha.

I have generated two different types of models, one based on the row-dominated model described in Tables 1 and 2, and a second MDS model that precisely follows the suggested application in Hubert and Golledge discussed earlier. First, I will describe the simulation results of the row-dominated model.

ROW-DOMINATED MODEL RESULTS

The parameters in the \mathbf{M} (Table 1) were fixed. Samples were generated by adding a known amount of error. The size of the error was determined by the size of K . K took on one of three relatively small values: 0.001, 0.01, and 0.1, so that in no case was the amount of error in the data overwhelming. Additionally, I generated data using three different N -sizes, one with $N = 5$ (the one described in Table 1), one with $N = 10$, and one with $N = 20$. For each combination of N -size and K -weight, I generated 1000 samples (\mathbf{A} in Table 1). For each sample, I tested the QAP statistic by permuting the matrix 1000 times for models based on $N = 10$ and $N = 20$; and for $N = 5$, the entire set of 120 permutations was used to test the statistic.

The performance of a statistical test should not depend on the particular arbitrary alpha level chosen. If the data are generated in accordance with the tested model, then the test should reject the model alpha fraction of the time, regardless of what alpha is. Specifically, 5 percent of the samples should reject the model at the alpha = 0.05 level; 10 percent of the samples should reject the model at the alpha = 0.10 level, and so on. I performed tests for three alpha levels on each sample; alpha = 0.05,

alpha = 0.10, and alpha = 0.20. Each test was based on both restricted and unrestricted permutations.

The results of these simulations are presented in Table 3. When $K = 0.001$ and $N = 5$, 187 of the 1000 samples were found to be significant using the unrestricted permutation test at the 0.05 level. Only 37 of the samples were found to be significant at the 0.05 level using the restricted permutation test. It is interesting to note that these proportions do not change dramatically as the amount of error increases for the $N = 5$ set. When $K = 0.1$, 165 of the samples are significant using the unrestricted permutation test, and 39 of the samples are significant using the restricted test.

While this insensitivity to error size may seem encouraging, it is misleading. This robustness does not reappear in samples where $N = 10$ or $N = 20$. More importantly, in none of the samples and under none of the tests does the probability of finding a significant result correspond to the alpha level chosen. When $N = 5$ and a criterion of 0.05 is used based on restricted permutations, the probability of finding a significant result is slightly less than alpha. When a criterion of 0.10 is used, the probability of finding a significant result is somewhat more than 0.10. In all other cases, the probability of a significant finding is far greater than alpha. Virtually all of the samples generated when $K = 0.01$ or greater and $N = 10$ or greater were significant at all three levels of alpha.

TABLE III
Results of simulations of row-dominated model. Proportion of simulated samples that were deemed significantly different from the reconstruction at the prescribed Alpha level

K	N	Alpha = 0.05		Alpha = 0.10		Alpha = 0.20		Number of Samples
		U	R	U	R	U	R	
0.001	5	0.187	0.037	0.592	0.168	0.982	0.549	1000
0.010	5	0.168	0.036	0.603	0.129	0.986	0.568	1000
0.100	5	0.165	0.039	0.647	0.167	0.991	0.614	1000
0.001	10	0.731	0.631	0.847	0.731	0.940	0.839	1000
0.010	10	0.999	0.930	1.000	1.000	1.000	1.000	1000
0.100	10	1.000	0.919	1.000	1.000	1.000	1.000	1000
0.001	20	0.653	0.614	0.682	0.651	0.729	0.686	1000
0.010	20	1.000	1.000	1.000	1.000	1.000	1.000	1000
0.100	20	1.000	1.000	1.000	1.000	1.000	1.000	1000

U = Unrestricted permutation tests

R = Restricted permutation tests

MDS MODEL RESULTS

As mentioned earlier, Hubert and Golledge suggested that this test could be used to determine whether a particular n -dimensional MDS solution of a matrix of interpoint distances adequately accounts for the original data. The procedure to test the MDS solution is somewhat more complicated than that used to test the row-dominated model, but the underlying logic is very similar.

