# Compressive Sensing, Lecture 3

## Yesterday

- Proof of RIP property for Gaussian matrices
- Proof of signal recovery for RIP and RIPless sensing
- Convex optimization

## Today

- Matrix rank minimization and nuclear norm
- Low rank plus sparse decomposition
- Transform invariant low-rank textures

# Linear matrix equations

## Problem

Recover matrix $\bar{X} \in \mathbb{R}^{n \times n}$ from $m \ll n^2$ linear measurements

$$b_k = \langle A_k, \bar{X} \rangle,\ k = 1, \ldots, m \rightsquigarrow b = \mathcal{A}(\bar{X}).$$

- In general this is impossible.
- Suppose we know $\mathrm{rank}(\bar{X}) = r \ll n$.
  Could we get by with fewer than $n^2$ measurements?

## Possible approach for low-rank $\bar{X}$

Take $m \ll n^2$ measurements $b = \mathcal{A}(\bar{X})$ and then solve

$$\begin{array}{ll} \min & \mathrm{rank}(X) \\ & \mathcal{A}(X) = b. \end{array}$$

# Applications

## Matrix completion ("netflix" problem)

- Preference matrix $M$.
- We only observe a small portion of its entries $M_{ij}$.
- Fill in missing entries of $M$.

## Sensor location

- $n$ locations in $\mathbb{R}^d$.
- We only measure a subset of pairwise distances.
- Find the locations.

## Linear system identification

- Dynamical linear system $\begin{cases} x(t+1) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$
- Find $A, B, C, D$ from observations of input $u(t)$ and output $y(t)$ for $t = 0, 1 \ldots, N$.

# Norms

In compressed sensing there are three key norms: $\ell_1, \ell_2, \ell_\infty$.

## Matrix norms

- Operator norm: $\|M\| := \max_{\|x\|_2=1} \|Mx\|_2 = \|\sigma(M)\|_\infty$
- Frobenius norm: $\|M\|_F := \sqrt{\sum_{i=1}^n \sum_{j=1}^n |M_{ij}|^2} = \|\sigma(M)\|_2$
- Nuclear norm $\|M\|_* := \|\sigma(M)\|_1$

Endow $\mathbb{R}^{n\times n}$ with the inner product: $\langle M, N\rangle = \mathrm{trace}(M^\mathsf{T} N)$.

With this inner product:

- The Frobenius norm is the Hilbert space norm.
- The nuclear norm is the dual of the operator norm.

# Rank minimization and convex relaxation

Nuclear norm heuristic for rank minimization problem (Fazel)

$$\begin{array}{ll} \min & \mathrm{rank}(X) \\ & \mathcal{A}(X) = b \end{array} \quad \rightsquigarrow \quad \begin{array}{ll} \min & \|X\|_* \\ & \mathcal{A}(X) = b \end{array}$$

Nuclear norm

$$\|X\|_* := \sum_{i=1}^{n} \sigma_i(X).$$

## Theorem (Fazel)

*The nuclear norm is the convex envelope of the rank function on* $\{M : \|M\| \leq 1\}$.

## Fact

Nuclear norm minimization can be cast as a semidefinite program.

# Dictionary vectors/matrices

Analogy between compressed sensing and low-rank recovery:

| | | |
|---|---|---|
| Parsimony concept | cardinality | rank |
| Hilbert space norm | Euclidean | Frobenius |
| Relaxed norm | $\ell_1$ | nuclear |
| Dual norm | $\ell_\infty$ | operator |
| Convex relaxation | linear programming | semidefinite programming |

The above can be seen in terms of the singular value map

$$X \mapsto \sigma(X).$$

# Deterministic approach: restricted isometry

## Restricted isometry property (RIP)

Given $\mathcal{A} : \mathbb{R}^{n\times n} \to \mathbb{R}^m$ and $k \in \{1, \dots, m\}$, the $k$-isometry constant $\delta_k$ is the smallest $\delta \geq 0$ such that

$$(1-\delta)\|X\|_F^2 \leq \|\mathcal{A}(X)\|_2^2 \leq (1+\delta)\|X\|_F^2$$

for all $X \in \mathbb{R}^{n\times n}$ with $\operatorname{rank}(X) \leq k$.

If $\delta_k < 1$, we say that $A$ satisfies the RIP with constant $\delta_k$.

