A Proof Checker for Teaching Classical Logic

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Abstract

In this paper, we consider a proof checker for classical logic based on earlier work by Andreas Abel, Bor-Yuh Evan Chang, and Frank Pfenning on the Tutch proof checking system for intuitionistic propositional logic. A formal system for this classical logic proof checker is given and soundness and completeness theorems are proven. The work is motivated by the desire to extend the domain of the Tutch proof checking system to other logics and by the desire to create a pedagogic tool for students learning classical logic.

1 Introduction

In the Fall 2000 semester, the Tutch proof checker was introduced by Andreas Abel, Bor-Yuh Evan Chang, and Frank Pfenning, as a tool for teaching intuitionistic propositional logic in their Constructive Logic course at Carnegie Mellon University. Other proof tutor systems, such as ETPS [3], CMU Proof Tutor [9], ProveEasy [4], and alfie [10], were available at that time. However, these systems usually required the user to construct a proof script containing tutor-specific commands for guiding the tutor to a correct proof. As a result there was an unintentional bias towards writing proof *scripts* in a specific tutor system rather than writing formal proofs. For this reason, Abel, Chang, and Pfenning chose to design Tutch, instead of using an existing proof tutor system as a pedagogic tool. [2]

Abel, Chang, and Pfenning reported significant positive results from a midterm student evaluation of Tutch—on a scale of 1 to 5, 15 of 26 students rated the helpfulness of Tutch as a 5, with an average of 4.28. Abel, Chang, and Pfenning credited the success of Tutch to its focus on writing proofs rather than proof scripts. Thus, Tutch proved to be a valuable pedagogic tool. [2]

The success of Tutch for intuitionistic logic suggests that extensions or complements to the original system would prove valuable. Because certain judgments are not provable in intuitionistic logic, such as the Law of Excluded Middle and Double Negation Elimination, we are motivated to consider a version of the Tutch proof checker for classical logic. We will refer to this version as Classical Tutch.

An additional, and more important, motivation for Classical Tutch is pedagogic. Anecdotally, students in the Spring 2006 offering of the Constructive Logic course at Carnegie Mellon University found the unit on classical logic to be one of the most difficult and expressed an interest in using a version of Tutch for this logic.

We acknowledge that extending Tutch to classical logic is not an original idea. Abel, Chang, and Pfenning included a classical mode with their original implementation of Tutch for intuitionistic logic [1]. However, this mode is not well-suited for use in the Constructive Logic course at Carnegie Mellon University because it deviates significantly from the classical logic system presented in the course (see Section 3). Thus, we are prompted to create Classical Tutch so that it adheres to the system that will be used in the course.

Classical Tutch is therefore our response to all of these motivations and is intended to complement the existing Tutch system.

The remainder of this paper is organized as follows. Section 2 gives a brief introduction to the syntax and terminology used in Tutch. Section 3 presents the classical logic on which Classical Tutch will be based. Section 4 describes a formal system for Classical Tutch. This system is then shown to be sound and complete in Sections 5 and 6, respectively. Following the proofs of soundness and completeness, we give an overview of related and future work. The Appendix contains additional details of the proofs that were omitted from the main narrative.

```
proof contra : (A => B) => ~B => ~A =
begin
[ A => B;
  [ ~B;
    [ A;
      B;
      F ];
    ~A ];
  ~B => ~A ];
(A => B) => ~B => ~A
end;
```

Figure 1: A Tutch proof of $(A \supset B) \supset \neg B \supset \neg A$ true.

```
proof dne : ~~A => A true =
begin
[ ~~A true;
  [ A false;
    ~A true;
    ~~A false;
    # ];
  A true ];
~~A => A true
end;
```

Figure 2: A Classical Tutch proof of $\neg\neg A \supset A$ true, Double Negation Elimination.

2 A Brief Introduction to Tutch

Before continuing, it is useful to give a brief introduction to Tutch. A Tutch proof consists of a name, a goal, and a proof. In the case of Figure 1, the name is `contra`, the goal is `(A => B) => ~B => ~A`, and the proof is the text between the `begin` and `end` keywords.

A Tutch proof consists of a series of steps, each of which is an assertion or a frame. An assertion is simply a proposition that stands for the truth judgment for that proposition. In this way, explicit use of truth judgment labels is avoided. A frame consists of a hypothesis and a proof surrounded by brackets. The first proposition following the left bracket is the hypothesis. This hypothesis can only be used in the proof that immediately follows and cannot be used after the right bracket. So, the brackets delimit the scope of the hypothesis.