TABLE IV
Example of calculations for simulation of MDS Model ($N = 5, K = 0.001$; value rounded to the nearest thousandths)

Set of given $\langle x, y \rangle$ coordinates in model:						
	0.899	0.467				
	0.370	0.455				
	0.887	0.986				
	0.842	0.982				
	0.438	0.312				
	0.000	0.529	0.519	0.517	0.486	Matrix of pairwise Euclidean distances in model
	0.529	0.000	0.741	0.707	0.158	
$M =$	0.519	0.741	0.000	0.045	0.810	
	0.517	0.707	0.045	0.000	0.782	
	0.486	0.158	0.810	0.782	0.000	
	0.000	0.530	0.520	0.517	0.486	"Observed" pairwise distances (= model + error)
	0.530	0.000	0.742	0.707	0.158	
$A =$	0.520	0.742	0.000	0.044	0.809	
	0.517	0.707	0.044	0.000	0.784	
	0.486	0.158	0.809	0.784	0.000	
Set of $\langle x, y \rangle$ coordinates obtained from MDS program on negative of A (Stress > 0.01)						
	0.209	-0.849				
	0.741	0.797				
	-1.069	0.259				
	-0.890	0.038				
	1.009	-0.169				
	0.000	1.731	1.692	1.366	1.050	Reconstruction matrix of pairwise Euclidean distances calculated from MDS solution above
	1.731	0.000	1.889	1.833	1.003	
$C =$	1.692	1.889	0.000	0.348	2.121	
	1.366	1.833	0.348	0.000	0.903	
	1.050	1.003	2.121	1.903	0.000	
	0.000	0.003	-0.040	-0.052	-0.179	Normalized "observed" data
	0.003	0.000	0.868	0.728	-1.523	
$A^* =$	-0.040	0.868	0.000	-1.991	1.146	
	-0.052	0.728	-1.991	0.000	1.040	
	-0.179	-1.523	1.146	1.040	0.000	

Table IV (Continued)

	0.000	0.457	0.382	-0.246	-0.854	Normalized reconstruction
	0.457	0.000	0.762	0.653	-0.945	
$C^* =$	0.382	0.762	0.000	-2.207	1.210	
	-0.246	0.653	-2.207	0.000	0.788	
	-0.854	-0.945	1.210	0.788	0.000	
	0.000	-0.454	-0.422	0.194	0.674	Difference in normalized matrices
	-0.454	0.000	0.107	0.075	-0.578	
$A^* - C^* =$	-0.422	0.107	0.000	0.216	-0.063	
	0.194	0.075	0.216	0.000	0.252	
	0.674	-0.578	-0.063	0.252	0.000	
Observed $r = 0.1831952$						
First permutation of A^*						
	0.000	-1.991	1.146	0.868	-0.040	A sample permutation of A^*
	-1.991	0.000	1.040	0.728	-0.052	
$A^{*'} =$	1.146	1.040	0.000	-1.523	-0.179	
	0.868	0.728	-1.523	0.000	0.003	
	-0.040	-0.052	-0.179	0.003	0.000	
$r_{A^{*'}, A^* - C^*} = 0.1203248$						

Table 4 provides a step-by-step account of the Hubert-Golledge procedure. First, I assume a "true" two-dimensional model of $\langle x, y \rangle$ coordinates. From these coordinates, I calculate the exact interpoint distances of all pairs of points. This represents the "true" interpoint distances ($= M$). I add a small amount of error to these distances to create the "observed" matrix of distances A . I then estimate or reconstruct the two-dimensional model from which A was generated by multiplying A by the scalar -1 (to turn A into a similarity matrix) and running it through an MDS program, extracting the $\langle x, y \rangle$ coordinates. Note that the stress for this solution is very low (less than 0.01), which is no surprise since only a small amount of error was added to the "true" distances. I then estimate or reconstruct M by calculating the Euclidean distances among the five points ($= C$). Both A and C are normalized to mean $= 0$ and variance $= 1$ and called A^* and C^* , respectively. The difference between them is calculated, and the correlation between A^* and $A^* - C^*$ is computed as the test statistic.

To create the reference distribution, A^* was permuted 1000 times, and the test statistic $r_{A^{*'}, A^* - C^*}$ was recalculated each time. The first permutation is provided at the bottom of Table 4. The rows and columns $\langle 1, 2, 3, 4, 5 \rangle$ were reordered as $\langle 5, 4, 1, 2, 3 \rangle$ in $A^{*'}$.

Again, I varied the N -size of the matrix and the error weight K to see if the test was sensitive to those parameters (see Table 5). For each of

the nine models, 1000 samples were drawn and tested using both the restricted and unrestricted permutation tests at the 0.05, 0.10, and 0.20 alpha levels.

