Lecture Notes on Pitch-Class Set Theory

Topic 4: Inversion

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Inversion

We have already seen in the notes on set classes that while in tonal theory, to invert a chord is to take the lowest note and put it higher up, in theory of 20^{th}-century music to invert a set is to flip it upside down.

Pitch Inversion

We have already seen that major triads and minor triads invert to each other. Inversion always involves some **axis**, a point around which we are inverting. To stick with familiar tonal chords, let's take the triad {E4 G#3 B4} and invert it around D4. E4 is a major second above D4, so it inverts to the note a major second below, C4. G#4 is an augmented fourth above, so it inverts to an augmented fourth below, Ab3. Finally, B4 is major sixth above D4, so it inverts to F3, a major sixth below, and our new set is {F3 Ab3 C4}.

We can do the same thing using numbers. {4 8 11} inverts through 2 to {-7 -4 0}. Now inversionally-paired numbers always sum to the same number:

E4+C4=4+0+4

G#5+Ab3=8+(-4)=4

B4+F3=11+(-7)=4

This number that inversionally-paired pitch classes always sum to is called the **index**, and we will designate the inversion as Ip n, where n is the index. <u>The index is always twice the axis.</u>

This is an important point to be clear about – <u>don't name the inversion using the axis!</u>

To see why the index is always twice the axis, consider various inversions of 0, C4. Suppose the axis of inversion is C#4, or 1. Under Ip1, C4 inverts to D4, or 0 to 2.

However far the axis is from 0, the inversionally-paired pitch class will always be the same distance on the other side of the axis. If the axis is Eb4, or 3, 0 will invert to Gb, or 6. Because the index is the sum of the inversionally-paired pitch classes, and because anything plus 0 is the same thing, the pitch that 0 inverts to is always the same as the index, and it is always numerically equal to twice the axis.

{C4 E4 G4} can be inverted to form any root-position minor triad in close spacing. What inversion sends {C4 E4 G4} to {E4 G4 B4}? We can solve this easily using numbers:

C4+B4=0+11=11

Ip11 is our inversion. It's always a good idea to check other notes to be sure that the math is right:

E4+G4=4+7=11

G4+E4=7+4=11

This means that our axis is 11/2, or 5.5. But what note is 5.5? 5.5 is the gap between 5 and 6, or between F4 and F#4. And this corresponds to the more intuitive way of doing inversions: if C4 inverts to B4, the middle point is the gap between F4 and F#4 – C4 is P4 below F, so a P4 and a half a semitone from the gap point, and B4 is a P4 above F#4, so again a P4 and a half a semitone from the gap point.

Working with Pitch Inversions

Suppose you are given a set of pitches and an inversion. It's easy to find the inversion of that set; just use the fact that inversionally-paired pitches. So if you want to apply Ip n to $\{x_1\ x_2\ x_3\}$, the inverted set $\{y_1\ y_2\ y_3\}$ is found as follows:

$x_1+y_1 = n$, so $y_1 = n - x_1$

$x_2+y_2 = n$, so $y_2 = n - x_2$

$x_3+y_3 = n$, so $y_3 = n - x_3$

If you prefer, you can also find the axis (half the index) and then work by finding the notes that are the equal distance on the other side of the axis.

Suppose you are given two sets that are inversions of one another and told to find the inversion. The trick is finding which notes pair with which. But this is easy, because the highest note in the higher set must invert to the lowest note in the lower set, the second-highest note in the higher set must invert to the second-lowest note in the lower set, etc. Then you just sum corresponding pitches to find the index.

Of course, you can also work more visually and intuitively to find the axis, and then double that to find the index.

Pitch-Class Inversion

We have already seen that pitch-class inversion corresponds to flipping the set around some axis on the circle. As was made clear in Examples 5 and 6 from the diagrams on set classes, the axis of inversion on the circle makes contact with the circle at two points, directly across from each other. This means that the two axis points are 6 away from each other. For convenience, we will name the axis using the lower number.

Note that this creates no problems for naming the inversion based on the index, which is twice the axis.

$2axis_2 = 2(axis_1+6) = 2axis_1+12 = 2axis_1$ because the 12 washes out mod 12

With pitch-class inversion, inversionally-paired pitch classes sum to the index, mod 12. We use I n to designate pitch-class inversions. Let's use Examples 5 and 6 from the set class notes again.

When {C E G} inverts to {C Ab F}, the axis points are 0 and 6, so the index is 0 (2*6=12 which is 0 mod 12); this pitch-class inversion is I_0.

(0+0) mod 12 = 0 mod 12 = 0

(4+8) mod 12 = 12 mod 12 = 0

(7+5) mod 12 = 12 mod 12 = 0

When {C E G} inverts to {E C A}, the axis points are 2 and 8, so the index is 4 (2*8=16 which is 4 mod 12); this inversion is I_4.

