

# School Choice with Controlled Choice Constraints: Hard Bounds versus Soft Bounds\*

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## Abstract

Controlled choice over public schools attempts giving options to parents while maintaining diversity, often enforced by setting feasibility constraints with *hard upper and lower bounds* for each student type. We demonstrate that there might not exist assignments that satisfy standard fairness and non-wastefulness properties; whereas *constrained non-wasteful* assignments which are *fair for same type students* always exist. We introduce a “controlled” version of the deferred acceptance algorithm with an improvement stage (CDAAI) that finds a Pareto optimal assignment among such assignments. To achieve *fair* (across all types) and *non-wasteful* assignments, we propose the control constraints to be interpreted as *soft bounds*—flexible limits that regulate school priorities. In this setting, a modified version of the deferred acceptance algorithm (DAASB) finds an assignment that is Pareto optimal among fair assignments while eliciting true preferences. CDAAI and DAASB provide two alternative practical solutions depending on the interpretation of the control constraints.

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# 1 Introduction

*School choice* policies are implemented to grant parents the opportunity to choose the school their child will attend. In order to create a diverse environment for students, school districts often implement *controlled school choice* programs providing parental choice while maintaining the racial, ethnic or socioeconomic balance at schools. Before school choice policies were in effect, children were assigned a public school in their immediate neighborhood. However, neighborhood-based assignment eventually led to socioeconomically segregated neighborhoods, as wealthy parents moved to the neighborhoods of their school of choice. Parents without such means had to send their children to their neighborhood schools, regardless of the quality or appropriateness of those schools for their children. As a result of these concerns, *controlled school choice programs* have become increasingly more popular across the United States. This paper provides a foundation for such programs and introduces two new algorithms with different desirable properties that can be readily adapted in practice.

There are many examples of controlled public school admission policies in the United States. In some school districts, control over student assignment is enforced by a court order. For instance, a Racial Imbalance Law that was passed in 1965 in Massachusetts, prohibits racial imbalance and discourages schools from having student enrollments that are more than 50 percent minority. After a series of legal decisions, the Boston Public Schools (BPS) was ordered to implement a controlled choice plan in 1975.<sup>1</sup> Although BPS has been relieved of legal monitoring, it still tries to achieve diversity across ethnic and socioeconomic lines at city schools (Abdulkadiroğlu, Pathak, Roth, and Sönmez 2005, 2006). Likewise, St. Louis and Kansas City, Missouri, must observe court-ordered racial desegregation guidelines for the placement of students in city schools.<sup>2</sup> In contrast, the White Plains Board of Education employ their nationally recognized Controlled Parents' Choice Program voluntarily.<sup>3</sup>

Other types of control are also present. In New York City, "Educational Option" (EdOpt) schools have to accept students across different ability ranges. In particular, 16 percent of students that attend an EdOpt school must score above grade level on the standardized English Language Arts test, 68 percent must score at grade level, and the remaining 16

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<sup>1</sup>See <http://boston.k12.ma.us/bps/assignmntfacts.pdf> for a brief history of student assignment in Boston.

<sup>2</sup>Similarly, Section 228.057 of Florida Statutes requires each school district in the state to design a choice plan. Section 228.057 emphasizes the importance of maintaining socioeconomic, demographic, and racial balance within each school.

<sup>3</sup>The reason behind initiating the choice program was the Board's "belief that balance of the racial and ethnic diversity of the schools' population would promote students' understanding, appreciation, and acceptance of persons of different racial, ethnic, social, and cultural backgrounds" (See <http://wpcsd.k12.ny.us/1info/index.html> for more detail). Cambridge, MA has a similar policy of control not only on racial diversity but on socioeconomic diversity as well.

percent must score below grade level (Abdulkadiroğlu, Pathak, and Roth, 2005).<sup>4</sup> Miami-Dade County Public Schools control for the socioeconomic status of students in order to diminish concentrations of low-income students at certain schools. Similarly, Chicago Public Schools diversify their student bodies by enrolling students in choice options at schools that are not the students' designated neighborhood schools.<sup>5</sup> Lastly, the Jefferson County School District has an assignment plan that requires elementary schools to have between 15 and 50 percent of their students coming from a particular geographic area inside the district that harbors the highest concentration of designated beneficiaries of the affirmative action policy.<sup>6</sup>

In general, a crucial policy of most school choice programs (not only controlled choice programs) is to give some students priority at certain schools. For example, some state and local laws require that students who live in the attendance area of a school must be given priority for that school over students who do not live in the school's attendance area; siblings of students already attending a school must be given priority; and students requiring a bilingual program must be given priority in schools that offer such programs. All these priority altering decisions, including the controlled choice, should be implemented while preserving the notion of *fairness*.

Following Abdulkadiroğlu and Sönmez (2003), we can define an assignment to be *fair* if there is no unmatched student-school pair where the student prefers the school to her assignment and she has higher priority than some other student who is assigned a seat at the school. In the context of school choice, there is *justified envy* if the assignment is not fair. Abdulkadiroğlu and Sönmez (2003) show that *the student proposing deferred acceptance algorithm* (also known as Gale-Shapley student optimal algorithm) finds the fair assignment which is preferred by every student to any other fair assignment. Moreover, revealing preferences truthfully is a weakly dominant strategy for every student in the preference revelation game in which students submit their preferences over schools first, and then the assignment is determined via the students proposing deferred acceptance algorithm (DAA) using the submitted preferences (Dubins and Freedman, 1981; Roth 1982).<sup>7</sup>

Abdulkadiroğlu and Sönmez (2003) and Abdulkadiroğlu (2005) consider a relaxed con-

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<sup>4</sup>There are similar constraints in other countries as well. For example in England, City Technology Colleges are required to admit a group of students from across the ability range and their student body should be representative of the community in the catchment area (Donald Hirsch, 1994, page 120).

<sup>5</sup>We refer the interested reader <http://www.buildingchoice.org> for an illuminating overview of inter-district school choice programs including possible desegregation guidelines.

<sup>6</sup>More details on this policy are present on the "No Retreat" brochure on Jefferson Country School District's website (<http://www.jefferson.k12.ky.us/Pubs/NoRetreatBro.pdf>).

<sup>7</sup>Although for schools it is not a weakly dominant strategy to truthfully reveal their preferences in DAA, Kojima and Pathak (2009) have recently shown under some regularity conditions that in DAA the fraction of participants that can gain from misreporting approaches zero as the market becomes large.

trolled choice problem by employing *type-specific quotas*. Control is imposed on the maximum number of students from each racial/ethnic group that a school can enroll. Their proposed solutions do not capture *controlled choice* to the fullest extent because they do not exclude segregated schools in fair assignments. For example, consider a school that can enroll 100 students with hard upper bound of 50 Caucasian students. In this case, a student body of 50 Caucasian students would not violate the maximum quota, yet the school is fully segregated. Such an assignment would be unacceptable for many school districts.

In order to provide a foundation for controlled school choice programs, a thorough analysis of fairness and controlled choice requires a substantial generalization of the model. Extending the model to fully capture controlled choice brings major difficulties. Following the laws of a state or the policies of a school choice program (or of the school district), an assignment is legally *feasible (under hard bounds)* (or politically acceptable) if both (i) every student is assigned to a public school and (ii) at each school the desegregation guidelines are respected. We incorporate these constraints in the definition of justified envy, thereby in the definition of fairness. The nature of controlled choice imposes that a student-school pair can cause a *justified envy* (or blocks) only if matching this pair does neither result in any unassigned student nor violate the controlled choice constraints at any school.

This raises the question of existence of fair and legally feasible assignments in the controlled school choice. We show that feasible student assignments which are fair may not exist. Due to this impossibility, either fairness needs to be weakened in order to respect legal constraints, or the interpretation of the legal constraints must be changed. We first focus on the case where we relax the notion of fairness (while maintaining the hard bounds). In this setting, a natural route is to allow envy only among students of the same type. Then, for example, only Caucasian students can justifiably envy other Caucasian students (but not any African-American students). It turns out that legally feasible assignments, which are fair for same types, may not exist if we require additionally non-wastefulness (Balinski and Sönmez, 1999). In our context, this condition requires that empty seats should not be wasted if students claim them while the legal constraints maintained. A positive result emerges if non-wastefulness is constrained: students can claim empty seats only if the resulting assignment does not cause any envy among students of the same type. In particular, a controlled version of the student proposing deferred acceptance algorithm followed by an improvement mechanism (CDAAI) finds for each controlled school choice problem a legally feasible assignment which is both fair for same types and constrained non-wasteful. The resultant assignment is also Pareto efficient among the set of assignments that are fair for same types and constrained non-wasteful. Unfortunately, CDAAI is not (dominant strategy) incentive compatible. Indeed, we show that it is impossible to elicit true preferences in

dominant strategies while maintaining fairness and the legal constraints.

Instead of relaxing the fairness notion, we can also re-interpret the legal constraints, which are reflected as upper and lower bounds (*floors* and *ceilings*, respectively) for each student type in the controlled school choice context. Most school districts administer floors and ceilings as hard bounds, so a theoretical analysis of such policies is inarguably important. However, applications of these hard bounds are quite paternalistic in the sense that assignments can be forced *despite* student preferences. That is, with this specification school districts end up not allowing students to take some available seats, even if there are no physical limitations. In contrast, we provide an alternative interpretation of these constraints as *soft bounds*. To be more explicit, school districts may adapt a dynamic priority structure, giving highest priority to student types who do not fill their floors, medium priority to student types who fill their floors and not their ceilings, and lowest priority to student types who fill their ceilings. Yet, schools can still admit fewer students than their floor or more than their ceiling as long as students with higher priorities do not veto this match. In Section 4, we consider this soft bounds view: control policies promote the desired balancing at schools, only when student preferences allow them to do so. In other words, soft bounds policies give the parents an opportunity to establish desired balancing at schools, but do not force them to achieve this balance.

With hard bounds, an assignment that is fair and non-wasteful might not exist even if fairness is restricted to students with same types. However, with soft bounds the existence of an assignment that is fair and non-wasteful is guaranteed. To show this, we consider the student-proposing *deferred acceptance algorithm with soft bounds* (DAASB), in which schools tentatively admit the set of students at each step with the dynamic priority structure implied by the soft bounds view of floors and ceilings. Furthermore, we provide the following connection between soft bounds and hard bounds. We show that all students weakly prefer the outcome of this algorithm (under soft bounds) to any assignment that is strongly fair across types and non-wasteful under hard bounds. In this sense, all students are better off with soft bounds.

To summarize, we demonstrate that it is impossible to eliminate envy across different types while fully respecting controlled school choice constraints, even though envy for same types can be eliminated. On the other hand, if one considers the soft bounds view, all the desirable properties of the deferred acceptance algorithm are restored: DAASB produces a fair and non-wasteful assignment (which is Pareto optimal among fair and non-wasteful assignments) and, furthermore, DAASB is group incentive compatible.

Although we focus on controlled school choice, all of our results equally apply to centralized matching programs where diversity constraints are wished to be implemented. For

instance, a college admissions office that wants to avoid completely segregated student bodies may use controlled policies. Other examples are entry-level labor markets where we may wish to exclude gender segregated worker groups meaning that for each firm there are both female and male workers among its hires. In labor markets we may even desire to control for both race and gender.

**Related Literature** In a recent paper, Kojima (2010) considers a model where there are two kinds of students (minority and majority) and only a quota for majority students. He investigates the consequences of such affirmative action policies and shows that these policies may hurt minority students, the purported beneficiaries. To overcome this shortcoming, Hafalir, Yenmez and Yildirim (2011) propose *affirmative action with minority reserves* in which schools give higher priority to minority students up to the point that the minorities fill the reserves. They consider both deferred acceptance and top trading cycles algorithms. They also perform simulations and conclude that minorities are on average better off with minority reserves while adverse effects on majorities are mitigated.

Abdulkadiroğlu (2010) considers the same control environment as in this paper but proposes different feasibility and fairness concepts. In particular, due to the non-existence of feasible and fair student assignments, he relaxes feasibility by not requiring that all students are enrolled at a school and then looks for fair assignments which are not dominated by any other fair assignment. Budish et al. (2011) consider expected assignments satisfying control constraints and determine when such expected assignments can be implemented by a lottery over deterministic assignments satisfying the control constraints.

In a recent paper, Kamada and Kojima (2010) study entry-level medical markets with regional caps: hospitals (or schools) are partitioned into regions and each region is controlled by a cap (or ceiling) determining the maximal number of students that can be assigned to the hospitals in that region. Similar to our context, they propose different stability notions like “strong stability” and “stability”. Some of their results have a similar flavor like ours: (i) strongly stable assignments do not exist (like fairness (for same types) and non-wastefulness are incompatible under hard bounds for school choice with control) and (ii) stable assignments exist (like fairness and non-wastefulness are compatible under soft bounds) and (iii) their “flexible deferred acceptance algorithm” finds a stable assignment and is incentive compatible (like DAASB finds a fair and non-wasteful assignment under soft bounds and is incentive compatible).

The paper is organized as follows. Section 2 formalizes controlled school choice problem and introduces our desirable criteria, namely fairness for same types and non-wastefulness. Section 3 shows that there may not exist any feasible assignment which is both fair for same

types and non-wasteful. Therefore, we constrain non-wastefulness and show that CDAAI always finds a feasible assignment which is both fair for same types, constrained non-wasteful and Pareto efficient among assignments with the same properties. Section 3 also studies incentive compatibility and shows that there may not exist any incentive compatible mechanism which is both fair for same types and constrained wasteful. In Section 4, we consider controlled school choice problem with soft bounds and show that an adaptation of the deferred acceptance algorithm achieves fairness and non-wastefulness under soft bounds, which is also incentive compatible. Section 5 concludes. All proofs are given in Appendix A.