TABLE V
Results of simulations of MDS model. Proportion of simulated samples that were deemed significantly different from the reconstruction at the prescribed Alpha level

K	N	Alpha = 0.05		Alpha = 0.10		Alpha = 0.20		Number of Samples
		U	R	U	R	U	R	
0.001	5	0.000	0.000	0.000	0.000	0.000	0.000	1000
0.010	5	0.000	0.000	0.002	0.000	0.008	0.001	1000
0.100	5	0.018	0.006	0.037	0.014	0.129	0.034	1000
0.001	10	0.000	0.000	0.000	0.000	0.000	0.000	1000
0.010	10	0.000	0.000	0.000	0.000	0.000	0.000	1000
0.100	10	0.013	0.000	0.218	0.012	0.919	0.191	1000
0.001	20	0.000	0.000	0.000	0.000	0.000	0.000	1000
0.010	20	0.002	0.000	0.002	0.002	0.002	0.002	1000
0.100	20	1.000	0.967	1.000	0.999	1.000	1.000	1000

U = Unrestricted permutation tests

R = Restricted permutation tests

The results in this simulation are even less encouraging than in the row-dominated model simulation. When $N = 5$, all of the tests for each of the error weights were far less than the prescribed alpha levels. When $N = 10$, only in the case where 1) the prescribed alpha was 0.20, 2) the error weight 0.1, and 3) the restricted permutation test was used, did the probability (= 0.191) of a significant finding approach alpha. When $N = 20$, the results are very unstable, with the probability of a significant finding being either very close to 0 or very close to 1 under all conditions.

DISCUSSION

In summary, in neither the row-dominated model nor in the MDS model did the test perform in accordance with the statistical interpretation of the test. In the row-dominated model, the QAP test of fit was largely too liberal, although not universally. In the MDS model, the results were either vastly too liberal or too conservative, depending on the N -size and the precise size of the small amount of error added.

Recall that Hubert and Golledge anticipated a possible positive or

liberal bias in the way these tests are constructed, suggesting that a significant finding “can be viewed as a necessary but not sufficient condition” for drawing the appropriate statistical conclusion. The results of the simulations reported in this paper suggest that an insignificant p -value is neither necessary nor sufficient for concluding that the data are “adequately reconstructed.” Because the test is sometimes too liberal, in cases where the data are disproportionately found to be significantly different from the reconstruction, one cannot claim that an insignificant result constitutes a necessary condition for concluding that the data are adequately reconstructed. Conversely, because the test is sometimes too conservative, in cases where the data are disproportionately found to be not significantly different from the reconstruction, one cannot claim that an insignificant result constitutes a sufficient condition for concluding that the data are adequately reconstructed. In short, an insignificant $r_{A^*, A^* - C^*}$ does not imply that the data are adequately reconstructed, nor does a significant $r_{A^*, A^* - C^*}$ imply that the data are not adequately reconstructed.

PERMUTATION TESTS AND PARAMETRIC SIMULATIONS

The QAP test is a permutation test, a member of a family of conditional statistical tests. The QAP test was designed for cases where parametric assumptions about the data are unknown. Hubert and Golledge are careful not to refer to a population from which the observations are sampled. In fact, no population is assumed; rather the data are assumed to comprise the population and hence no assumptions about sampling from a population are necessary.

On the other hand, the models I have used to test the behavior of this test are stochastic, parametric, non-conditional models. One might argue that my simulations have not truly tested the kinds of models the QAP test was designed for, since my parametric models are not conditional models.

The argument is insufficient. The reason for using conditional tests is that it does not require assumptions about the error terms in the population. When parametric assumptions are untenable, the nonparametric tests are used because they are not dependent on such assumptions. Thus, the nonparametric tests are more general; they apply to situations in which the parametric tests may not apply. Simply because the nonparametric test is applicable to a wider range of population characteristics and assumptions than the parametric case does not mean that the nonparametric test is inapplicable if one happens to know the parametric nature of the population. A more general test must certainly work in a specific case. It seems to me that a minimal criterion for the adequacy of a nonparametric test is that it should behave appropriately in a well-defined parametric case.

