Lecture Notes on Pitch-Class Set Theory

Topic 5: Invariants

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Invariants

We have now learned four operations (also called transformations), two that work on pitches (pitch transposition and pitch inversion) and two that work on pitch classes (pitch-class transposition and pitch-class inversion). When you perform an operation on one set, X, to get another set, Y, sometimes X and Y have elements in common; if so, these are called **invariants**.

To take a simple example, suppose we take a one-octave run of a whole-tone scale starting on C4, and operate on it with Tp 2, transposing it up a whole step. The original set, X, and its transposition, Y, are shown in Example 1. Using pitch numbers with C4=0, the sets are as follows.

X = {0 2 4 6 8 10 12}

Y = {2 4 6 8 10 12 14}

When Tp2 operates on X, the resultant set, Y, has 6 pitches in common with X, so we say that this operation leaves 6 elements invariant. Equivalently, we could say that X and Y both have a subset Z = {2 4 6 8 10 12}.[1] We could then say that Tp2 creates an invariant subset of cardinality 6. These are all different ways of saying basically the same thing; you may encounter these phrasings (and similar ones) in your theory reading.

When working with pitch classes, it's possible to have the entire set held invariant under some transformation. For example, if set is an 048, an augmented triad, then the whole set will be invariant under T_4 and T_8. For example, if our set is {D F# Bb}, T_4 is {F# Bb D} and T_8 is {Bb D F#}. These are all the same set because pitch-class sets are unordered.

[1] If A is a set, B is a subset of A if B contains only elements that A also contains.

Working with Invariants

Given two sets, it's easy to find whether there are any invariants. It's more interesting to try to figure out what operations might produce invariant subsets, given just one set. This is most easily done working with subsets.

Suppose we are given some set, X, and we want to find transformations that create invariant subsets of some given cardinality. To make this more concrete, say our set is {C C# Eb E F F# A B}. X is shown in Example 2.

Suppose that we are looking for transformations that create invariant subsets of cardinality 3. This means that the transformed set has at least three pitch-classes that are also in the original set. Rather than thinking about the whole set, which would be overwhelming, we can focus just on the trichords that it contains. What we want to find is some trichord A contained in X, where some transposition or inversion of A is also contained in X. If A and T_nA are both subsets of X, then, because T_nX will necessarily contain T_nA, T_nA will be an invariant subset of X under T_n.

Let's deal with transpositions first. We're looking for some trichord, A, within X, such that X also includes some transposition of A. This is relatively easy, visually. Suppose we start with {B C C#}. It's easy to see that there's no other run of three pitch-classes in a row anywhere else in X. But what about {C C# Eb}? Picture the bracket that connects these pitch classes, and imagine rotating it around the circle. You will find that the bracket lines up with the set in two other places, at the subsets {Eb E F#} and {F# G A}. This means that T_3 sends {C C# Eb} to {Eb E F#}, and it also sends {Eb E F#} to {F# G A}. In addition, T_6 sends {C C# Eb} to {F# G A}.

This may lead us to observe just how many "diminished triads" there are in X: {C Eb F#}, {C# E G}, {Eb F# A}. {F# A C}, {A C Eb}. (If we care about larger invariant subsets, this could lead to the observation that under T_3, the only pitch classes *not* held invariant are G and B.) Depending on the task we have been set (find a certain number of subsets, find a certain number of transformations, find all invariant subsets, etc.) we might or might not be done at this point – but we've seen how you would go about answering questions of this sort.

Now let's deal with inversions. To find invariant trichords under inversion, pick trichords, invert them, and see if the inversions are found in the original set. For example, {C C# Eb} inverts through an axis between C# and D to {C D Eb} – the outer notes are kept fixed, and the inner note moves from being next to the lower note to being next to the upper note. If we visualize the bracket that connects these notes, we can imagine rotating it around the circle, and we find that it lines up with three subsets: {A B C}, {C# Eb E} and {E F# G}.

Then we have to figure out which inversions pair these subsets. Here the equivalence of I_n with T_nI helps us. T_0I {C C# Eb} is {A B C} – as luck would have it, this is one of the sets we wanted to find. Now we just need to find the transpositions that send {A B C} to {C# Eb E} and {E F# G}. These are T_4 and T_7. This means that the inversions that leave these trichords invariant are I_0, I_4 and I_7.

Remember that these operations are understood to apply to the entire set, X. Taking one of our examples, if A={C C# Eb}, because X contains both A and I_0A, I_0X will also contain I_0A; I_0A is therefore an invariant subset. Do you see why subsets make finding invariants easier? X contains eight pitch-classes, so it would be rather tedious to find each inversion and transposition and then count up how many pitch classes they had in common.

It's also possible to find invariant subsets visually, although there's less of a step-by-step path to that. It's done by looking for symmetries within the set. For example, since it's relatively easy to visualize reflections through the vertical axis, you may already have noticed that I_0 doesn't just leave that one trichord invariant; much of the set is symmetrical around the C-F# axis, in fact all pitch classes except E and G. The two spots where there are two adjacent pitch classes might also catch our eye; if we find the reflection that sends these to each other, the reflection through the F-B axis, we will see that the only pitch-class not held invariant under I_{10} is C.

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Examples for Topic 5: Invariants

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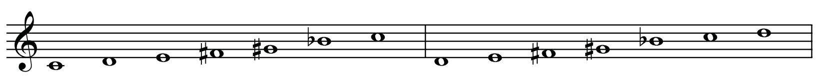
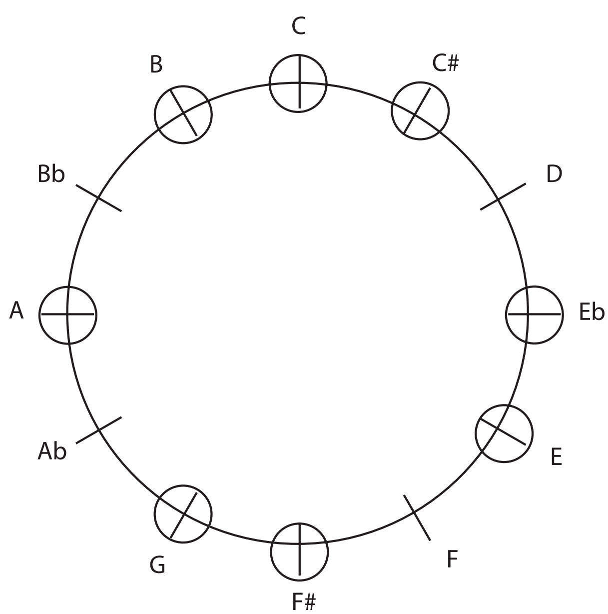


Example 1



Example 2