## 2 Controlled School Choice

A **controlled school choice problem** or simply a **problem** consists of the following:

1. a finite set of students  $S = \{s_1, \dots, s_n\}$ ;
2. a finite set of schools  $C = \{c_1, \dots, c_m\}$ ;
3. a capacity vector  $q = (q_{c_1}, \dots, q_{c_m})$ , where  $q_c$  is the capacity of school  $c \in C$  or the number of seats in  $c \in C$ ;
4. a students' preference profile  $P_S = (P_{s_1}, \dots, P_{s_n})$ , where  $P_s$  is the strict preference relation of student  $s \in S$  over  $C$ , i.e.,  $cP_s c'$  means that student  $s$  strictly prefers school  $c$  to school  $c'$ ;
5. a schools' priority profile  $\succ_C = (\succ_{c_1}, \dots, \succ_{c_m})$ , where  $\succ_c$  is the strict priority ranking of school  $c \in C$  over  $S$ ;  $s \succ_c s'$  means that student  $s$  has higher priority than student  $s'$  to be enrolled at school  $c$ ;
6. a type space  $T = \{t_1, \dots, t_k\}$ ;
7. a type function  $\tau : S \rightarrow T$ , where  $\tau(s)$  is the type of student  $s$ ;
8. for each school  $c$ , two vectors of type specific constraints  $\underline{q}_c^T = (q_c^{t_1}, \dots, q_c^{t_k})$  and  $\bar{q}_c^T = (\bar{q}_c^{t_1}, \dots, \bar{q}_c^{t_k})$  such that  $\underline{q}_c^t \leq \bar{q}_c^t \leq q_c$  for all  $t \in T$ , and  $\sum_{t \in T} \underline{q}_c^t \leq q_c \leq \sum_{t \in T} \bar{q}_c^t$ .

Here,  $\underline{q}_c^t$  is the minimal number of slots that school  $c$  must by law allocate to type  $t$  students, called **the floor for type  $t$  at school  $c$** , whereas  $\bar{q}_c^t$  is the maximal number of slots that school  $c$  is allowed by law to allocate to type  $t$  students, called **the ceiling for type  $t$  at school  $c$** . The same model is studied by Abdulkadiroğlu (2010).

In summary, a **controlled school choice problem** is given by

$$\left( S, C, (q_c)_{c \in C}, P_S, \succ_C, T, \tau, (\underline{q}_c^T, \bar{q}_c^T)_{c \in C} \right).$$

When everything except  $P_S$  remains fixed, we simply refer to  $P_S$  as a controlled school choice problem.

The set of types may represent different characteristics of students such as: (i) race; (ii) socioeconomic status (determined by free or reduced-price lunch eligibility); or (iii) the district where the student lives. Controlled choice constraints are imposed by law or the policies of a state (via desegregation orders), and the school choice program has to comply with these constraints.<sup>8</sup>

An **assignment**  $\mu$  is a function from the set  $C \cup S$  to the set of all subsets of  $C \cup S$  such that

- i.  $\mu(s) \in C$  for every student  $s$ ;
- ii.  $|\mu(c)| \leq q_c$  and  $\mu(c) \subseteq S$  for every school  $c$ ;
- iii.  $\mu(s) = c$  if and only if  $s \in \mu(c)$ .

In words,  $\mu(s)$  denotes the school that student  $s$  is assigned;  $\mu(c)$  denotes the set of students that are assigned school  $c$ ; and  $\mu^t(c)$  denotes the students of type  $t$  that are assigned to school  $c$ , i.e.,  $\mu^t(c) = \mu(c) \cap S_t$  where  $S_t \equiv \{s \in S : \tau(s) = t\}$  is the set of all type  $t$  students.

Given two assignments  $\mu$  and  $\mu'$ , we say that  $\mu$  **Pareto dominates**  $\mu'$  if all students weakly prefer  $\mu$  to  $\mu'$  and  $\mu \neq \mu'$ . Similarly, we say that an assignment  $\mu$  is **Pareto optimal (or Pareto efficient)** among the assignments satisfying certain properties if there is no assignment which both satisfies these properties and Pareto dominates  $\mu$ .

A set of students  $S' \subseteq S$  **respects (capacity and controlled choice) constraints at school**  $c$  if  $|S'| \leq q_c$  and for every type  $t \in T$ ,  $\underline{q}_c^t \leq |\{s \in S' : \tau(s) = t\}| \leq \bar{q}_c^t$ . An assignment  $\mu$  **respects constraints** if for every school  $c$ ,  $\mu(c)$  respects constraints at  $c$ , i.e., for every type  $t$  we have

$$\underline{q}_c^t \leq |\mu^t(c)| \leq \bar{q}_c^t.$$

As outlined before, the law of many states in the United States requires students to be assigned to schools such that (i) at each school the constraints are respected and (ii) each

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<sup>8</sup>Ehlers (2010) also considers the case when students have multi-dimensional types and when control constraints are imposed in terms of percentages. He demonstrates that these extensions are easily accommodated to controlled school choice problems.

student is enrolled at a public school. An assignment  $\mu$  is **(legally or politically) feasible (under hard bounds)** if  $\mu$  respects constraints and every student is assigned to a school. Later in Section 4, we are going to re-interpret assumption (i) and study controlled choice with soft bounds.<sup>9</sup>

Obviously, a controlled school choice problem does not have a feasible solution if there are not enough students of a certain type to fill the minimal number of slots required by law for that type at all schools. Therefore, we assume that the number of students of any type is bigger than the sum of the floors for that type at all schools, i.e., for each  $t \in T$ ,  $|S_t| \geq \sum_{c \in C} q_c^t$ . Similarly, in order not to leave any student unassigned we need to have enough slots for each student type, that is  $|S_t| \leq \sum_{c \in C} \bar{q}_c^t$ .<sup>10</sup> From now on we assume that the legal constraints at schools are such that a legally feasible assignment exists. In Appendix B we show that the existence of a feasible assignment is equivalent to finding a solution of the so-called *transportation problem* (Nemhauser and Wolsey, 1999) which can be done in polynomial time. If no feasible assignments exist, then the laws are not compatible with each other and either they need to be modified (and this issue is out of this paper’s scope) or we may reconsider the controlled choice constraints (which we discuss in Section 4).

What are desirable properties of feasible assignments in controlled school choice problems? The following notions are the natural adaptations of their counterparts in standard two-sided matching (without type constraints).

The first requirement is that whenever a student prefers an empty slot to the school assigned to her, the legal constraints are violated when assigning the empty slot to this student while keeping all other assignments unchanged.<sup>11</sup>

We say that **student  $s$  justifiably claims an empty slot at school  $c$  under the feasible assignment  $\mu$**  if

$$(nw1) \quad cP_s\mu(s) \text{ and } |\mu(c)| < q_c,$$

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<sup>9</sup>According to law, every student has a right to attend a public school. Hence, we assume that all students are acceptable to every school. Moreover, we consider the case when students have to give a full ranking of all schools. This is because if students are allowed to give shorter lists and admission process requires them to be assigned to a school in their lists, students could simply include only their favorite schools. This clearly may result in non-existence of feasible assignments. Here, students can still prefer their outside options (going to a private school, or being homeschooled) to their assigned schools, nonetheless, they are required to rank all schools.

<sup>10</sup>Note that these constraints are not sufficient for the existence of a feasible assignment. For example, consider the problem consisting of three schools and three students. Each student has a different type. The capacities are all equal to 1, the floors are all equal to zero, and the ceilings are given by  $\bar{q}_{c_1}^{t_1} = \bar{q}_{c_1}^{t_2} = \bar{q}_{c_1}^{t_3} = 1$ ,  $\bar{q}_{c_2}^{t_1} = \bar{q}_{c_2}^{t_2} = 0$  and  $\bar{q}_{c_2}^{t_3} = 1$ , and  $\bar{q}_{c_3}^{t_1} = \bar{q}_{c_3}^{t_2} = 0$  and  $\bar{q}_{c_3}^{t_3} = 1$ . There does not exist a feasible assignment because student  $s_1$  or student  $s_2$  has to be left unassigned if the constraints at schools  $c_2$  and  $c_3$  are respected.

<sup>11</sup>This requirement is in the spirit of the property “non-wastefulness” introduced by Balinski and Sönmez (1999).

(nw2)  $\underline{q}_{\mu(s)}^{\tau(s)} < |\mu^{\tau(s)}(\mu(s))|$ , and

(nw3)  $|\mu^{\tau(s)}(c)| < \bar{q}_c^{\tau(s)}$ .

Here (nw1) means student  $s$  prefers an empty slot at school  $c$  to the school assigned to him; (nw2) means that the floor of student  $s$ 's type is not binding at school  $\mu(s)$ ; and (nw3) means that the ceiling of student  $s$ 's type is not binding at school  $c$ . Hence, under (nw1-3) student  $s$  can be assigned to an empty slot at the better school  $c$  without changing the assignments of the other students and violating the constraints at any school. A feasible assignment  $\mu$  is **non-wasteful** if no student justifiably claims an empty slot at any school.

A well studied requirement of the literature is fairness or no-envy (Foley, 1967).<sup>12</sup> In school choice student  $s$  envies student  $s'$  when  $s$  prefers the school at which  $s'$  is enrolled, say school  $c$ , to her school. However, the nature of controlled school choice imposes the following (legal) constraints: Envy is justified only when

- (i) student  $s$  has higher priority to be enrolled at school  $c$  than student  $s'$ ,
- (ii) student  $s$  can be enrolled at school  $c$  without violating controlled choice constraints (at all schools) by removing  $s'$  from  $c$ , and
- (iii) student  $s'$  can be enrolled at another school without violating constraints by removing  $s'$  from  $c$  in favor of  $s$ .

We say that **student  $s$  justifiably envies student  $s'$  at school  $c$  under the feasible assignment  $\mu$**  if there exists another feasible assignment  $\mu'$  such that

(f1)  $\mu(s') = c$ ,  $cP_s\mu(s)$  and  $s \succ_c s'$ ,

(f2)  $\mu'(s) = c$ ,  $\mu'(s') \neq c$ , and  $\mu'(\hat{s}) = \mu(\hat{s})$  for all  $\hat{s} \in S \setminus \{s, s'\}$ .

Because  $\mu'$  is feasible, (f2) simply says that  $(\mu(c) \setminus \{s'\}) \cup \{s\}$  respects the controlled choice constraints at school  $c$  and student  $s'$  can be enrolled at school  $c' = \mu'(s')$  such that  $(\mu(c') \setminus \{s\}) \cup \{s'\}$  respects the controlled choice constraints at  $c'$ ; in other words assigning  $s$  a slot at  $c$ ,  $s'$  a slot at  $c'$ , and keeping all the other assignments intact does not violate any controlled choice constraint at any school. A feasible assignment  $\mu$  is **fair across types (or fair)** if no student justifiably envies any student.

We also consider a weaker version of envy (and fairness) where envy is justified only if both the envying student and the envied student are of the same type. If this is the case,

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<sup>12</sup>See for example Tadenuma and Thomson (1991), for an excellent survey, also see Thomson (forthcoming), Thomson (2000) and Young (1995).

then (ii) and (iii) are always true since then the envying student and the envied student can simply exchange schools. More formally, we say that **student  $s$  justifiably envies student  $s'$  of the same type at school  $c$  under the feasible assignment  $\mu$**  if

(f1\*)  $\mu(s') = c$ ,  $cP_s\mu(s)$  and  $s \succ_c s'$ , and

(f2\*)  $\tau(s) = \tau(s')$ .

In (f1\*), student  $s'$  is enrolled at school  $c$  and both student  $s$  prefers school  $c$  to his assigned school  $\mu(s)$  and student  $s$  has higher priority to be enrolled at school  $c$  than student  $s'$ . By (f2\*), student  $s$  and student  $s'$  are of the same type. Then we obtain a feasible assignment when students  $s$  and  $s'$  exchange their slots, i.e., choose  $\mu'$  as follows:  $\mu'(s) = \mu(s')$ ,  $\mu'(s') = \mu(s)$ , and  $\mu'(\hat{s}) = \mu(\hat{s})$  for all  $\hat{s} \in S \setminus \{s, s'\}$ . The assignment  $\mu'$  is feasible because  $s$  and  $s'$  are of the same type and  $\mu$  was feasible. A feasible assignment  $\mu$  is **fair for same types** if no student justifiably envies any student who is of the same type.

### 3 Controlled School Choice with Hard Bounds

Our first result shows the difficulty in finding assignments that satisfy the legal constraints together with other desirable properties such as fairness and non-wastefulness by establishing two benchmark incompatibility results (even though we assumed that feasible assignments exist).

**Theorem 1** (i) *The set of feasible assignments that are fair across types may be empty in a controlled school choice problem.*

(ii) *The set of feasible assignments that are both fair for same types and non-wasteful may be empty in a controlled school choice problem.*

The proof of Theorem 1 is provided in Appendix A; and it is by means of examples.

In contrast to the literature on matching, our impossibility result is *not* obtained by violating the responsiveness condition (or “substitutability”) of schools’ preferences over sets of students, but by controlled choice. Clearly Theorem 1 is a negative result. We will see later that the answer is affirmative to both (i) the existence of feasible assignments which are fair for same types and (ii) the existence of feasible and non-wasteful assignments. Hence, in controlled school choice problems we may retain fairness for same types or non-wastefulness while giving up the other requirement.

Giving up completely either fairness for same types or non-wastefulness may not be satisfactory for a controlled school choice program. We hence keep fairness for same types and weaken non-wastefulness to the following criterion.

We say that a feasible assignment  $\mu$  is **constrained non-wasteful** if: student  $s$  justifiably claims an empty slot at school  $c$  under  $\mu$  implies that the assignment  $\mu'$  (where  $\mu'(s) = c$  and  $\mu'(s') = \mu(s')$  for all  $s' \in S \setminus \{s\}$ ) is not fair for same types.

If the feasible assignment  $\mu$  is fair for same types and constrained non-wasteful, then the above definition is equivalent to the requirement that whenever a student  $s$  of type  $t$  justifiably claims an empty slot at school  $c$  under  $\mu$ , then some other type  $t$  student  $s'$  justifiably envies student  $s$  at school  $c$  under the assignment  $\mu'$  (where  $\mu'$  is defined as above).

The idea of feasible assignments which are both fair for same types and constrained non-wasteful is similar to the one of “bargaining sets”: if a type  $t$  student  $s$  has an objection to  $\mu$  because  $s$  claims an empty slot at  $c$ , then there will be a counterobjection once  $s$  is assigned to  $c$  since some other type  $t$  student will then justifiably envy  $s$  at  $c$ . Roughly speaking, an outcome belongs to the “bargaining set” if and only if for any objection to the outcome there exists a counterobjection.<sup>13</sup>

We show that the set of feasible assignments which are both fair for same types and constrained non-wasteful is non-empty in a controlled school choice problem. To show this we propose a controlled version of the student proposing deferred acceptance algorithm (DAA). Recall that in the classical algorithm of Gale and Shapley (1962) students are tentatively admitted to schools, and at any step students who are not matched simultaneously propose to schools to which they did not propose yet. Then each school considers the new proposals and students who were tentatively admitted from the previous step, and tentatively admits the most preferred students among these. The other students are rejected permanently. If there is no rejection, then the algorithm ends and all the current tentative assignments are made permanent.