# Low-rank recovery and RIP property (Recht-Parrilo-Fazel)

**Theorem**

*Assume $\bar{X} \in \mathbb{R}^{n \times n}$ satisfies rank$(\bar{X}) \leq r$.*

- *If $\delta_{2r}(\mathcal{A}) < 1$ then $\bar{X}$ can be recovered (via, e.g., rank minimization) from $b = \mathcal{A}(\bar{X})$.*
- *If $\delta_{4r}(\mathcal{A}) < \sqrt{2} - 1$ then the nuclear norm solution recovers $\bar{X}$.*

RIP for randomly generated matrices:

**Theorem**

*If $\mathcal{A}$ is Gaussian, then $\mathcal{A}$ satisfies $\delta_{4r} < \sqrt{2} - 1$ if $m \geq Crn$ for some suitable constant $C$.*

The proofs of the above are extensions of the analogous proofs for RIP approach to compressed sensing.

# A probabilistic approach

Probabilistic approach:

- Fix $\bar{X} \in \mathbb{R}^{n \times n}$ with $\text{rank}(\bar{X}) = r \ll n$.
- Pick random Gaussian $\mathcal{A} : \mathbb{R}^{n \times n} \to \mathbb{R}^m$ and put $b = \mathcal{A}(\bar{X})$.
- Let $\hat{X} := \text{argmin}_X \{\|X\|_* : \mathcal{A}(X) = b\}$.

### Theorem (Candès and Recht)

*If $m \geq r(6n - 5r)$ for $\beta > 1$, then recovery is exact with probability at least $1 - 2e^{(1-\beta)n/8}$.*

# Matrix completion

## Problem

Assume $M$ low rank and observe a subset of entries. Recover $M$.

- This is certainly an undetermined system of matrix equations.
- Unfortunately RIP fails in most interesting cases.

## Model

$\Omega$ uniform random subset of $\{1, \ldots, n\} \times \{1, \ldots, n\}$.

# When is matrix completion possible?

### Bad cases

Observe that if

$$M = e_1 v^*$$

then recovery is not possible from a random small set of entries.

Likewise if

$$M = uv^*$$

where $u, v$ are sparse vectors.

### Coherence

- In the above cases the rows and/or columns of $M$ are aligned with the basis vectors.
- Coherence is a measure of this kind of alignment.

# Incoherence

Assume $M$ has singular value decomposition

$$M = U\Sigma V^*,$$

and let $r = \mathrm{rank}(M)$.

## Coherence parameter

Smallest $\mu > 0$ such that for $i = 1, \ldots, n$

$$\|U^* e_i\|_2^2 \leq \frac{\mu r}{n}, \quad \|V^* e_i\|_2^2 \leq \frac{\mu r}{n}$$

and

$$|UV^*|_{ij} \leq \frac{\mu r}{n^2}$$

# Incoherence and matrix completion

Consider the nuclear norm heuristic

$$\begin{array}{ll} \min & \|X\|_* \\ & X_{ij} = M_{ij}, \;\; (i,j) \in \Omega. \end{array}$$

## Theorem (Candès & Recht)

*Assume rank*$(M) = r$ *and* $\Omega$ *is a random set of size* $m$. *If*

$$m \geq C\mu r(1+\beta)\log^2 n$$

*then the solution to the nuclear norm heuristic is exact with probability at least*

$$1 - n^{-\beta}.$$

# Theoretical limits of matrix completion

The above result is nearly optimal:

## Theorem (Candès & Tao)

*No method can ensure recovery with high probability if*

$$m \lesssim \mu \cdot nr \cdot \log n.$$

## Neat connection with random graph theory

- For successful matrix recovery, the adjacency graph defined by entries in $\Omega$ must be connected.
- Given a bipartite graph, how many random edges should we pick to get a single component?

# More general result

Assume $\{A_1, \ldots, A_{n^2}\}$ orthonormal basis of $\mathbb{R}^{n \times n}$ and $M = U\Sigma V^*$.

## Coherence

$M$ has coherence $\mu$ with respect to $\{A_1, \ldots, A_{n^2}\}$ if either

$$\max_k \|A_k\|^2 \leq \frac{\mu}{n}$$

or

$$\max_k \|P_U A_k\|^2 \leq \frac{\mu r}{n}, \ \max_k \|A_k P_V\|^2 \leq \frac{\mu r}{n}, \ \max_k |\langle A_k, UV^* \rangle \leq \frac{\mu r}{n^2}.$$

## Theorem (Gross)

*Exact recovery with probability at least* $1 - n^{-\beta}$ *if*

$$m \geq C\mu r(1+\beta) \log^2 n.$$

# Low rank + sparse decomposition

## Separation problem

Suppose $M = L_0 + S_0$ where $L_0$ low rank and $S_0$ sparse.
If we observe $M$, could we recover $L_0$ and $S_0$?