INTERPRETING $r_{A^*, A^* - C^*}$

What does the probability associated with the QAP test of fit mean? How might it be reasonably interpreted? I have argued that it is not the probability of observing this test statistic value given a null hypothesis that $\rho = 0$, as demonstrated logically earlier. And the simulations demonstrate that it clearly is not the probability of observing the statistic value given a model and assuming the observations are born from the model with a little error added.

If I were to give the Hubert-Golledge fit statistic an interpretation, I would say it was descriptive, not inferential. For example, R -square is a perfectly good measure of fit; and R -square of 0.8 is a better fit than an R -square of 0.2. If the p -value derived from the QAP test of fit is 0.1, I might be able to say, all else being equal, that the reconstruction is a better fit than if the p -value had been 0.01. I would never ascribe an inferential or probabilistic interpretation to my R -square measure. Similarly, I cannot ascribe a reasonable inferential or probabilistic interpretation to my QAP test results.

CONCLUDING REMARKS

This paper has attempted to focus attention on the problem of using QAP to test whether data are fit well by a particular reconstruction. It was shown logically that the underlying null hypothesis in the QAP test is not reasonable. Moreover, the simulations demonstrate that the QAP goodness-of-fit test is inappropriate as a statistical test to the class of models that Hubert and Golledge recommended.

In statistical terms, parametric tests of goodness-of-fit are well defined. They stipulate both a set of parameters and error distributions around those parameter values. The non-parametric QAP permutation procedure never asks the question how are the observations likely to be distributed around the population parameters. If one has no idea how the outcomes are distributed, one has little hope of answering a question about the probability of observing any particular outcome.

In conclusion, I would like to emphasize the point that QAP has opened up possibilities for the testing of hypotheses that were previously untestable. Much has been written about the distribution of r (or the raw cross-product index) under various conditions (Dietz, 1983; Mielke, 1978; 1979; Faust and Romney, 1985; Krackhardt, 1988; 1991; Romney and Weller, 1989). More attention should be placed on the conditions under which a permutation model is reasonable or interpretable, given the structure of the data. This paper hopes to make a start in that direction.

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APPENDIX:

$$\text{Proof of } r_{A, A^* - C^*} = \sqrt{\frac{1 - r_{A, C}}{2}}$$

1. By construction:

$$A^* = \frac{A - \bar{X}_A}{SD_A} \quad C^* = \frac{C - \bar{X}_C}{SD_C}$$

$$\text{Therefore, } \bar{X}_{A^*} = \bar{X}_{C^*} = 0 \text{ and } \text{Var}(A^*) = \text{Var}(C^*) = 1$$

$$2. \quad r_{A^*, A^* - C^*} = \frac{\text{Cov}(A^*, A^* - C^*)}{\sqrt{\text{Var}(A^*) \text{Var}(A^* - C^*)}}$$

$$\begin{aligned} \text{Cov}(A^*, A^* - C^*) &= \text{Var}(A^*) - \text{Cov}(A^*, C^*) \\ &= 1 - \text{Cov}(A^*, C^*) \end{aligned}$$

$$\begin{aligned} \text{Var}(A^* - C^*) &= \text{Var}(A^*) + \text{Var}(C^*) - 2 \text{Cov}(A^*, C^*) \\ &= 2 - 2 \text{Cov}(A^*, C^*) \\ &= 2(1 - \text{Cov}(A^*, C^*)) \end{aligned}$$

$$\begin{aligned} \therefore r_{A^*, A^* - C^*} &= \frac{1 - \text{Cov}(A^*, C^*)}{\sqrt{2(1 - \text{Cov}(A^*, C^*))}} \\ &= \sqrt{\frac{1 - \text{Cov}(A^*, C^*)}{2}} \end{aligned}$$

$$3. r_{A^*, C^*} = \frac{\text{Cov}(A^*, C^*)}{\sqrt{\text{Var}(A^*) \text{Var}(C^*)}} = \text{Cov}(A^*, C^*)$$

$$\text{by substitution, } r_{A^*, A^* - C^*} = \sqrt{\frac{1 - r_{A^*, C^*}}{2}}$$

4. Since \mathbf{A} is linear function of \mathbf{A}^* , and \mathbf{C} is linear function of \mathbf{C}^*

$$r_{A^*, A^* - C^*} = r_{A, A^* - C^*} \text{ and } r_{A^*, C^*} = r_{A, C}$$

By substitution:

$$r_{A, A^* - C^*} = \sqrt{\frac{1 - r_{A, C}}{2}}$$