Stage 1 of our algorithm is reminiscent of the DAA but it has three important differences. First, proposals cannot be simultaneous. When several students propose simultaneously, it may be infeasible to tentatively admit them at the same time. In our controlled student proposing deferred acceptance algorithm, proposals are sequential (say according to when the applications were received): similar to McVitie and Wilson (1970) at each step one student, who is not tentatively assigned to a school, proposes to the most preferred school

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<sup>13</sup>In a paper subsequent to Ehlers (2010), Alcalde and Romero-Medina (2011) weakened stability in a similar fashion in school choice problems without constraints in order to improve efficiency of stable assignments. Kesten (2010) also proposes a method for the latter.

which has not rejected him yet.

Second, when tentatively accepting a student we need to make sure that the rest of the students can be assigned feasibly. In other words, we check whether there is some feasible assignment such that all tentative assignments can be made permanent. In the standard DAA, we only check the feasibility of assignments at schools who receive proposals at that step, without looking ahead.

Third, since we only require fairness for same types, in our algorithm, a student cannot make a tentatively admitted student of another type be rejected by a school. In other words, if there are no empty seats, students can only claim the tentative seats of the students of their own types.

Stage 2 of our algorithm is an improvement stage where we look for Pareto improvements within the class of assignments that are fair for same types. This is similar to Erdil and Ergin (2008) in spirit, however, we have to make sure that the improvement is done so that the legal constraints are respected—which is the main difficulty.

To be more explicit, roughly, Stage 1 works as follows: For an order of students, students make proposals to schools one by one. Students can propose to a school only if there can be a feasible assignment following from this proposal. All acceptances are tentative, but a student can make a tentatively admitted student to be permanently rejected only when the latter student is of the same type of the former student. Even though the initial stage finds an assignment that is fair for same types and constrained non-wasteful, the assignment does not have to be Pareto efficient among such assignments which is shown in Example 1 below. Therefore, in Stage 2 we improve the matches of students in a systematic way to get an assignment with the additional Pareto efficiency property. A more formal definition of this algorithm is in order.

## **Controlled Student Proposing Deferred Acceptance Algorithm with Improvement (CDAAI)**

### **Stage 1: Initial Assignment**

Start: Fix an order of the students, in which they are allowed to make proposals to schools, say  $s_1 - s_2 - \dots - s_n$ . We will always define a tentative assignment  $\nu$  (which also allows students to be unassigned). The tentative assignment is such that it is possible to allocate the unassigned students to schools such that the resulting assignment is feasible. Let  $\mathcal{F}$  denote the set of all feasible assignments and  $\nu_0$  be the empty assignment, i.e.,  $\nu_0(s) = s$  for all  $s \in S$ . Let  $P_S$  be a controlled school choice problem.

1. Let student  $s_1$  apply to the school which is ranked first under  $P_{s_1}$ , say  $c_1$ . If there is some  $\mu \in \mathcal{F}$  such that  $\mu(s_1) = c_1$ ,<sup>14</sup> then set  $\nu_1(s_1) = c_1$  and  $\nu_1(s) = \nu_0(s) = s$  for all  $s \in S \setminus \{s_1\}$ ; otherwise  $s_1$  is rejected by school  $c_1$  and we set  $\nu_1 = \nu_0$ .
- ⋮
- $k$ . If there is some student  $s$  such that  $\nu_{k-1}(s) = s$  ( $s$  is unassigned), then student  $s$  did not yet apply to all the schools which are acceptable to him. Let  $s$  be the student with minimal index among such students. Let  $c$  be the school which is the most preferred under  $P_s$  among schools that have not rejected  $s$  yet.
  - (i) If there is  $\mu \in \mathcal{F}$  such that  $\mu(s) = c$  and  $\mu(s') = \nu_{k-1}(s')$  for all students  $s'$  satisfying  $\nu_{k-1}(s') \neq s'$ , then student  $s$  justifiably claims an empty slot at school  $c$  under  $\nu_{k-1}$ . Then we set  $\nu_k(s) = c$  and  $\nu_k(s') = \nu_{k-1}(s')$  for all  $s' \in S \setminus \{s\}$  (Appendix B provides an algorithm that checks in polynomial time whether student  $s$  can tentatively be assigned to school  $c$ );
  - (ii) If (i) is not true, but there exists a type  $\tau(s)$  student such that  $s$  justifiably envies that student at school  $c$  under  $\nu_{k-1}$ , then we do the following. Let  $s'$  be the student who has the lowest priority under  $\succ_c$  among all students of type  $\tau(s)$  who are tentatively admitted at school  $c$  under  $\nu_{k-1}$ . Then we set  $\nu_k(s) = c$ ,  $\nu_k(s') = s'$ , and  $\nu_k(s'') = \nu_{k-1}(s'')$  for all  $s'' \in S \setminus \{s, s'\}$ , i.e., school  $c$  permanently rejects  $s'$  and tentatively admits  $s$ ; and
  - (iii) Otherwise (if (i) and (ii) are not true) we set  $\nu_k = \nu_{k-1}$  and student  $s$  is rejected by school  $c$ .

End: Stage 1 ends at a Step  $x$  where  $\nu_x(s) \neq s$  for all  $s \in S$ . Let  $\mu \equiv \nu_x$  be the tentative assignment.

## Stage 2: Improvement

1. We construct a graph for assignment  $\mu$  as follows. For each school  $c$ , we create  $k$  nodes enumerated as  $c(t_1), \dots, c(t_k)$ . Therefore, for each school  $c$  and type  $t_i$  we have a node denoted by  $c(t_i)$ . If there is an empty seat in school  $c$ , that is if  $|\mu(c)| < q_c$ , then we create an additional node  $c(t_0)$  representing the empty seats.

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<sup>14</sup>We do not need to characterize  $\mathcal{F}$ , but instead we need to check for any partial assignment, which specifies the matches for a subset of students, whether there exists an assignment that has the same matches defined in the partial assignment. The algorithm that we provide in Appendix B checks whether such assignments exist in polynomial time.

2. For each student type  $t$  and school  $c$ , we consider all type  $t$  students who would prefer to be matched with  $c$  rather than their current assignments, i.e.,  $\{s \in S_t : cP_s\mu(s)\}$ . If this set is empty, then we do nothing. Otherwise, if this set is non-empty, then we consider the student in this set with the highest priority according to  $\succ_c$ . Let  $s$  be this student. Then  $\mu(s)(t)$  points to  $c(t)$ . In addition,  $\mu(s)(t)$  also points to  $c(t')$  such that  $s$  can be admitted to  $c$  by replacing a student of type  $t'$  (that is, if  $|\mu^{t'}(c)| > \underline{q}_c^{t'}$  and  $|\mu^t(c)| < \bar{q}_c^t$ ). If there is an empty seat in  $c$  (i.e., if  $|\mu(c)| < q_c$ ), then  $\mu(s)(t)$  also points to  $c(t_0)$  if  $s$  can take that seat without violating the feasibility constraints in  $c$ , (that is, if  $|\mu^t(c)| < \bar{q}_c^t$ .)
3. For each school  $c$  with an empty seat,  $c(t_0)$  points to all  $c'(t)$  where  $c' \neq c$  and  $t \neq t_0$  such that a student of type  $t$  can be expelled from  $c'$  without violating the feasibility constraints in  $c'$  (that is, if  $|\mu^t(c')| > \underline{q}_{c'}^t$ .)
4. If there exists no cycle in the graph then stop. Otherwise, if there exists a cycle in this graph, then we rematch students associated with each node in the cycle.<sup>15</sup> Redefine  $\mu$  to be the new assignment and go back to Step 1 of Stage 2.

In Stage 1 of CDAAI students with smaller indices are allowed to propose first (and students may be indexed according to when their applications were received by the controlled school choice program). However, it is easy to verify that the order, in which students are allowed to propose, is irrelevant for the conclusion of Theorem 2. Therefore, at each step alternatively we may randomly choose a student from the students who are not tentatively admitted to any school. This randomization of the CDAAI ensures that the algorithm becomes anonymous. Then using Roth and Rothblum (1999) and Ehlers (2008) it can be shown that in a low information environment it is a weakly dominant strategy for each student to submit his true ranking. Unfortunately, in contrast to McVitie and Wilson's sequential version of DAA, CDAAI may yield different outcomes for different orders. For instance, in the example used to prove part (i) of Theorem 1, CDAAI finds  $\mu_1$  when student  $s_1$  proposes first in Step 1 and it finds  $\mu_3$  when student  $s_2$  proposes first in Step 1 instead.

The assignment found by CDAAI may be wasteful because in the example used to prove part (i) of Theorem 1, the algorithm finds  $\mu_1$  when  $s_1$  proposes in Step 1 and student  $s_2$

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<sup>15</sup>For example, suppose that there exists a cycle  $c_1(t_1) \rightarrow c_2(t_2) \rightarrow \dots \rightarrow c_k(t_k) \rightarrow c_1(t_1)$ . Then there exists  $s_i \in \mu^{t_i}(c_i)$  for  $i \in \{1, \dots, k\}$  such that  $s_i \succ_{c_{i+1}} s'$  for all  $s' \in \{s \in S_{t_i} : c_{i+1}P_s\mu(s)\}$  where  $c_{k+1} \equiv c_1$ . The improvement algorithm then outputs assignment  $\mu'$  such that  $\mu'(s_i) = c_{i+1}$  for  $i \in \{1, \dots, k\}$  and  $\mu'(s) = \mu(s)$  for all  $s \in S \setminus \{s_1, \dots, s_k\}$ . Similarly, if the cycle involves node  $c(t_0)$ , i.e., if the cycle is  $c(t_0) \rightarrow c_2(t_2) \rightarrow \dots \rightarrow c_k(t_k) \rightarrow c(t_0)$  then  $c_2$  loses a student and  $c$  gains a new student whereas other schools have the same number of students at the end of the algorithm.

justifiably claims an empty slot at school  $c_3$  under  $\mu_1$ . However, we show that it finds a feasible assignment satisfying some other desirable properties.

**Theorem 2** *For any controlled school choice problem CDAAI yields a feasible assignment that is fair for same types and constrained non-wasteful. Moreover, the assignment produced by CDAAI is Pareto optimal among such assignments.*

The proof of Theorem 2 is provided in Appendix A. In the proof, we initially show that the first stage of CDAAI results in an assignment that is feasible, fair for same types, and constrained non-wasteful. Then we show that the final assignment produced at the end of the improvement stage is *constrained efficient* (i.e., it is Pareto optimal in the class of assignments that are fair for same types). In fact, Lemma 5 in this proof can be used to show that the assignment produced by CDAAI is weakly Pareto optimal, i.e., that there exists no other feasible assignment which all students strictly prefer to the outcome of CDAAI.

The improvement stage of CDAAI corrects for the unnecessary efficiency loss caused by the initial assignment. Indeed, the outcome of the initial stage can be Pareto dominated by an assignment that is fair for same types and constrained non-wasteful. Therefore, in the improvement stage, we rematch students to obtain an assignment that is Pareto efficient among assignments that are fair for same types. To demonstrate this, we provide the following example.

**Example 1.** *An illustration of CDAAI.* Consider the following example with six students  $\{s_1, s_2, s_3, s_4, s_5, s_6\}$ , four schools  $\{c_1, c_2, c_3, c_4\}$  and three student types  $\{t_1, t_2, t_3\}$  such that  $\tau(s_1) = \tau(s_3) = t_1$ ,  $\tau(s_2) = \tau(s_5) = t_2$ , and  $\tau(s_4) = \tau(s_6) = t_3$ . Schools  $c_1, c_3, c_4$  have capacities of two and  $c_2$  has a capacity of one. The only effective control constraints are  $\underline{q}_{c_1}^{t_3} = 1$  and  $\bar{q}_{c_3}^{t_3} = \bar{q}_{c_3}^{t_2} = 0$  (all other floors are zero and all other ceilings are equal to quotas). For all schools, student priorities are the same and given as follows; for all  $c \in C$ ,

$$s_3 \succ_c s_5 \succ_c s_1 \succ_c s_2 \succ_c s_4 \succ_c s_6.$$

For students  $s \in \{s_1, s_4, s_5, s_6\}$  the preferences are  $c_1 P_s c_2 P_s c_3 P_s c_4$ ; whereas for students  $s \in \{s_2, s_3\}$  the preferences are  $c_2 P_s c_1 P_s c_3 P_s c_4$ . This information is summarized in Table 1.

TABLE 1.

	$P_{s_1} = P_{s_4} = P_{s_5} = P_{s_6}$	$P_{s_2} = P_{s_3}$	$\succ_{c_1} = \succ_{c_2} = \succ_{c_3} = \succ_{c_4}$
	$c_1$	$c_2$	$s_3$
	$c_2$	$c_1$	$s_5$
	$c_3$	$c_3$	$s_1$
	$c_4$	$c_4$	$s_2$
			$s_4$
			$s_6$
capacities	$q_{c_1} = 2$	$q_{c_2} = 1$	$q_{c_3} = 2$
effective ceilings	$\bar{q}_{c_3}^{t_2} = 0$	$\bar{q}_{c_3}^{t_3} = 0$	$q_{c_4} = 2$
effective floors	$\underline{q}_{c_1}^{t_3} = 1$		

Suppose that students make proposals in the following order:  $s_1 - s_2 - s_3 - s_4 - s_5 - s_6$ . Let us apply CDAAI. To illustrate how Stage 1 of CDAAI works, we show some of the earlier steps:

- $s_1$  applies to  $c_1$  and gets admitted to  $c_1$ ;
- $s_2$  applies to  $c_2$  and gets admitted to  $c_2$ ;
- $s_3$  cannot get admitted to  $c_2$  since there is no empty seat in  $c_2$ , and there is no student admitted to  $c_2$  who has the same type as  $s_3$ ,  $s_3$  is rejected by  $c_2$ ;
- $s_3$  cannot get the empty seat in  $c_1$  (because then it is not possible to fill the floor for  $t_3$  in  $c_1$ ), but since  $s_3$  has higher priority than  $s_1$  at  $c_1$ ,  $s_1$  is rejected by  $c_1$  and  $s_3$  is admitted to  $c_1$ ;
- $s_1$  cannot get admitted to  $c_2$  since there is no empty seat in  $c_2$  and there is no student admitted to  $c_2$  who has the same type as  $s_1$ ,  $s_3$  is rejected by  $c_2$ ;
- $\vdots$

Hence, at the end of Stage 1 of CDAAI algorithm, we obtain the assignment  $\mu$ :

$$\mu = \begin{pmatrix} c_1 & c_2 & c_3 & c_4 \\ \{s_3, s_4\} & s_5 & s_1 & \{s_2, s_6\} \end{pmatrix}.$$

In Stage 2 of the CDAAI algorithm, instead of creating nodes for every possible type-school pair, we can create nodes only for types that are already present in the school. Therefore, we create seven nodes in the directed graph:  $c_1(t_1)$ ,  $c_1(t_3)$ ,  $c_2(t_2)$ ,  $c_3(t_1)$ ,  $c_3(t_0)$ ,  $c_4(t_2)$ , and  $c_4(t_3)$ . We determine the edges as defined in CDAAI. The graph is depicted in Figure 1.