To clarify these ideas, we will now walk through the proof in Figure 1. Line 3 begins a frame with hypothesis `A => B`, which stands for the judgment $A \supset B$ true. Lines 4 and 5 begin two more frames with hypotheses `~B` and `A`, respectively. Line 6 asserts `B`, which stands for the judgment B true. This line is justified by implication elimination with the hypotheses from lines 3 and 5. Line 7 asserts `F`, which is justified by implication elimination with lines 4 and 6. Note that line 7 closes the innermost frame, yielding the hypothetical judgment A true $\vdash \bot$ true. Line 8 asserts `~A`, which is justified by implication introduction with the frame closed on line 7. Line 9 asserts `~B => ~A`, which is justified by implication introduction with the frame closed on line 8. Line 10 asserts `(A => B) => ~B => ~A`, which is justified by implication introduction with the frame closed on line 9. The assertion on line 10 matches the goal and so the proof is complete.

To allow for a smooth transition by users from Tutch to Classical Tutch, we retain the concept of a frame and make only minor, but necessary, syntactic changes. These changes are the addition of truth and falsity judgment labels to each proof step and goal, and the introduction of `#` as the syntax for the contradiction judgment.

$$
\begin{array}{rcl}
A, B & ::= & I \\
& | & \top \\
& | & \bot \\
& | & \neg A \\
& | & A \wedge B \\
& | & A \vee B \\
& | & A \supset B \\
& | & A \setminus B
\end{array}
$$

Figure 3: Propositions in classical logic.

$$
\begin{array}{rcl}
H & ::= & A \text{ true} \\
& | & A \text{ false}
\end{array}
$$

$$
\begin{array}{rcl}
J & ::= & H \\
& | & \#
\end{array}
$$

Figure 4: Truth, falsity, and contradiction judgments in classical logic.

Figure 2 gives an example proof of Double Negation Elimination in Classical Tutch. The slight syntactic changes and strong similarity to Tutch are evident here.

3 Classical Logic Formal System

We now give the classical logic system that is presented in the Constructive Logic course at Carnegie Mellon University. We begin by describing the syntax, followed by a description of the judgments used and their corresponding inference rules.

3.1 Syntax

Figure 3 describes well-formed propositions in classical logic. I represents identifiers. A proposition may be an atomic proposition, truth, falsehood, the negation of a proposition, or the conjuction, disjunction, implication, or the difference of two propositions. Note that we write the difference of B and A as $B \setminus A$.

Figure 4 defines well-formed judgments in classical logic. The truth judgment A true states that the proposition A is true. The falsity judgment A false states that the proposition A is false. An H, then, is a truth or falsity judgment. The contradiction judgment # states that there exists an inconsistency in the reasoning. A J, then, is a truth, falsity, or contradiction judgment.

Figure 5 describes classical logic contexts as a list of Hs. This justifies the distinction between Hs and Js; we do not allow contradiction judgments as entries in contexts.

$$
\begin{array}{rcl}
\Delta & ::= & \cdot \\
& | & \Delta, H
\end{array}
$$

Figure 5: Classical logic contexts.

Judgment	Name	Explanation
$H \in \Delta$	Context Membership	H is a member of Δ
$\Delta \vdash J$	Entailment	Context Δ entails J

Figure 6: A summary of the two judgments for classical logic.

$$\frac{}{H \in \Delta, H}\ \text{mem}F \qquad \frac{H \in \Delta}{H \in \Delta, H'}\ \text{mem}S$$

Figure 7: The inference rules for the context membership judgment.

3.2 Judgments

Figure 6 gives a summary of the two judgments used in defining classical logic. The first of these is the context membership judgment, $H \in \Delta$, which states that H is an entry in the context Δ. The second of these judgments is the entailment judgment, $\Delta \vdash J$, which states that J can be deduced from some (or none) of the entries in the context Δ. The context membership judgment is used to define the entailment judgment as will be seen in Figure 8.

Figure 7 gives the inference rules for the context membership judgment. H is by definition an entry in the context Δ, H (mem$F$ rule[1]). When H is an entry in Δ, H is an entry in Δ, H' (mem$S$ rule[2]).

Figure 8 gives the inference rules for the entailment judgment. This presentation of classical logic adheres to the one given in the Spring 2006 Constructive Logic course at Carnegie Mellon University and also follows that of [6]. At the risk of being redundant, we now restate the inference rules without the notation, for the purpose of providing intuition.