(0+4) mod 12 = 4 mod 12 = 4

(4+0) mod 12 = 4 mod 12 = 4

(7+9) mod 12 = 16 mod 12 = 4

Working with Pitch-Class Inversions

Suppose you are given a set and asked to find a specified inversion. This is very straightforward to do mathematically, based on the inversionally-paired pitch classes summing to the index mod 12. So if you want to apply I_n to $\{x_1\ x_2\ x_3\}$, the inverted set $\{y_1\ y_2\ y_3\}$ is found as follows:

$(x_1+y_1) \bmod 12 = n$, so $y_1 = (n - x_1) \bmod 12$

$(x_2+y_2) \bmod 12 = n$, so $y_2 = (n - x_2) \bmod 12$

$(x_3+y_3) \bmod 12 = n$, so $y_3 = (n - x_3) \bmod 12$

If you are asked to notate specific pitches, you are free to choose any register for the pitches that belong to the pitch classes of the inverted set.

If you prefer to work visually, you can plot the given set on the pitch circle, find the axis by taking half of the index, and then reflect the given set through the axis.

The more challenging task is to find which inversion relates two given sets. This is difficult because we are dealing with pitch classes, and therefore it is not meaningful to talk about either a higher set or the highest member within one of the sets. The problem is figuring out which pitch classes are paired with which through the inversion.

Often it helps to plot the sets on circles; it may be easy to see the axis. For larger sets, if it is hard to see the axis, see if there are any interval classes that occur only once within the set. Since inversion preserves interval class, you know that the pitch class on the further clockwise side of the interval (the "later" side of the interval) will correspond to the further counter-clockwise pitch class (the "earlier" side) in the other set.

For example, suppose you are given the chords shown in example 1, and asked to find the pitch-class inversion that relates them. The pitch classes are charted on the circle in Example 2. There are two unique interval classes, i.c. 1 and i.c. 5. Using the i.c. 1, the pitch class on the earlier side of the i.c. in set 1 is F#, and this must correspond to the pitch class on the later side in set 2, Bb.

At this point we can work visually, in which case we see that the axis that reflects F# and Bb to each other runs through 8. If the axis is 8, the index is 2*8 mod 12, or 16 mod 12, or 4. Example 3 confirms visually that these sets are related through I_4, with axis points of 2 and 8.

We can also get to the same result using sums. (Use at least two sets of paired pitch classes to guard against errors.)

$6+10 \bmod 12 = 16 \bmod 12 = 4$

$7+9 \bmod 12 = 16 \bmod 12 = 4$

$10+6 \bmod 12 = 16 \bmod 12 = 4$

$3+1 \bmod 12 = 4 \bmod 12 = 4$

Let's work one more example. Suppose we are given the chords shown in Example 4 and asked to consider them as pitch-class sets, finding the inversion that relates them. This is a harder set to work with, because none of the intervals between adjacent pitch classes are unique. The only unique interval class is the i.c. 6 between B and F in set 1, but because the tritone symmetrically divides the circle, we can't speak meaningfully of "earlier" and "later" sides of the interval. Instead we need to work more visually, going for related features. One way in is to observe that though there are two i.c. 3's between adjacent pitch classes, the i.c. 3 between B and D in set 1 must correspond to the i.c. 3 between Eb and F# in set 2, because in each case there are i.c. 1's surrounding the i.c. 3's on each side. Then the early side of the interval in set 1, B, must correspond to the late side in set 2, F#. The axis of inversion that pairs these pitch classes runs through the gap between G# and A; this is 8.5, so the index is 2*8.5 mod 12, or 5. We confirm this with the sum 11+6 mod 12 = 17 mod 12 = 5. So the inversion is I_5.

T_n/T_nI Types

We have already seen that the various inversions are all transpositions of one another. One might ask, then, why bother keeping track of all these different inversions? Why not just make one inversion operation, I, which inverts with an axis of C/F#, and whose index is 0? Let that be the inversion, and then get "other inversions" by transposing the results of the one inversion. In fact, that's exactly the way some people deal with inversion.

The inversion of a set A is I(A). If we then transpose, we get $T_nI(A)$. We invert first and transpose. The order of the T and the I reflects this – think of it as successive operations on A; $T_nI(A)$ is shorthand for $T_n[I(A)]$, which makes it visually clearer that you invert first and then transpose.

As it happens, the "n" in T_nI is the same as the "n" in I_n. (That's a misleading phrasing – this isn't at all coincidental, but rather one of the reasons that we name inversions by the index and not the axis.) Let's see why this is so.

Let's say we're inverting some pitch-class set, and let's call one of the pitch classes x. To find $I_n(x)$ we go through the following steps.

$$(x + I_n(x)) \bmod 12 = n$$

$$I_n(x) = (n - x) \bmod 12$$

To find $T_nI(x)$ we go through a longer process:

$$(x + I(x)) \bmod 12 = 0$$

$$I(x) = -x \bmod 12$$

$$T_nI(x) = (I(x) + n) \bmod 12 = (-x + n) \bmod 12 = (n - x) \bmod 12$$

Either way we get the same result.

This gives rise to the last of our names for set classes, mentioned in passing in the notes on set classes. Because set classes treat transpositions and inversions as equivalent, we can think of them as the class of all transpositions (T_n's) and inversions (T_nI's) of some set. As a result, set classes are sometimes called T_n/T_nI types.