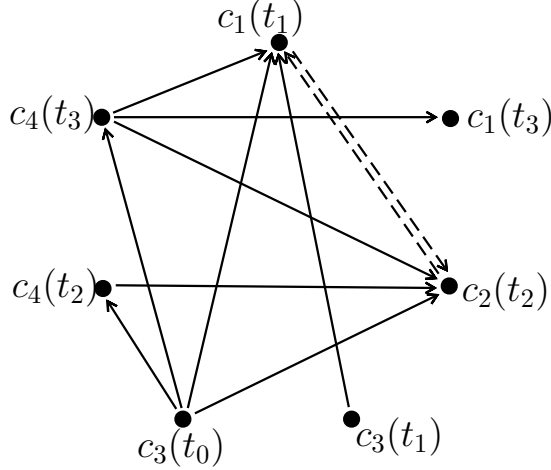


Figure 1: The graph in the improvement stage of CDAAI.

The only cycle in this graph is  $c_1(t_1) \rightarrow c_2(t_2) \rightarrow c_1(t_1)$ . Hence, we rematch students associated with each node in the cycle, so  $s_3$  is matched to  $c_2$  and  $s_5$  is matched with  $c_1$ . Note that both  $s_3$  and  $s_5$  prefer their new schools to old schools. The new assignment  $\mu'$  is given by:

$$\mu' = \begin{pmatrix} c_1 & c_2 & c_3 & c_4 \\ \{s_4, s_5\} & s_3 & s_1 & \{s_2, s_6\} \end{pmatrix}.$$

If we apply the improvement algorithm to this new assignment, we can confirm that there are no cycles, hence  $\mu'$  is the resulting assignment of the CDAAI algorithm. Therefore,  $\mu'$  is fair for same types, constrained non-wasteful, and Pareto efficient among such assignments.

### 3.1 Incentives

Apart from students' preferences all components of a controlled school choice problem are exogenously determined (like the capacities of the schools) or given by law (like the priority rankings and the controlled choice constraints). The only information which is private are students' preferences over schools. They need to be stated by the students to the school choice program. Since students must be assigned schools for any possible reported profile,

the program has to be based on a mechanism selecting an assignment for each possible problem. In a controlled school choice program the mechanism should respect the legal constraints imposed by the state. A mechanism is **(legally) feasible** if it selects a feasible assignment for any reported student profile.

Any program would like to elicit the true preferences from students. If students misreport, then the assignment chosen by the program is based on *false* preferences and may be highly unfair for the *true* preferences.

Avoiding this problem means constructing a mechanism where no student has ever an incentive to misrepresent her true preference for any preferences reported by the other agents. Any mechanism which makes truthful revelation of preferences a dominant strategy for each student is called **(dominant strategy) incentive compatible**. A feasible mechanism is **fair across types** if it selects for any controlled school choice problem a feasible assignment that is fair for same types whenever such an assignment exists. Analogously we define **fair for same types**, **non-wastefulness** and **constrained non-wastefulness**, respectively, for a mechanism.

In contrast to the school choice problems studied in the previous literature, it is impossible to construct a mechanism that is incentive compatible, fair for same types and constrained non-wasteful while respecting the diversity constraints given by law. Therefore, it is impossible to choose for each profile an order in which students propose in CDAAI such that the mechanism is incentive compatible. A similar result holds for fairness across types.

**Theorem 3** *(i) In controlled school choice there is no feasible mechanism that is dominant strategy incentive compatible, fair for same types and constrained non-wasteful.*

*(ii) In controlled school choice there is no feasible mechanism that is dominant strategy incentive compatible and fair across types.*

The proof of Theorem 3 is provided in Appendix A, where we give examples to prove the non-existence results.

**Remark 1** *The non-existence of feasible mechanisms, which are incentive compatible, fair for same types and (constrained) non-wasteful, shows that controlled school choice is not equivalent to the college admissions problem. In all models of school choice studied so far it was possible to connect the school choice problem to the college admissions problem and show that DAA is a mechanism which is non-wasteful, fair, and incentive compatible. This was due to the absence of diversity constraints (the floors) which are present in controlled choice.*

*In college admissions, any mechanism which is incentive compatible for students chooses the extreme of the lattice of stable assignments which students prefer over any other stable*

assignment. In controlled school choice there is not always a unique candidate for a feasible assignment that is fair for same types and (constrained) non-wasteful.<sup>16</sup> This provides additional reason for Theorem 3, i.e., for the non-existence of feasible mechanisms that are incentive compatible, fair for same types and (constrained) non-wasteful.

**Remark 2** Theorem 3 implies that for any order of the students CDAAI is not incentive compatible. Due to this fact students may misrepresent their preferences over schools. Now if the students play a Nash equilibrium (NE), what are the properties of the outcome (or the assignment) of any NE? It is easy to see that the outcome of any NE must be constrained non-wasteful.<sup>17</sup> Unfortunately, the outcome of a NE may not be fair for same types according to students' true preferences.<sup>18,19</sup>

Any controlled school choice program must give up constrained non-wastefulness or fairness for same types to achieve incentive compatibility. Does an existence result reemerge if we give up exactly one of our two basic requirements, namely constrained non-wastefulness or fairness for same types?

Since in real life often the number of available seats is approximately the same as the number of students, potential justified claims of empty seats occur less frequently than potential justified envy. Hence, a school choice program may be ready to give up constrained non-wastefulness while retaining fairness for same types and incentive compatibility. We will demonstrate that this weakening results in existence.<sup>20</sup>

**Example 2.** A feasible mechanism that is both fair for same types and incentive compatible. Fix a feasible assignment, say  $\mu$ . We relate any controlled school choice problem with a college admissions problem in the following way: break any school  $c$  into  $k$  schools  $\{c(t_1), \dots, c(t_k)\}$  where  $|T| = k$  and  $c(t)$  is the part of school  $c$  filling slots with students of type  $t$ . The capacity of school  $c(t)$  is  $q_{c(t)} = |\mu^t(c)|$  and the preference of  $c(t)$  ranks only students of

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<sup>16</sup>In the example used to prove (i) of Theorem 1,  $\mu_1$  and  $\mu_3$  are the only feasible assignments which are fair for same types and constrained non-wasteful. Student  $s_1$  strictly prefers  $\mu_3$  to  $\mu_1$  whereas student  $s_2$  strictly prefers  $\mu_1$  to  $\mu_3$ .

<sup>17</sup>Otherwise a student would justifiably claim an empty slot and after assigning him this empty slot the resulting assignment is fair for same types. Then this student profits from changing his preference such that he proposes to this school before proposing to the school to which he is assigned to.

<sup>18</sup>Ehlers (2010) provides an explicit example.

<sup>19</sup>In school choice problems without control and legal constraints, Ergin and Sönmez (2006) consider revelation games induced by the Boston school choice mechanism and DAA.

<sup>20</sup>Giving up fairness for same types also results in existence. A serial dictatorship (which is used frequently for the allocation of indivisible objects) is a feasible mechanism which is both non-wasteful and incentive compatible. A serial dictatorship orders the set of students alphabetically, say  $s_1, s_2, \dots$ , and  $s_n$ . Then for any problem, first student  $s_1$  picks the feasible assignments which he most prefers, second student  $s_2$  picks the assignments, which he most prefers, among the remaining feasible assignments and so on. This mechanism is fair only if each school's priority ranking is identical with the alphabetical order of the students.

type  $t$  acceptable, in the same order as  $\succ_c$ . Note that some slots are wasted at school  $c$  if  $|\mu(c)| < q_c$ . Any student replaces on his preference school  $c$  by  $|T|$  copies of  $c$  in the order  $c(t_1), c(t_2), \dots, c(t_k)$ . Then determine the student optimal assignment of this related problem. Because (i) all students rank all schools as acceptable, (ii) for any type  $t$  there are exactly  $\sum_{c \in C} q_{c(t)} = \sum_{c \in C} |\mu^t(c)| = |S_t|$  slots available and (iii) any school  $c(t)$  ranks acceptable exactly all students of type  $t$ , the student optimal assignment  $\bar{\mu}$  of the related problem satisfies for all types  $t$  and all schools  $c$ ,  $\bar{\mu}^t(c(t)) \subseteq S_t$  and  $|\bar{\mu}(c(t))| = q_{c(t)} = |\mu^t(c)|$ .

Thus the feasibility of  $\mu$  implies that the student optimal assignment of the related problem is a feasible assignment of the controlled school choice problem. We know that DAA is incentive compatible. Furthermore, the stability of the student optimal assignment in the related problem implies that there is no student envying justifiably another student of the same type. Thus the “related” mechanism is a feasible mechanism which is both fair for same types and incentive compatible. The mechanism is constrained non-wasteful only if the initial assignment  $\mu$  filled all available slots at each school. Furthermore the mechanism is fair (across types) only if all students are of the same type.

Observe that the above mechanism is “rigid”: in Example 2 for each type  $t$ , the slots, which will be filled with type- $t$  students, are exogenously given by the feasible assignment  $\mu$ . This inflexibility was the price for incentive compatibility of this mechanism. In general this price includes giving up Pareto optimality because due to the inflexibility all students may be strictly better off with another feasible assignment compared to the assignment chosen by the mechanism in Example 2. Note that this inefficiency stems from the rigidity of the mechanism and not necessarily from the waste of empty seats. In the next section, we overcome this inefficiency by providing a different interpretation of the ceilings and floors.

## 4 Controlled School Choice with Soft Bounds

Some school districts administer floors and ceilings as hard bounds, so a theoretical analysis of such policies is inarguably important. In the previous sections, we accommodate this constraint by considering an assignment infeasible if it assigns less than  $\underline{q}_c^t$  or more than  $\bar{q}_c^t$  number of type  $t$  students to school  $c$ . However, applications of these hard bounds are quite paternalistic in the sense that assignments can be forced *despite* student preferences. In contrast, in this section we view these bounds as *soft bounds*. In controlled school choice with soft bounds, school districts adapt a dynamic priority structure: giving highest priorities to the student types who have not filled their floors; medium priorities to the student types who have filled their floors, but not filled their ceilings; and lowest priorities to student types

who have filled their ceilings. Yet, schools can still admit fewer students than their floors or more than their ceilings as long as students with higher priorities do not veto this match. With this view, there are no feasibility constraints as long as school quotas are not exceeded (and our approach below can be used for situations where no feasible assignment exists). All controlled choice concerns are embedded in the schools' choice functions. A formal discussion is in order.

In controlled school choice with soft bounds, any assignment that matches at most  $q_c$  number of students to school  $c$  is **feasible under soft bounds**. An assignment  $\mu$  is **non-wasteful under soft bounds** if for any student  $s$  and any school  $c$ ,  $cP_s\mu(s)$  implies  $|\mu(c)| = q_c$ . Previously, non-wastefulness required student  $s$  to be matched with  $c$  without violating ceilings and floors, which is not required anymore. Furthermore, an assignment  $\mu$  **removes justifiable envy under soft bounds** if for any student  $s$  and any school  $c$  such that  $cP_s\mu(s)$  with  $\tau(s) = t$ , we have both  $|\mu^t(c)| \geq \underline{q}_c^t$  and  $s' \succ_c s$  for all  $s' \in \mu^t(c)$ , and either

(i)  $|\mu^t(c)| \geq \bar{q}_c^t$  and  $s' \succ_c s$  for all  $s' \in \mu(c)$  such that  $|\mu^{\tau(s')}(c)| > \bar{q}_c^{\tau(s')}$ , or

(ii)  $\bar{q}_c^t > |\mu^t(c)| \geq \underline{q}_c^t$ , and

(a)  $|\mu^{t'}(c)| \leq \bar{q}_c^{t'}$  for all  $t' \in T \setminus \{t\}$ , and

(b)  $s' \succ_c s$  for all  $s' \in \mu(c)$  such that  $\bar{q}_c^{\tau(s')} \geq |\mu^{\tau(s')}(c)| > \underline{q}_c^{\tau(s')}$ .

Less formally, an assignment removes justifiable envy under soft bounds if a student  $s$  of type  $t$  cannot attend a favorable school  $c$ , then type  $t$  students fill their floor in  $c$  and  $c$  prefers all type  $t$  students that it has been assigned to  $s$ . Moreover, either  $c$  has admitted more than its ceiling of type  $t$  students, and all students with types exceeding their ceilings are preferred to  $s$ ; or  $c$  has admitted more than its floor, but not more than its ceiling of type  $t$  students, there are no students with types exceeding their ceilings, and all students with types exceeding their floors are preferred to  $s$ .

Finally, an assignment  $\mu$  is **fair under soft bounds** if it removes justifiable envy under soft bounds.

With hard bounds, no assignment that is fair and non-wasteful exists even if fairness is restricted to students with the same types (Theorem 1). However, with soft bounds we guarantee the existence of an assignment that is non-wasteful and fair under soft bounds. To show this, we consider the student-proposing deferred acceptance algorithm with soft bounds, defined below. We will first give the general version where schools use choice functions at each step to reject and tentatively admit students and formalize the choice functions afterwards.

## Deferred Acceptance Algorithm with Soft Bounds (DAASB)

**Step 1** Start with the assignment in which no student is matched. Each student  $s$  applies to her first-choice school. Let  $S_{c,1}$  denote the set of students who applied to school  $c$ . School  $c$  accepts the students in  $Ch_c(S_{c,1})$  and rejects the rest.

**Step  $k$**  Start with the tentative assignment obtained at the end of step  $k - 1$ . Each student  $s$  who got rejected at step  $k - 1$  applies to her next-choice school. Let  $S_{c,k}$  denote the set of students who either were tentatively matched to  $c$  at the end of step  $k - 1$ , or applied to school  $c$  in this step. Each school accepts the students in  $Ch_c(S_{c,k})$  and rejects the rest. If there are no rejections, then stop.