A proposition is true under a context if the truth judgment for that proposition is an entry in the given context (hyp$T$). The hyp$F$ rule is analogous. Note that the context membership judgment is used to define these two rules.

Truth is true under any context ($\top T$). Analogously, falsehood is false under any context ($\bot F$).

The negation of a proposition is true under a context if the proposition is false under the same context ($\neg T$). The $\neg F$ rule is symmetric.

The conjunction of two propositions is true under a context if both propositions are true under that context ($\wedge T$). The conjunction of two propositions is false under a context if either of the propositions is false under that context ($\wedge F_L$ and $\wedge F_R$).

The disjunction of two propositions is true under a context if either of the propositions is true under that context ($\vee T_L$ and $\vee T_R$). The disjunction of two proposition is false under a context if both propositions are false under that context ($\vee F$). The reader should note the symmetry between the rules for conjunction and disjunction.

The implication of two propositions is true under a context if the right-hand proposition is true under that context extended with the truth judgment for the left-hand proposition ($\supset T$). The implication of two proposition is false under a context if the left-hand proposition is true under that context and the right-hand proposition is false under that context ($\supset F$).

The difference of two propositions is true under a context if the right-hand proposition is false under that context and the left-hand proposition is true under that context ($\setminus T$). The difference of two propositions is false under a context if the left-hand proposition is false under that context extended with the falsity judgment for the right-hand proposition ($\setminus F$). The reader should note the symmetry between the rules for implication and difference.

A contradiction exists under a context if there exists a proposition that is both true and false under that context ($\#I$). A proposition is true under a context if there exists a contradiction under that context extended with the falsity judgment for that proposition ($\#ET$). A proposition is false under a context if there exists a contradiction under that context extended with the truth judgment for that proposition ($\#EF$).

[1] The F in mem$F$ stands for "found," because the entry has been found in the context.
[2] The S in mem$S$ stands for "search," because the entry is searched for in the context.

$$\frac{A \text{ true} \in \Delta}{\Delta \vdash A \text{ true}}\ \text{hyp}T \qquad \frac{A \text{ false} \in \Delta}{\Delta \vdash A \text{ false}}\ \text{hyp}F$$

$$\frac{}{\Delta \vdash \top \text{ true}}\ \top T \qquad \frac{}{\Delta \vdash \bot \text{ false}}\ \bot F$$

$$\frac{\Delta \vdash A \text{ false}}{\Delta \vdash \neg A \text{ true}}\ \neg T \qquad \frac{\Delta \vdash A \text{ true}}{\Delta \vdash \neg A \text{ false}}\ \neg F$$

$$\frac{\Delta \vdash A \text{ true} \quad \Delta \vdash B \text{ true}}{\Delta \vdash A \wedge B \text{ true}}\ \wedge T \qquad \frac{\Delta \vdash A \text{ false}}{\Delta \vdash A \wedge B \text{ false}}\ \wedge F_L \qquad \frac{\Delta \vdash B \text{ false}}{\Delta \vdash A \wedge B \text{ false}}\ \wedge F_R$$

$$\frac{\Delta \vdash A \text{ true}}{\Delta \vdash A \vee B \text{ true}}\ \vee T_L \qquad \frac{\Delta \vdash B \text{ true}}{\Delta \vdash A \vee B \text{ true}}\ \vee T_R \qquad \frac{\Delta \vdash A \text{ false} \quad \Delta \vdash B \text{ false}}{\Delta \vdash A \vee B \text{ false}}\ \vee F$$

$$\frac{\Delta, A \text{ true} \vdash B \text{ true}}{\Delta \vdash A \supset B \text{ true}}\ \supset T \qquad \frac{\Delta \vdash A \text{ true} \quad \Delta \vdash B \text{ false}}{\Delta \vdash A \supset B \text{ false}}\ \supset F$$

$$\frac{\Delta \vdash A \text{ false} \quad \Delta \vdash B \text{ true}}{\Delta \vdash B \setminus A \text{ true}}\ \setminus T \qquad \frac{\Delta, A \text{ false} \vdash B \text{ false}}{\Delta \vdash B \setminus A \text{ false}}\ \setminus F$$

$$\frac{\Delta \vdash A \text{ true} \quad \Delta \vdash A \text{ false}}{\Delta \vdash \#}\ \#I \qquad \frac{\Delta, A \text{ false} \vdash \#}{\Delta \vdash A \text{ true}}\ \#ET \qquad \frac{\Delta, A \text{ true} \vdash \#}{\Delta \vdash A \text{ false}}\ \#EF$$

Figure 8: The inference rules for the entailment judgment.