## Applications

- Robust PCA
- Latent variable detection
- Video surveillance

## Nuclear & $\ell_1$ heuristic (PCP)

$$\begin{array}{ll} \min & \|L\|_* + \lambda \cdot \|S\|_1 \\ & L + S = M. \end{array}$$

# Recovery theorems

Low rank plus sparse separation:

- It is not possible in certain cases, e.g, if a matrix is both low rank and sparse.
- Recovery statements depend on matrix coherence.

### Theorem

*Suppose* $M = L_0 + S_0 \in \mathbb{R}^{n \times n}$ *where*

$$\textit{rank}(L_0) \leq \frac{\rho_r n}{\mu}, \;\; \|S_0\|_0 \leq \rho_s n^2.$$

*Then PCP succeeds with probability* $1 - \frac{C}{n^{10}}$ *for* $\lambda = 1/\sqrt{n}$.

# Matrix completion with corrupted data

Suppose entries of $M$ may be both missing and corrupted.
Can we recover $M$?

Extended PCP

$$\begin{array}{ll} \min & \|L\|_* + \lambda \cdot \|S\|_1 \\ & L_{ij} + S_{ij} = M_{ij}, \;\; (i,j) \in \Omega. \end{array}$$

## Theorem

- $L_0 \in \mathbb{R}^{n \times n}$, *rank*$(L_0) \leq \frac{\rho_r n}{\mu \log^2 n}$
- *$\Omega$ random set of size $cn^2$, where $c \in (0,1)$.*
- *Each observed entry is corrupted with probability $\tau \leq \tau_s$.*

*Then PCP succeeds with probability $1 - \frac{C}{n^{10}}$ for $\lambda = \frac{1}{\sqrt{cn}}$.*

# Relationship with matrix completion

Suppose we have no corruption.

MC: recovery via

$$\begin{array}{ll} \min & \|L\|_* \\ & L_{ij} = M_{ij}, \;\; (i,j) \in \Omega. \end{array}$$

PCP: recovery via

$$\begin{array}{ll} \min & \|L\|_* + \lambda \cdot \|S\|_1 \\ & L_{ij} + S_{ij} = M_{ij}, \;\; (i,j) \in \Omega. \end{array}$$

Under suitable conditions both yield the same answer.

The second one is a robust version of the first one.

# Transform invariant low-rank textures (TILT)

## Zhang-Ganesh-Liang-Ma

Suppose a low rank matrix is both corrupted and misaligned.
Can we recover it?

## Model

$$M \circ \tau = L_0 + S_0$$

$L_0$: low rank, $S_0$: sparse, $\tau$: parametric deformation.

## Problem

Given $M$, can we find the above decomposition?

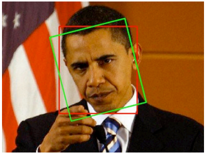
## Approach

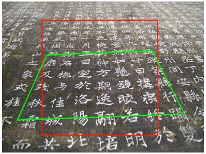
Find $L, S, \tau$ that solves

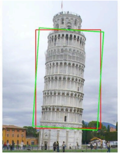
$$\begin{array}{rl} \min & \|L\|_* + \lambda \cdot \|S\|_1 \\ & L + S = M \circ \tau. \end{array}$$

# Examples

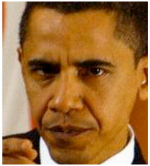
red windows indicate the original input

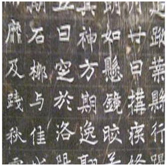


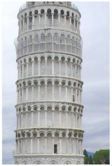




green windows indicate texture found







# Main references for today's material

- Slides for this minicourse:
`http://andrew.cmu.edu/user/jfp/UNencuentro`


- B. Recht, M. Fazel, P. Parrilo, "Guaranteed Minimum-Rank Solutions of Linear Matrix Equations via Nuclear Norm Minimization," *SIAM Review,* vol 52, no 3, pp. 471–501, 2010.

- V. Chandrasekaran, S. Sanghavi, P.A. Parrilo and A. Willsky , "Rank-Sparsity Incoherence for Matrix Decomposition," *SIAM Journal on Optimization,* vol. 21, issue 2, pp. 572–596, 2011

- E. J. Candès, X. Li, Y. Ma, and J. Wright, "Robust Principal Component Analysis?," *Journal of ACM,* vol 58, no 3, article no 11, 2011.

- Z. Zhang, A. Ganesh, X. Liang, and Y. Ma, "TILT: Transform Invariant Low-rank Textures," To Appear in International Journal of Computer Vision.