Here, the choice function for school  $c$  depends on quota  $q_c$ , floors  $\underline{q}_c^T$ , and ceilings  $\bar{q}_c^T$  as described above. However, we are going to take these parameters as given and simplify the notation by omitting them. To define the choice function more formally, given  $\tilde{S} \subseteq S$ , let  $Ch_c(\tilde{S}, q_c, (q_c^t)_{t \in T})$  be the subset of students  $\tilde{S}' \subseteq \tilde{S}$  that includes the highest ranked students in  $\tilde{S}$  according to  $\succ_c$  such that there are no more than  $q_c$  students in total and  $q_c^t$  students of type  $t$ . In addition, let

$$\begin{aligned} Ch_c^{(1)}(\tilde{S}) &\equiv Ch_c(\tilde{S}, q_c, (\underline{q}_c^t)_{t \in T}), \\ Ch_c^{(2)}(\tilde{S}) &\equiv Ch_c(\tilde{S} \setminus Ch_c^{(1)}(\tilde{S}), q_c - |Ch_c^{(1)}(\tilde{S})|, (\bar{q}_c^t - \underline{q}_c^t)_{t \in T}), \text{ and} \\ Ch_c^{(3)}(\tilde{S}) &\equiv Ch_c(\tilde{S} \setminus (Ch_c^{(1)}(\tilde{S}) \cup Ch_c^{(2)}(\tilde{S})), q_c - |Ch_c^{(1)}(\tilde{S}) \cup Ch_c^{(2)}(\tilde{S})|, (q_c - \bar{q}_c^t)_{t \in T}). \end{aligned}$$

Intuitively,  $Ch_c^{(1)}(\tilde{S})$  is the set of students chosen with the highest priorities among  $\tilde{S}$  without exceeding the floor of each student type,  $Ch_c^{(2)}(\tilde{S})$  is the set of remaining students chosen from  $\tilde{S}$  with the highest priorities without exceeding the ceilings, and  $Ch_c^{(3)}(\tilde{S})$  is the set of students chosen above the ceilings. Finally,  $Ch_c(\tilde{S}) \equiv Ch_c^{(1)}(\tilde{S}) \cup Ch_c^{(2)}(\tilde{S}) \cup Ch_c^{(3)}(\tilde{S})$  is the set of students chosen from  $\tilde{S}$ . It is apparent from this formulation that schools dynamically give highest priorities to the student types who have not filled their floors; medium priorities to the student types who have filled their floors, but not filled their ceilings; and lowest priorities to student types who have filled their ceilings.

DAASB terminates when there are no new applications. At each step of the algorithm, there is at least one student rejected. Hence, the algorithm ends in finite time. Furthermore, we establish that the well-known properties of the deferred acceptance algorithm continue to hold.

**Theorem 4** *For any controlled school choice problem, DAASB yields a feasible under soft bounds assignment that is fair under soft bounds and non-wasteful under soft bounds. Moreover, in the assignment produced by DAASB each student is matched with the best outcome among the set of all such assignments.*

The proof of Theorem 4 is provided in Appendix A.

Even though CDAAI fails to satisfy incentive compatibility, DAASB satisfies a stronger version of incentive compatibility: An assignment mechanism  $\phi$  (choosing for any profile an assignment) is **group (dominant strategy) incentive compatible** if for any group of students  $\hat{S} \subseteq S$ , for any profile  $P_S$  there exists no  $P'_S$  such that  $\phi_s(P'_S, P_{S \setminus \hat{S}}) P_s \phi_s(P_S)$  for all  $s \in \hat{S}$ . If a mechanism is group incentive compatible, then there exists no group of students who can jointly change their preference profiles to make each student in the group better off.

**Theorem 5** *DAASB is group dominant strategy incentive compatible.*

The proof of Theorem 5 is provided in Appendix A. It is an application of Hatfield and Kojima (2009).

In addition, we establish a Pareto dominance relation between the outcome of DAASB and non-wasteful assignments that also satisfy another fairness notion which is stronger than fairness across types. This result gives us an interesting connection between hard bounds and soft bounds for controlled school choice.

This fairness notion is defined as follows. A student  $s$  weakly-envies student  $s'$  when  $s$  prefers the school at which  $s'$  is enrolled, say school  $c$ , to her school. However, the nature of controlled school choice imposes the following (legal) constraints: Weak-envy is justified only when

- (i) student  $s$  has higher priority to be enrolled at school  $c$  than student  $s'$ ,
- (ii) student  $s$  can be enrolled at school  $c$  without violating controlled choice constraints by removing  $s'$  from  $c$ .<sup>21</sup>

An assignment is **strongly-fair across types** if no student justifiably weakly-envies any student. Now, we proceed with the formal result.

**Theorem 6** *Suppose that  $\mu$  is a feasible assignment that is strongly-fair across types and non-wasteful. Then all students weakly prefer the outcome of DAASB to  $\mu$ .*

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<sup>21</sup>Recall that envy is justified only with the additional requirement that  $s'$  can be enrolled at another school without violating constraints.

The proof of Theorem 6 is provided in Appendix A. Therefore, if feasible assignments that are strongly-fair across types and non-wasteful exist, then the outcome of DAASB (weakly) Pareto dominates *all* such assignments. In such situations all students are weakly better off under soft bounds than under hard bounds.<sup>22</sup>

Although we have required students to submit full rankings of all schools, it is easy to check that all our results under soft bounds remain true if students are only required to submit partial rankings of the schools which they strictly prefer to their outside options (like private schooling). Hence, the desirable properties of DAASB continue to hold in this case.

## 5 Conclusion

Although there is a large literature in education evaluating and estimating the effects of segregation across schools on students' achievements (Hanushek, Kain, and Rivkin (2002), Guryan (2004), Card and Rothstein (2005), and others),<sup>23</sup> and on how to measure segregation and how to determine optimal desegregation guidelines,<sup>24</sup> none of these papers discusses the problem of *how in practice* to assign students to schools while complying with these desegregation guidelines. This is exactly what the first part of our paper does.

Without controlled choice, the student proposing deferred acceptance algorithm eliminates any justified envy and makes truthful revelation of preferences a dominant strategy for students (Abdulkadiroğlu and Sönmez, 2003). Once controlled choice constraints are imposed as hard bounds it may be impossible to eliminate any justified envy. However, justified envy can be eliminated only among students of the same type.

We demonstrate that controlled choice comes with a price, especially when the bounds are taken as inflexible hard bounds. The alternative view of soft bounds has benefits over hard bounds. It results in much attractive fairness, efficiency, and incentive properties. The downside of it is that desired diversity in schools is achieved only when student preferences are also in line with them. If the school districts' objectives are not very paternalistic, and only giving an opportunity to parents to achieve diversity is good enough, they should go with soft bound policies. Otherwise the choice is hard bounds, along with the prices coming with it.<sup>25</sup>

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<sup>22</sup>The corresponding result for assignments that are fair across types and non-wasteful does not hold. An example showing the contrary is available from the authors.

<sup>23</sup>We will refer the interested reader to Echenique, Fryer, and Kaufman (2006) for an illuminating account of this literature.

<sup>24</sup>School segregation can be purely racial or, as in Echenique, Fryer and Kaufman (2006), school segregation is measured according to the spectral segregation index of Echenique and Fryer (2006) which uses the intensity of social interactions among the members of a group (see also Cutler and Glaeser (1997)).

<sup>25</sup>One benefit of hard bounds is that, it is straightforward to check whether they are implemented or not.

# Appendix A: Proofs

In this Appendix, we provide the omitted proofs.

## Proof of Theorem 1

The proof for both parts is by means of an example. For part (i) consider the following problem consisting of three schools  $\{c_1, c_2, c_3\}$  and two students  $\{s_1, s_2\}$ . Each school has a capacity of two ( $q_c = 2$  for all schools  $c$ ). All students are of the same type  $t$ . The ceiling of type  $t$  is equal to two at all schools ( $\bar{q}_c^t = 2$  for all schools  $c$ ). School  $c_1$  has a floor for type  $t$  of  $\underline{q}_{c_1}^t = 1$ . All other floors are equal to zero. The schools' priorities are given by  $s_2 \succ_{c_1} s_1$ ,  $s_2 \succ_{c_2} s_1$  and  $s_1 \succ_{c_3} s_2$ . The students' preferences are given by  $c_2 P_{s_1} c_3 P_{s_1} c_1 P_{s_1} s_1$  and  $c_3 P_{s_2} c_2 P_{s_2} c_1 P_{s_2} s_2$ . This information is summarized in Table 2.

TABLE 2.

	$\succ_{c_1}$	$\succ_{c_2}$	$\succ_{c_3}$	$P_{s_1}$	$P_{s_2}$
	$s_2$	$s_2$	$s_1$	$c_2$	$c_3$
	$s_1$	$s_1$	$s_2$	$c_3$	$c_2$
				$c_1$	$c_1$
				$s_1$	$s_2$
capacities	$q_{c_1} = 2$	$q_{c_2} = 2$	$q_{c_3} = 2$		
ceiling for $t$	$\bar{q}_{c_1}^t = 2$	$\bar{q}_{c_2}^t = 2$	$\bar{q}_{c_3}^t = 2$		
floor for $t$	$\underline{q}_{c_1}^t = 1$	$\underline{q}_{c_2}^t = 0$	$\underline{q}_{c_3}^t = 0$		

Next we determine the set of assignments which are feasible for this problem. Feasibility requires that student  $s_1$  or student  $s_2$  is assigned school  $c_1$  and all students are enrolled at a school. Therefore,

$$\begin{array}{ccc}
 \mu_1 = \begin{pmatrix} c_1 & c_2 & c_3 \\ s_1 & s_2 & \emptyset \end{pmatrix} & \xrightarrow{s_2 \text{ claims } c_3} & \mu_2 = \begin{pmatrix} c_1 & c_2 & c_3 \\ s_1 & \emptyset & s_2 \end{pmatrix}, \\
 s_2 \text{ envies } s_1 \uparrow & & \downarrow s_1 \text{ envies } s_2 \\
 \mu_4 = \begin{pmatrix} c_1 & c_2 & c_3 \\ s_2 & s_1 & \emptyset \end{pmatrix} & \xleftarrow{s_1 \text{ claims } c_2} & \mu_3 = \begin{pmatrix} c_1 & c_2 & c_3 \\ s_2 & \emptyset & s_1 \end{pmatrix}
 \end{array}$$

If the districts use soft bounds, schools are not guaranteed to have diverse student bodies. Hence parents can question whether these control policies are appropriately applied.

and  $\mu_5 = \begin{pmatrix} c_1 & c_2 & c_3 \\ \{s_1, s_2\} & \emptyset & \emptyset \end{pmatrix}$  are the only assignments which are feasible. Now (as indicated above)

- (i)  $\mu_1$  is wasteful because  $s_2$  justifiably claims an empty slot at  $c_3$ ,
- (ii)  $\mu_2$  is not fair for same types because  $s_1$  justifiably envies  $s_2$  at  $c_3$ ,
- (iii)  $\mu_3$  is wasteful because  $s_1$  justifiably claims an empty slot at  $c_2$ ,
- (iv)  $\mu_4$  is not fair for same types because  $s_2$  justifiably envies  $s_1$  at  $c_2$ ; and
- (v)  $\mu_5$  is wasteful because  $s_1$  justifiably claims an empty slot at  $c_2$ .

Hence there is no feasible assignment which is both fair for same types and non-wasteful.

For part (ii) consider the following problem consisting of three schools  $\{c_1, c_2, c_3\}$  and three students  $\{s_1, s_2, s_3\}$ . Each school has a capacity of one ( $q_c = 1$  for all schools  $c$ ). The type space consists of two types  $t_1$  and  $t_2$ . Students  $s_1$  and  $s_2$  are of type  $t_1$  whereas student  $s_3$  is of type  $t_2$ . For all types the ceiling is equal to one at all schools ( $\bar{q}_c^t = 1$  for all types  $t$  and all schools  $c$ ). School  $c_1$  has a floor for type  $t_1$  of  $\underline{q}_{c_1}^{t_1} = 1$ . All other floors are equal to zero. The schools' priorities are given by  $s_2 \succ_{c_1} s_1 \succ_{c_1} s_3$ ,  $s_2 \succ_{c_2} s_1 \succ_{c_2} s_3$  and  $s_1 \succ_{c_3} s_2 \succ_{c_3} s_3$ . The students' preferences are given by  $c_2 P_{s_1} c_3 P_{s_1} c_1 P_{s_1} s_1$ ,  $c_3 P_{s_2} c_2 P_{s_2} c_1 P_{s_2} s_2$  and  $c_2 P_{s_3} c_3 P_{s_3} c_1 P_{s_3} s_3$ . This information is summarized in Table 3.

TABLE 3.

	$\succ_{c_1}$	$\succ_{c_2}$	$\succ_{c_3}$	$P_{s_1}$	$P_{s_2}$	$P_{s_3}$
	$s_2$	$s_2$	$s_1$	$c_2$	$c_3$	$c_2$
	$s_1$	$s_1$	$s_2$	$c_3$	$c_2$	$c_3$
	$s_3$	$s_3$	$s_3$	$c_1$	$c_1$	$c_1$
				$s_1$	$s_2$	$s_3$
capacities	$q_{c_1} = 1$	$q_{c_2} = 1$	$q_{c_3} = 1$			
ceiling for $t_1$	$\bar{q}_{c_1}^{t_1} = 1$	$\bar{q}_{c_2}^{t_1} = 1$	$\bar{q}_{c_3}^{t_1} = 1$			
floor for $t_1$	$\underline{q}_{c_1}^{t_1} = 1$	$\underline{q}_{c_2}^{t_1} = 0$	$\underline{q}_{c_3}^{t_1} = 0$			
ceiling for $t_2$	$\bar{q}_{c_1}^{t_2} = 1$	$\bar{q}_{c_2}^{t_2} = 1$	$\bar{q}_{c_3}^{t_2} = 1$			
floor for $t_2$	$\underline{q}_{c_1}^{t_2} = 0$	$\underline{q}_{c_2}^{t_2} = 0$	$\underline{q}_{c_3}^{t_2} = 0$			

Next we determine the set of assignments which are both feasible and fair across types for this problem. Feasibility requires that student  $s_1$  or student  $s_2$  is assigned school  $c_1$  and all

students are enrolled at a school. Therefore,

$$\begin{array}{ccc}
\mu_1 = \begin{pmatrix} c_1 & c_2 & c_3 \\ s_1 & s_2 & s_3 \end{pmatrix} & \begin{array}{c} s_2 \text{ envies } s_3 \\ \longrightarrow \end{array} & \mu_2 = \begin{pmatrix} c_1 & c_2 & c_3 \\ s_1 & s_3 & s_2 \end{pmatrix}, \\
s_2 \text{ envies } s_1 \uparrow & & \downarrow s_1 \text{ envies } s_2 \\
\mu_4 = \begin{pmatrix} c_1 & c_2 & c_3 \\ s_2 & s_1 & s_3 \end{pmatrix} & \begin{array}{c} \longleftarrow \\ s_1 \text{ envies } s_3 \end{array} & \mu_3 = \begin{pmatrix} c_1 & c_2 & c_3 \\ s_2 & s_3 & s_1 \end{pmatrix}
\end{array}$$

are the only assignments which are feasible. Now (as indicated above)

- (i)  $\mu_1$  is not fair across types because  $s_2$  justifiably envies  $s_3$  at  $c_3$ ,
- (ii)  $\mu_2$  is not fair across types because  $s_1$  justifiably envies  $s_2$  at  $c_3$ ,
- (iii)  $\mu_3$  is not fair across types because  $s_1$  justifiably envies  $s_3$  at  $c_2$ , and
- (iv)  $\mu_4$  is not fair across types because  $s_2$  justifiably envies  $s_1$  at  $c_2$ .

Hence there is no assignment which is both feasible and fair across types. ■

## Proof of Theorem 2

Let  $P_S$  be a controlled school choice problem and  $\mu$  be the assignment that the assignment stage of CDAAI finds for  $P_S$ . We first show that (a)  $\mu$  is feasible, (b)  $\mu$  is fair for same types, and (c)  $\mu$  is constrained non-wasteful.

For (a) it suffices to show at Step  $k$ , any student, who is unassigned under  $\nu_{k-1}$ , did not yet propose to all schools on his preference. Suppose that  $\nu_{k-1}(s) = s$  and student  $s$  proposed to all schools before.

Let student  $s$  have been on tentatively admitted at a school, say school  $c$ , until Step  $h$ . Then at Step  $h$  another student  $s'$  proposed to  $c$  and school  $c$  rejected  $s$ . Given that  $s'$  is unassigned at Step  $h-1$  and both  $s'$  and  $s$  have the same type, then there were other schools  $c'$  which could have given  $s'$  an empty slot keeping all the other matches of  $\nu_h$  unchanged. But  $s$  did not apply to any of those empty slots before and afterwards (because otherwise he would have received that slot). Therefore, this is impossible.

If student  $s$  was never tentatively admitted at a school, then let  $h$  be the step where student  $s$  applied to his most preferred school. Since  $s$  is rejected at Step  $h$ ,  $s$  could not justifiably claim an empty slot at his most preferred school. But then there were no  $\mu' \in \mathcal{F}$  such that  $\mu'(s') = \nu_{h-1}(s')$  for all  $s' \in S \setminus \{s\}$  with  $\nu_{h-1}(s') \neq s'$ . But then  $\nu_{h-1}$  is an impossible tentative assignment at Step  $h-1$ , which contradicts the definition of CDAAI.

For (b), suppose that  $\mu$  is not fair for same types. Then there is a student  $s$  who justifiably envies student  $s'$  at school  $c$  under  $\mu$  and both students  $s$  and  $s'$  are of the same type. Let  $s'$  have lowest priority in  $\mu(c)$  among the students who are of type  $\tau(s)$ . Since  $cP_s\mu(s)$ , student  $s$  applied to school  $c$  at some step, say Step  $k$ .

If  $\nu_k(s) = c$ , then by  $\mu(s) \neq c$ , student  $s$  was later rejected by school  $c$  because some student of type  $\tau(s)$  applied to school  $c$  and had higher priority than  $s$  under  $\succ_c$ . Now it is impossible that student  $s'$  was tentatively admitted at school  $c$  later because  $s'$  must have had higher priority than  $s$  and we have  $s \succ_c s'$ .

If  $\nu_k(s) \neq c$ , then (i) was not possible at Step  $k$ , i.e.,  $s$  could not justifiably claim an empty slot at school  $c$  under  $\nu_{k-1}$ . Since (ii) was neither possible, all students of type  $\tau(s)$  in  $\nu_{k-1}(c)$  had higher priority than  $s$ . Now it is again impossible that student  $s'$  was tentatively admitted at school  $c$  later because  $s'$  must have had higher priority than  $s$  and we have  $s \succ_c s'$ .

It may be that student  $s'$  later justifiably claimed an empty slot at school  $c$ . This is also impossible because at Step  $x$  given a tentative assignment  $\nu_x$ , for each school  $c$  and each type, the students of that type admitted at the school only increases, i.e., it is not possible that  $s'$  claims an empty slot later whereas  $s$  could not do that earlier.

For (c), suppose that  $\mu$  is not constrained non-wasteful. Then a student  $s$  justifiably claims an empty slot at school  $c$  under  $\mu$  and  $\mu'$  (where  $\mu'(s) = c$  and  $\mu'(s') = \mu(s')$  for all  $s' \in S \setminus \{s\}$ ) is fair for same types. Since  $s$  justifiably claims an empty slot at school  $c$ , we have  $cP_s\mu(s)$  and  $s$  must have proposed to  $c$ , say at Step  $k$ , before proposing to  $\mu(s)$ . The following is true in CDAAI: once a student is tentatively admitted to a school, then the student can only be rejected by this school if another student of the same type is admitted. Therefore, for all types  $t$  and all schools  $c'$  we have

$$|\nu_{k-1}^t(c')| \leq |\mu^t(c')|. \quad (1)$$

Now by the feasibility of  $\mu$  and  $s$ 's justified claim of an empty slot at  $c$  under  $\mu$ , at Step  $k$  there was a feasible assignment  $\hat{\mu}$  such that  $\hat{\mu}(s) = c$  and  $\hat{\mu}(\hat{s}) = \nu_{k-1}(\hat{s})$  for all  $\hat{s}$  such that  $\nu_{k-1}(\hat{s}) \neq \hat{s}$ . Hence,  $\nu_k(s) = c$  and  $s$  was tentatively assigned to  $c$  at Step  $k$ . Since  $\mu(s) \neq c$ , at a later step, say Step  $k'$ , school  $c$  rejected student  $s$  and admitted a student  $s'$ . Then student  $s'$  must be of the same type as  $s$  and at Step  $k'$  (i) was not true, i.e., student  $s'$  could not justifiably claim an empty slot at school  $c$  at Step  $k'$ . But then by the same property (1) for Step  $k'$  no student of type  $\tau(s)$  can justifiably claim an empty slot at school  $c$  under  $\mu$ , a contradiction to  $s$ 's justified claim of an empty slot at  $c$  under  $\mu$ .

Hence, we established that the assignment  $\mu$  produced at the end of Stage 1 of CDAAI

is feasible, fair for same types, and constrained non-wasteful.

We now prove that the assignment produced at the end of improvement stage is “constrained efficient” (in the sense that it is Pareto optimal in the class of assignments that are fair for same types) in steps by showing the following lemmas.

**Lemma 1** *The assignment produced by the improvement stage is feasible.*

**Proof.** For  $t, t' \in T$ , each node  $c(t)$  only points to a node  $c'(t')$  when a type  $t'$  student can be fired from school  $c'$  and a type  $t$  student can be admitted to  $c'$  without violating the feasibility conditions in school  $c'$ . Thus, when we execute a cycle consisting of such nodes we get a feasible assignment. On the other hand, suppose that we execute a cycle containing  $c(t_0)$ . Let the cycle include the following path  $c'(t) \rightarrow c(t_0) \rightarrow c''(t')$ . Since  $c'(t)$  is pointing  $c(t_0)$ , then a type  $t$  student can take an empty seat in  $c$  without violating feasibility constraints. Similarly, since  $c(t_0)$  is pointing  $c''(t')$  a type  $t'$  student can be fired from school  $c''$ . Therefore, the assignment produced is feasible. ■

**Lemma 2** *If  $\mu'$  is the assignment produced by improvement stage, then  $\mu'$  is fair for same types.*

**Proof.** Suppose otherwise that  $\mu'$  is not fair for same types. Therefore, there exist students  $s$  and  $s'$  of the same type such that  $s'$  justifiably envies  $s$ :  $\tau(s) = \tau(s')$ ,  $c \equiv \mu'(s)P_{s'}\mu'(s')$ , and  $s' \succ_c s$ . There are two cases depending on whether  $\mu(s) = \mu'(s)$ .

- $\mu(s) = \mu'(s)$ : For student  $s'$ , let  $R_{s'}$  be the weak order associated with  $P_{s'}$ . Since  $\mu$  is fair and  $s' \succ_c s$ , we have  $\mu(s')R_{s'}c$ . Since the improvement stage improves the match of every student or keeps it the same, we get  $\mu'(s')R_{s'}\mu(s')$ . Therefore,  $\mu'(s')R_{s'}c$  which is a contradiction.
- $\mu(s) \neq \mu'(s)$ : In this case,  $s$  must have matched with  $c$  in the improvement stage. In order to have a node for type  $\tau(s)$  in  $\mu(s)$  point to any node for school  $c$ ,  $s$  must have the highest priority among type  $\tau(s)$  students who prefer  $c$  to their current assignments. This implies that  $s \succ_c s'$ , a contradiction.

In both cases, we get a contradiction. The conclusion follows. ■

**Lemma 3** *Suppose that  $\mu$  is a feasible assignment that is fair for same types, which is also Pareto efficient among such assignments. Then  $\mu$  is constrained non-wasteful.*

**Proof.** Suppose, otherwise, that  $\mu$  violates constrained non-wastefulness. Then there exists a student  $s$  and school  $c$  with an empty seat such that the assignment in which school  $c$  admits student  $s$  without changing the matches of any other student is fair for same types. This gives a contradiction to Pareto efficiency. ■

**Lemma 4** *Suppose that  $\mu$  is an assignment that is fair for same types and Pareto efficient among such assignments. Then  $\mu$  is also Pareto efficient among assignments that are fair for same types and constrained non-wasteful.*

**Proof.** This follows from the fact that the set of assignments that are fair for same types is a superset of the set of assignments that are fair for same types and constrained non-wasteful. The conclusion follows from Lemma 3 and the fact that if  $\mu$  is Pareto efficient in a bigger set, then it is also going to be Pareto efficient in a smaller set. ■

**Lemma 5** *Let  $\mu$  be an assignment that is fair for same types, which is not Pareto efficient among such assignments. Then there exists a cycle in the graph described in the improvement stage, and hence the assignment produced by the improvement stage is different than  $\mu$ .*

**Proof.** Let  $\mu'$  be an assignment that is fair for same types, which Pareto dominates  $\mu$ . Consider the graph associated with  $\mu$  described in the improvement stage. We are going to show that there exists a cycle in this graph. To do this, we split the analysis whether there exists a school  $c$  such that  $|\mu(c)| \neq |\mu'(c)|$ .

**Case 1:** (There exists  $c$  such that  $|\mu(c)| \neq |\mu'(c)|$ .) Since the number of assigned students is the same in both  $\mu$  and  $\mu'$ , there exists  $c$  such that  $|\mu'(c)| > |\mu(c)|$ . Hence, there exists a type  $t_i$  such that there are more type  $t_i$  students in  $\mu'(c)$  compared to  $\mu(c)$ . Hence, in the graph associated with  $\mu$  there exists a school  $c^{(1)}$  such that  $c^{(1)}(t_i)$  is pointing  $c(t_0)$ . If the floor of type  $t_i$  in  $c^{(1)}$  is not binding in  $\mu$ , then  $c(t_0)$  is also pointing  $c^{(1)}(t_i)$ . Therefore, there exists a cycle and we are done. Suppose otherwise that the floor of type  $t_i$  in  $c^{(1)}$  is binding at  $\mu$ . Let  $s \in \mu^{t_i}(c^{(1)})$  be the student with highest priority according to  $\succ_c$  among  $\mu^{t_i}(c^{(1)})$ . Either  $s$  is matched with  $c$  in  $\mu'$ , or  $s$  is not matched with it, then  $s$  must have been matched with a better school in  $\mu'$  since  $\mu'$  is fair. Both imply that there exists a student of type  $t_i$  who is in  $\mu'(c^{(1)})$  but not in  $\mu(c^{(1)})$  since  $\mu'$  is feasible and that the number of type  $t_i$  students in  $\mu^{t_i}(c^{(1)})$  is only at the floor level, i.e.,  $|\mu^{t_i}(c^{(1)})| = \underline{q}_{c^{(1)}}^{t_i}$ . Therefore, there exists a school  $c^{(2)}$  such that  $c^{(2)}(t_i)$  is pointing to  $c^{(1)}(t_i)$ . By a similar argument, we see that either  $c(t_0)$  is pointing to  $c^{(2)}(t_i)$  or that there exists a school  $c^{(3)}$  such that  $c^{(3)}(t_i)$  is pointing to  $c^{(2)}(t_i)$ . Since there is a finite number of schools, by mathematical induction, we see that there exists a positive number  $j$  such that  $c(t_0)$  is pointing to  $c^{(j)}(t_i)$  and for every  $l = 1, \dots, j$   $c^{(l)}(t_i)$  is

pointing to  $c^{(l-1)}(t_i)$ . Hence, there exists a cycle of type  $t_i$  nodes and a node for an empty seat.

**Case 2:** (For all  $c$ ,  $|\mu(c)| = |\mu'(c)|$ .) In this case, since  $\mu \neq \mu'$  there exist a type  $t_i$  student  $s$  and school  $c$  such that  $s \in \mu'(c) \setminus \mu(c)$ . Therefore, in the graph for  $\mu$ ,  $c(t_i)$  is being pointed by  $c^{(1)}(t_i)$  for some  $c^{(1)} \in C$ . If there exists a type  $t_i$  student in  $\mu'(c^{(1)}) \setminus \mu(c^{(1)})$  then there exists another node  $c^{(2)}(t_i)$  pointing to  $c^{(1)}(t_i)$ . Suppose otherwise that there exists no such student. Moreover, the type  $t_i$  student with the highest priority in  $\mu^{t_i}(c^{(1)})$  must have been matched with a new school in  $\mu'$  (either  $c$  or another one) that she prefers over  $c$  since  $\mu'$  is fair for same types. Therefore, we get that  $|\mu^{t_i}(c^{(1)})| > \underline{q}_{c^{(1)}}^{t_i}$ ; and that  $|\mu'(c^{(1)}) \setminus \mu(c^{(1)})| > 0$  since  $|\mu(c^{(1)})| = |\mu'(c^{(1)})|$ . Consider a type  $t_j$  such that there exists a student  $s'$  of type  $t_j$  such that  $s' \in \mu'(c^{(1)}) \setminus \mu(c^{(1)})$ . Hence, there exists a node  $c^{(2)}(t_j)$  pointing to  $c^{(1)}(t_i)$  since the number of type  $t_i$  students exceed their floor. We continue in this fashion constructing a path in the associated graph for  $\mu$ . Since there exists a finite number of nodes we see that this path must be a cycle. This completes the argument.<sup>26</sup> ■

Now we establish the result using above lemmas. Let  $\mu$  be the initial assignment produced by CDAAI after Stage 1. Suppose that  $\mu$  is not Pareto efficient among assignments which are fair for same types. Then there exists a cycle in the graph associated with  $\mu$  (Lemma 5); and one application of the improvement scheme (here the improvement scheme refers to steps 1-4 of the improvement stage without repeatedly applying it) produces an assignment, say  $\mu^1$ , that is fair for same types (Lemma 2). If  $\mu^1$  is not Pareto efficient among assignments that are fair for same types, then we can reapply the improvement scheme. Let  $\mu^k$  be the assignment produced after the  $k$ -th application of the improvement scheme, which must be fair for same types (Lemma 2). We continue applying the scheme until we get a Pareto efficient assignment among assignments that are fair for same types. This happens in a finite time, since each application of the scheme is a Pareto improvement. Therefore, there exists  $k$  such that  $\mu^k$  is an assignment that is fair for same types and also Pareto efficient among such assignments. Then,  $\mu^k$  is also constrained non-wasteful (Lemma 3). In addition,  $\mu^k$  is also Pareto efficient among assignments that are fair for same types and constrained non-wasteful (Lemma 4). ■

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<sup>26</sup>The proof of Case 1 of Lemma 5 can be used to show that the output of CDAAI, say  $\mu$ , is weakly Pareto optimal: if not, there exists a feasible assignment  $\mu'$  such that  $\mu'(s)P_s\mu(s)$  for all  $s \in S$ . But now  $\mu'(c) \cap \mu(c) = \emptyset$  and as in Case 1 it can be shown that there exists a cycle in the graph associated with  $\mu$  described in the improvement stage (and  $\mu$  cannot be the final output of CDAAI).

### Proof of Theorem 3

The proof for both parts is by means of an example. For part (i) consider the following problem consisting of three schools  $\{c_1, c_2, c_3\}$  and two students  $\{s_1, s_2\}$ . Each school has a capacity of two ( $q_c = 2$  for all schools  $c$ ). The type space consists of a single type  $t$ , i.e., both students are of the same type  $t$ . The ceiling for type  $t$  is equal to two for each school ( $\bar{q}_c^t = 2$  for all schools  $c$ ). School  $c_1$  has a floor for type  $t$  of  $\underline{q}_{c_1}^t = 1$  and both other schools have a floor of 0 for type  $t$ . Schools  $c_1$  and  $c_2$  give higher priority to student  $s_2$  whereas school  $c_3$  gives higher priority student  $s_1$ . The students' preferences are given by  $c_2 P_{s_1} c_1 P_{s_1} c_3 P_{s_1} s_1$  and  $c_3 P_{s_2} c_1 P_{s_2} c_2 P_{s_2} s_2$ . This information is summarized in Table 4.

TABLE 4.

	$\succ_{c_1}$	$\succ_{c_2}$	$\succ_{c_3}$	$P_{s_1}$	$P_{s_2}$
	$s_2$	$s_2$	$s_1$	$c_2$	$c_3$
	$s_1$	$s_1$	$s_2$	$c_1$	$c_1$
				$c_3$	$c_2$
				$s_1$	$s_2$
capacities	$q_{c_1} = 2$	$q_{c_2} = 2$	$q_{c_3} = 2$		
ceiling for $t$	$\bar{q}_{c_1}^t = 2$	$\bar{q}_{c_2}^t = 2$	$\bar{q}_{c_3}^t = 2$		
floor for $t$	$\underline{q}_{c_1}^t = 1$	$\underline{q}_{c_2}^t = 0$	$\underline{q}_{c_3}^t = 0$		

Next we determine the set of feasible assignments. Feasibility requires that one of the students is assigned school  $c_1$  and each student is assigned a school. Then it is straightforward to verify that

$$\mu_1 = \begin{pmatrix} c_1 & c_2 & c_3 \\ s_1 & \emptyset & s_2 \end{pmatrix}, \mu_2 = \begin{pmatrix} c_1 & c_2 & c_3 \\ s_1 & s_2 & \emptyset \end{pmatrix}$$

$$\mu_3 = \begin{pmatrix} c_1 & c_2 & c_3 \\ s_2 & \emptyset & s_1 \end{pmatrix}, \mu_4 = \begin{pmatrix} c_1 & c_2 & c_3 \\ s_2 & s_1 & \emptyset \end{pmatrix}, \mu_5 = \begin{pmatrix} c_1 & c_2 & c_3 \\ \{s_1, s_2\} & \emptyset & \emptyset \end{pmatrix}$$

is the set of all feasible assignments.

It is easy to check that  $\mu_1$  and  $\mu_4$  are the only feasible assignments which are both fair for same types and constrained non-wasteful for this controlled school choice problem. Note that under  $P_S$ ,

- (i)  $\mu_2$  and  $\mu_5$  are not constrained non-wasteful since  $s_2$  justifiably claims an empty slot at  $c_3$  under both  $\mu_2$  and  $\mu_5$  and the resulting assignment  $\mu_1$  is fair for same types, and
- (ii)  $\mu_3$  is not constrained non-wasteful since  $s_1$  justifiably claims an empty slot at  $c_2$  under  $\mu_3$  and the resulting assignment  $\mu_4$  is fair for same types.

Any feasible mechanism which is both fair for same types and constrained non-wasteful must select either the assignment  $\mu_1$  or the assignment  $\mu_4$ . We will show that in each case there is a student who profitably manipulates the mechanism.

*Case 1:* The mechanism selects  $\mu_1$ .

Under  $\mu_1$  student  $s_1$  is assigned school  $c_1$ . We will show that student  $s_1$  gains by misreporting his true preference. Suppose that student  $s_1$  states the (false) preference  $P'_{s_1}$  given by  $c_2 P'_{s_1} c_3 P'_{s_1} c_1 P'_{s_1} s_1$ , and student  $s_2$  were to report his true preference  $P_{s_2}$ . Keeping all other components of the above problem fixed, in the new problem the students' preferences are  $P'_S = (P'_{s_1}, P_{s_2})$ .

In the new problem under  $\mu_1$  student  $s_1$  justifiably envies student  $s_2$  at school  $c_3$  since (f1)  $\mu_1(s_1) = c_1$ ,  $c_3 P'_{s_1} c_1$  and  $s_1 \succ_{c_3} s_2$ , and (f2)  $\tau(s_1) = \tau(s_2)$ . Note that under  $P'_S$ ,

- (i)  $\mu_1$  and  $\mu_2$  are not fair for same types, and
- (ii)  $\mu_3$  and  $\mu_5$  are not constrained non-wasteful since  $s_1$  justifiably claims an empty slot at  $c_2$  under both  $\mu_3$  and  $\mu_5$  and the resulting assignment  $\mu_4$  is fair for same types.

Thus, the unique feasible assignment, which is both fair for same types and non-wasteful for the new problem, is  $\mu_4$ . Hence, any feasible mechanism, which is both fair for same types and constrained non-wasteful, must select the assignment  $\mu_4$  for the new problem. Under  $\mu_4$  student  $s_1$  is assigned school  $c_2$  which is strictly preferred to  $c_1$  under the true preference  $P_{s_1}$ . Thus student  $s_1$  does better by stating  $P'_{s_1}$  than by stating his true preference  $P_{s_1}$ , and the mechanism is not incentive compatible.

*Case 2:* The mechanism selects  $\mu_4$ .

Under  $\mu_4$  student  $s_2$  is assigned school  $c_1$ . Similarly as in Case 1 we will show that student  $s_2$  gains by misreporting his preference. Suppose that student  $s_2$  states the (false) preference  $P'_{s_2}$  given by  $c_3 P'_{s_2} c_2 P'_{s_2} c_1 P'_{s_2} s_2$ , and student  $s_1$  were to report his true preference  $P_{s_1}$ . Keeping all other components of the above problem fixed, in the new problem the students' preferences are  $P'_S = (P_{s_1}, P'_{s_2})$ .

In the new problem under  $\mu_4$  student  $s_2$  justifiably envies student  $s_1$  at school  $c_2$  since (f1)  $\mu_4(s_2) = c_1$ ,  $c_2 P'_{s_2} c_1$  and  $s_2 \succ_{c_2} s_1$ , and (f2)  $\tau(s_2) = \tau(s_1)$ . Note that under  $P'_S$ ,

- (i)  $\mu_4$  is not fair for same types,
- (ii)  $\mu_2$  and  $\mu_5$  are not constrained non-wasteful since  $s_2$  justifiably claims an empty slot at  $c_3$  under both  $\mu_2$  and  $\mu_5$  and the resulting assignment  $\mu_1$  is fair for same types, and

(iii)  $\mu_3$  is not constrained non-wasteful since  $s_1$  justifiably claims an empty slot at  $c_1$  under  $\mu_3$  and the resulting assignment  $\mu_5$  is fair for same types.

The unique feasible assignment, which is both fair for same types and constrained non-wasteful for the new problem, is  $\mu_1$ . Hence, any feasible mechanism, which is both fair for same types and constrained non-wasteful, must select the assignment  $\mu_1$  for the new problem. Under  $\mu_1$  student  $s_2$  is assigned school  $c_3$  which is strictly preferred to  $c_1$  under the true preference  $P_{s_2}$ . Thus student  $s_2$  does better by stating  $P'_{s_2}$  than by stating his true preference  $P_{s_2}$ , and the mechanism is not incentive compatible.<sup>27</sup>

For part (ii) we use the example in the proof of Theorem 1 part (ii). Recall the problem consisting of three schools  $\{c_1, c_2, c_3\}$  and three students  $\{s_1, s_2, s_3\}$ . Each school has a capacity of one ( $q_c = 1$  for all schools  $c$ ). The type space consists of two types  $t_1$  and  $t_2$ . Students  $s_1$  and  $s_2$  are of type  $t_1$  whereas student  $s_3$  is of type  $t_2$ . For all types the ceiling is equal to one at all schools ( $\bar{q}_c^t = 1$  for all types  $t$  and all schools  $c$ ). School  $c_1$  has a floor for type  $t_1$  of  $\underline{q}_{c_1}^{t_1} = 1$ . All other floors are equal to zero. The schools' priorities are given by  $s_2 \succ_{c_1} s_1 \succ_{c_1} s_3$ ,  $s_2 \succ_{c_2} s_1 \succ_{c_2} s_3$  and  $s_1 \succ_{c_3} s_2 \succ_{c_3} s_3$ . The students' preferences are given by  $c_2 P_{s_1} c_1 P_{s_1} c_3 P_{s_1} s_1$ ,  $c_3 P_{s_2} c_1 P_{s_2} c_2 P_{s_2} s_2$  and  $c_2 P_{s_3} c_3 P_{s_3} c_1 P_{s_3} s_3$ . This information is summarized in Table 5.

TABLE 5.

	$\succ_{c_1}$	$\succ_{c_2}$	$\succ_{c_3}$	$P_{s_1}$	$P_{s_2}$	$P_{s_3}$
$s_2$	$s_2$	$s_2$	$s_1$	$c_2$	$c_3$	$c_2$
$s_1$	$s_1$	$s_1$	$s_2$	$c_1$	$c_1$	$c_3$
$s_3$	$s_3$	$s_3$	$s_3$	$c_3$	$c_2$	$c_1$
				$s_1$	$s_2$	$s_3$
capacities	$q_{c_1} = 1$	$q_{c_2} = 1$	$q_{c_3} = 1$			
ceiling for $t_1$	$\bar{q}_{c_1}^{t_1} = 1$	$\bar{q}_{c_2}^{t_1} = 1$	$\bar{q}_{c_3}^{t_1} = 1$			
floor for $t_1$	$\underline{q}_{c_1}^{t_1} = 1$	$\underline{q}_{c_2}^{t_1} = 0$	$\underline{q}_{c_3}^{t_1} = 0$			
ceiling for $t_2$	$\bar{q}_{c_1}^{t_2} = 1$	$\bar{q}_{c_2}^{t_2} = 1$	$\bar{q}_{c_3}^{t_2} = 1$			
floor for $t_2$	$\underline{q}_{c_1}^{t_2} = 0$	$\underline{q}_{c_2}^{t_2} = 0$	$\underline{q}_{c_3}^{t_2} = 0$			

Next we determine the set of assignments which are both feasible and fair across types for this problem. Feasibility requires that student  $s_1$  or student  $s_2$  is assigned school  $c_1$  and all students are enrolled at a school. If student  $s_1$  is assigned school  $c_1$ , then  $s_2$  needs to be

<sup>27</sup>Using the same example and the same proof, one can easily check that for any feasible mechanism which is fair for same types and constrained non-wasteful, no feasible and incentive compatible mechanism Pareto dominates this mechanism. The latter means that for any problem the mechanism chooses an assignment which is weakly preferred by all students to the assignment chosen by the first mechanism.

assigned school  $c_3$  since otherwise  $s_3$  is assigned school  $c_3$ ,  $s_2$  school  $c_2$ , and  $s_2$  justifiably envies  $s_3$  at  $c_3$ . Similarly, if student  $s_2$  is assigned school  $c_1$ , then  $s_1$  needs to be assigned school  $c_2$  since otherwise  $s_3$  is assigned school  $c_2$ ,  $s_1$  school  $c_3$ , and  $s_1$  justifiably envies  $s_3$  at  $c_2$ . Now it is straightforward to verify that

$$\mu = \begin{pmatrix} c_1 & c_2 & c_3 \\ s_1 & s_3 & s_2 \end{pmatrix} \text{ and } \bar{\mu} = \begin{pmatrix} c_1 & c_2 & c_3 \\ s_2 & s_1 & s_3 \end{pmatrix}$$

are the only assignments which are both feasible and fair across types for this problem.

Any mechanism which is both feasible and fair across types must select either the assignment  $\mu$  or the assignment  $\bar{\mu}$ . We will show that in each case there is a student who profitably manipulates the mechanism.

*Case 1:* The mechanism selects  $\mu$ .

Under  $\mu$  student  $s_1$  is assigned school  $c_1$ . We will show that student  $s_1$  gains by misreporting his true preference. Suppose that student  $s_1$  states the (false) preference  $P'_{s_1}$  given by  $c_2 P'_{s_1} c_3 P'_{s_1} c_1 P'_{s_1} s_1$ , and all other students were to state their true preferences. Keeping all other components of the above problem fixed, in the new problem the students' preferences are  $P'_S = (P'_{s_1}, P_{s_2}, P_{s_3})$ .

In the new problem under  $\mu$  student  $s_1$  justifiably envies student  $s_2$  at school  $c_3$  through the feasible assignment

$$\mu' = \begin{pmatrix} c_1 & c_2 & c_3 \\ s_2 & s_3 & s_1 \end{pmatrix}$$

since  $\mu(s_1) = c_1$ ,  $c_3 P'_{s_1} c_1$  and  $s_1 \succ_{c_3} s_2$ . Now it is straightforward to verify that the unique feasible and fair across types assignment of the new problem is  $\bar{\mu}$ . Thus any mechanism, which is both feasible and fair across types, must select the assignment  $\bar{\mu}$  for the new problem. Under  $\bar{\mu}$  student  $s_1$  is assigned school  $c_2$  which is strictly preferred to  $c_1$  under the true preference  $P_{s_1}$ . Thus student  $s_1$  is better off by stating  $P'_{s_1}$  than by stating his true preference  $P_{s_1}$ , and the mechanism is not incentive compatible.

*Case 2:* The mechanism selects  $\bar{\mu}$ .

Under  $\bar{\mu}$  student  $s_2$  is assigned school  $c_1$ . Similarly as in Case 1 we will show that student  $s_2$  gains by misreporting his preference. Suppose that student  $s_2$  states the (false) preference  $P'_{s_2}$  given by  $c_3 P'_{s_2} c_2 P'_{s_2} c_1 P'_{s_2} s_2$ , and all other students were to state their true preferences. Keeping all other components of the above problem fixed, in the new problem the students' preferences are  $P'_S = (P_{s_1}, P'_{s_2}, P_{s_3})$ .

In the new problem under  $\bar{\mu}$  student  $s_2$  justifiably envies student  $s_1$  at school  $c_2$  through the feasible assignment

$$\bar{\mu}' = \begin{pmatrix} c_1 & c_2 & c_3 \\ s_1 & s_2 & s_3 \end{pmatrix}$$

since  $\bar{\mu}(s_2) = c_1$ ,  $c_2 P'_{s_2} c_1$  and  $s_2 \succ_{c_2} s_1$ . Now it is straightforward to verify that  $\mu$  is the unique feasible assignment which is fair across types for the new problem. Thus any mechanism, which is both feasible and fair across types, must select the assignment  $\mu$  for the new problem. Under  $\mu$  student  $s_2$  is assigned school  $c_3$  which is strictly preferred to  $c_1$  under the true preference  $P_{s_2}$ . Thus student  $s_2$  does better by stating  $P'_{s_2}$  than by stating his true preference  $P_{s_2}$ , and the mechanism is not incentive compatible. ■

## Proof of Theorem 4

This follows from Theorem 6.8 in Roth and Sotomayor (1990), since choice functions are substitutable.  $Ch_c$  satisfies **substitutability** if for any group of students  $\tilde{S}$  that contains students  $s$  and  $s'$  ( $s \neq s'$ ),  $s \in Ch_c(\tilde{S})$  implies  $s \in Ch_c(\tilde{S} \setminus \{s'\})$ . To show substitutability, note that if  $s \in Ch_c^{(1)}(\tilde{S})$ , then  $s \in Ch_c^{(1)}(\tilde{S} \setminus \{s'\})$ . Otherwise, if  $s \in Ch_c^{(i)}(\tilde{S})$  for  $i = 2$  or  $i = 3$ , then either  $s' \notin \bigcup_{j=1}^i Ch_c^{(j)}(\tilde{S})$  and  $Ch_c^{(i)}(\tilde{S}) = Ch_c^{(i)}(\tilde{S} \setminus \{s'\})$  or  $s' \in \bigcup_{j=1}^i Ch_c^{(j)}(\tilde{S})$  and  $(\bigcup_{j=1}^i Ch_c^{(j)}(\tilde{S})) \setminus \{s'\} \subseteq \bigcup_{j=1}^i Ch_c^{(j)}(\tilde{S} \setminus \{s'\})$ . Therefore,  $Ch_c$  is substitutable for every  $c$ , and the claim holds. ■

## Proof of Theorem 5

The proof is an application of either Martínez et al. (2004) or Hatfield and Kojima (2009). Here, we provide the argument following the latter.

Hatfield and Kojima (2009) show, in a many-to-one matching model with contracts, that if the choice functions of schools satisfy substitutability and the law of aggregate demand, then the student proposing deferred acceptance algorithm is group incentive compatible. School  $c$ 's preferences satisfy **law of aggregate demand** if for any  $S'' \subseteq S' \subseteq S$ , we have  $|Ch_c(S'')| \leq |Ch_c(S')|$ .

The law of aggregate demand is satisfied in our setup since if  $|Ch_c(S'')| = q_c$  then  $|Ch_c(S')| = q_c$ . Moreover, if  $|Ch_c(S'')| < q_c$  then either  $|Ch_c(S')| = |Ch_c(S'')|$  if  $S'' = S'$  or  $|Ch_c(S')| < |Ch_c(S'')|$  if  $|S''| < |S'|$ .

Our setup can be trivially embedded in the many-to-one matching model with contracts of Hatfield and Kojima (2009), so the conclusion follows. ■

## Proof of Theorem 6

Let  $\mu$  be a feasible assignment that is strongly-fair across types and non-wasteful. Since  $\mu$  is a feasible assignment, for every school  $c$  and student type  $t$  we have  $\underline{q}_c^t \leq |\mu^t(c)| \leq \bar{q}_c^t$ . Together with strong-fairness across types, this implies that  $\mu$  is fair under soft bounds. If  $\mu$  is also non-wasteful under soft bounds, the conclusion follows from Theorem 4. Suppose otherwise that  $\mu$  violates non-wastefulness under soft bounds. This means that there exist a school  $c$  and a student  $s$  such that  $cP_s\mu(s)$  and  $|\mu(c)| < q_c$ . Whenever there exists such a pair we apply the following algorithm to improve students' matches. Note that this algorithm is equivalent to the school-proposing deferred acceptance algorithm if  $\mu$  is the assignment in which no agent is matched.<sup>28</sup>

**Step 1** For school  $c$  defined above, find  $S^1 \equiv \{s \in S : cP_s\mu(s)\}$ . Among the students in  $S^1$  first match the highest ranked students according to  $\succ_c$  until the ceilings are filled. Then match the best students according to  $\succ_c$  up to the capacity or until  $S^1$  is exhausted. Define  $\mu_1$  to be the new assignment.

**Step k** If there is no school with an empty seat that a student prefers to her match in  $\mu_{k-1}$ , then stop. Otherwise consider one such school, say  $c_k$ . Let  $S^k \equiv \{s \in S : c_kP_s\mu_{k-1}(s)\}$ . Among the students in  $S^k$  first match the highest ranked students according to  $\succ_{c_k}$  until the floors are filled. Then match the highest ranked students according to  $\succ_{c_k}$  until the ceilings are filled. Finally, match the best students if there are more students and seats available. Define  $\mu_k$  to be the new assignment.

This algorithm ends in finite time since it improves the match of at least one student at every step of the algorithm. Let  $\hat{\mu}$  denote the assignment produced by this algorithm. It is clear that  $\hat{\mu}$  is non-wasteful under soft bounds. We further claim that  $\hat{\mu}$  removes justifiable envy under soft bounds.

Consider a student  $s$  and school  $c$  such that  $cP_s\hat{\mu}(s)$ . Let  $\tau(s) = t$ . For any student  $s' \in \mu^t(c)$ , either  $s'$  was already matched with  $c$  in strongly-fair across types assignment  $\mu$  which implies  $s' \succ_c s$ , or that  $s'$  got matched with  $c$  in the above algorithm which also implies  $s' \succ_c s$ . Now we split the rest of the analysis depending on whether type  $t$  students fill their floors or ceilings.

**Case 1** ( $|\hat{\mu}^t(c)| \geq \bar{q}_c^t$ ): Consider  $s' \in \hat{\mu}(c)$  such that  $|\hat{\mu}^{\tau(s')}(c)| > \bar{q}_c^{\tau(s')}$ . Since  $\mu$  is feasible it must be that some type  $\tau(s')$  students got matched with  $c$  in the above algorithm. Moreover, such students must have lower priority compared to other type  $\tau(s')$  students who were matched with  $c$  in  $\mu$ . In addition, type  $\tau(s')$  students who get matched with  $c$  in the

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<sup>28</sup>This is similar to the vacancy-chain dynamics studied in Blum, Roth and Rothblum (1997).

above algorithm has a descending priority with respect to the order they were matched. The last type  $\tau(s')$  student who got matched with  $c$  must have a higher priority than  $s$  since type  $\tau(s')$  has already filled their ceilings and student  $s$  is not admitted to  $c$  in this step even though she wants to switch to  $c$ . This implies that every student of type  $\tau(s')$  is preferred to  $s$ .

**Case 2** ( $\bar{q}_c^t > |\hat{\mu}^t(c)| \geq \underline{q}_c^t$ ): In this case, for any  $t' \in T \setminus \{t\}$  we cannot have  $|\hat{\mu}^{t'}(c)| > \bar{q}_c^{t'}$ : At least one student of type  $t'$  must have been matched with  $c$  during the above algorithm since  $\mu$  is feasible. Consider the last student of type  $t'$  who got matched with  $c$ . At the stage when this student got matched with  $c$ , since  $s$  is not matched with  $c$ , it must be that type  $t$  students have filled their ceilings. Later on some type  $t$  students in  $c$  must have matched with other schools, so that type  $t$  students do not fill their ceilings in school  $c$  at the end of the algorithm. After the step when type  $t$  students do not fill their floors anymore, type  $t$  students can be admitted without violating school  $c$ 's quota. Since  $s$  is not matched with  $c$ , and that type  $t$  students do not fill their ceilings at the end of the algorithm, we get a contradiction. Therefore,  $|\hat{\mu}^{t'}(c)| \leq \bar{q}_c^{t'}$ .

To complete the argument for Case 2, consider type  $t'$  such that  $\bar{q}_c^{t'} \geq |\hat{\mu}^{t'}(c)| > \underline{q}_c^{t'}$ . Let  $s'$  be the student in  $\hat{\mu}^{t'}(c)$  with the least priority among type  $t'$  students. If  $\mu(s') = c$  and  $|\mu^{t'}(c)| > \underline{q}_c^{t'}$  then  $s' \succ_c s$  since  $\mu$  is fair. If  $\mu(s') = c$  and  $|\mu^{t'}(c)| = \underline{q}_c^{t'}$  then at least one type  $t'$  student must be matched with  $c$  during the above algorithm. But this gives a contradiction since that student prefers  $c$  to her match in  $\mu$  and she has a higher priority than  $s'$ . Finally, if  $\mu(s') \neq c$ , then  $s'$  has been matched with  $c$  during the above algorithm. If at the stage when  $s'$  is admitted, type  $t$  students do not fill their ceilings then  $s' \succ_c s$ . Otherwise, if type  $t$  students fill their ceiling at that stage, then some of these students must have matched with other schools later in the algorithm. Since  $s$  is not matched with  $c$ , and that type  $t$  students do not fill their ceilings at the end of the algorithm, we get a contradiction. Therefore, in all of the possibilities we conclude  $s' \succ_c s$ .

Thus,  $\hat{\mu}$  removes justifiable envy under soft bounds. Hence,  $\hat{\mu}$  is both fair under soft bounds and non-wasteful under soft bounds. Since under DAASB all students are matched to the best outcome among such assignments, the conclusion follows. ■

## Appendix B: Feasibility Checking Algorithm

Below we provide an algorithm to check in (i) of any step in Stage 1 of CDAAI whether the proposing student can be assigned an empty slot at the school she proposes to. Note that for finding a feasible assignment only student types (not student names) matter.

We want to check whether there exists a feasible assignment such that the number of

type  $t$  students in school  $c$  is  $x_c^t$ . Let  $|S_t| = y^t$ . Given a type allocation vector  $y = (y^t)_{t \in T}$ , a quota vector  $q = (q_c)_{c \in C}$ , and floor and ceiling matrices  $\underline{q} = \{\underline{q}_c^t\}_{t \in T, c \in C}$ , and  $\bar{q} = \{\bar{q}_c^t\}_{t \in T, c \in C}$ , a type assignment matrix  $\{x_c^t\}_{t \in T, c \in C}$  is feasible if,

- (i) for all  $t \in T$ , we have  $\sum_{c \in C} x_c^t = y^t$ ,
- (ii) for all  $c \in C$ , we have  $\sum_{t \in T} x_c^t \leq q_c$ , and
- (iii) for all  $t \in T$  and  $c \in C$ , we have,  $\underline{q}_c^t \leq x_c^t \leq \bar{q}_c^t$ .

First, the floors can be reduced to zero by defining a new variable  $\hat{x}_c^t \equiv x_c^t - \underline{q}_c^t$ . The rest of the constraints then can be written in terms of  $\hat{x}_c^t$ . The reduced set of constraints corresponds to the so-called *transportation problem*, which is well-known in the operations research literature. It is a network flow on a bipartite graph, and the linear programming relaxation (allowing  $x$  to be non-integer) provides a feasible *integer* solution in polynomial time (note that  $y^t$ ,  $q_c$ ,  $\underline{q}_c^t$ , and  $\bar{q}_c^t$  are integers). Note that we can also use the *transportation problem* to check whether in any controlled school choice problem the set of feasible assignments is non-empty or not.<sup>29</sup>

Hence, we can do the feasibility check in the first stage of CDAAI in polynomial time for any preassignment  $\nu$  that specifies the assignment of a subset of students  $\tilde{S}$  as follows ( $s \in \tilde{S} \iff \nu(s) \neq s$ ). We consider the rest of the students  $S \setminus \tilde{S}$  and update the feasibility constraints as follows:  $y'(t) \equiv y(t) - |\tilde{S}_t|$ ,  $q'_c \equiv q_c - |\{s \in \tilde{S} : \nu(s) = c\}|$ , and  $\bar{q}'_c \equiv \bar{q}_c^t - |\{s \in \tilde{S}_t : \nu(s) = c\}|$ . In the reduced market, all students in  $\tilde{S}$  are assumed to be matched with schools according to  $\nu$ , and the constraints are updated accordingly.

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<sup>29</sup>For more information, see Nemhauser and Wolsey (1999) or Reed and Leavengood (2002). We thank Fatma Kilinc-Karzan, Ersin Korpeoglu, R. Ravi, and Willem-Jan van Hove for discussions.

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