3.3 Additional Comments

Now that the classical logic that is used in the Constructive Logic course at Carnegie Mellon University has been presented, it is possible to see why the classical mode for Tutch is unsuitable for use in the course. This classical logic system uses truth, falsity, and contradiction judgments, rather than a single truth judgment. Furthermore, the inference rules used by this system are significantly different from those of intuitionistic propositional logic. For these reasons, it is necessary to create a new version of Tutch, Classical Tutch, for this specific system.

4 Classical Tutch Formal System

Now that we have seen the presentation of classical logic on which Classical Tutch is based, we are prepared to describe the Classical Tutch formal system. We begin with a description of the syntax and follow with a description of the judgments used and their corresponding inference rules.

4.1 Syntax

Classical Tutch makes use of the same syntax for propositions and truth, falsity, and contradiction judgments as classical logic (recall Figs. 3 and 4). This is natural because we claim and later prove that Classical Tutch and classical logic are related.

In Figure 9, we give the possible forms of Classical Tutch context entries. An entry, γ, may be an H or a hypothetical judgment, where the hypothesis is an H and the conclusion is a J. This allows the conclusion, but not the hypothesis, to be a contradiction judgment, providing further justification for the distinction between Hs and Js. These hypothetical entries allow for the use of frames in Classical Tutch proofs. We do not allow a contradiction judgment to be a non-hypothetical entry because this is not in keeping with the classical logic presented in Section 3. Classical Tutch contexts, Γ, are then a list of entries.

$$
\begin{array}{rcl}
\gamma & ::= & H \\
& | & H \vdash J
\end{array}
$$

$$
\begin{array}{rcl}
\Gamma & ::= & \cdot \\
& | & \Gamma, \gamma
\end{array}
$$

Figure 9: Classical Tutch context entries and contexts.

$$
\begin{array}{rcl}
P, P' & ::= & \texttt{hence } J \\
& | & \texttt{hence } H; P \\
& | & [H; P]; P'
\end{array}
$$

Figure 10: Classical Tutch proofs.

Recall the description of contexts for classical logic given in Figure 5. We must differentiate between contexts for classical logic, Δ, and those for Classical Tutch, Γ, because Classical Tutch contexts can have hypothetical entries. The reader should be aware that a classical logic context is a Classical Tutch context, but that the converse does not hold.

Figure 10 gives the structure of Classical Tutch proofs. A proof may be either a hence-step that proves some J, a hence-step that proves some H followed by a proof, or a frame followed by a proof. A hence-step may not prove a J if it is followed by a proof; the judgments proved by these steps are included in the context that is used in checking the proof that follows, and Classical Tutch contexts cannot contain J entries. This intuition will become clearer when the inference rule for checking a frame followed by a proof is given.

4.2 Judgments

Figure 11 gives a summary of the three judgments used in formalizing Classical Tutch. The first of these is the context membership judgment, $\gamma \in \Gamma$, which states that γ is a member of Γ. This judgment is used in defining the single step judgment. The second of these is the single step judgment, $\Gamma \to J$, which states that J is justified by applying a single rule of classical logic to some (or none) of the entries in Γ. This judgment is used to define the valid proof judgment. The third of these is the valid proof judgment, $\Gamma \vdash P : J$, which states that P is a valid proof of J, given Γ. The context Γ is a collection of judgments that may be assumed to hold in P. The valid proof judgment is the central judgment for Classical Tutch and is used to define the appropriate notions of soundness and completeness.

Figure 12 gives the inference rules for the context membership judgment. Because each classical logic context is a Classical Tutch context and each H is a Classical Tutch context entry, this new formulation of context membership is simply a generalization of context membership from Figure 7.

Figure 13 presents the inference rules for the single step judgment. They are analogous in number and structure to the inference rules for the entailment judgment of classical logic (see Fig. 8). There is exactly one single step rule for each rule of the entailment judgment and the subderivations for each pair of

Judgment	**Name**	**Explanation**
$\gamma \in \Gamma$	Context Membership	γ is a member of Γ
$\Gamma \to J$	Single Step	J is justified by applying a rule to entries in Γ
$\Gamma \vdash P : J$	Valid Proof	P proves J, given Γ

Figure 11: A summary of the three judgments for Classical Tutch.

$$\frac{}{\gamma \in \Gamma, \gamma}\ \text{mem}F \qquad \frac{\gamma \in \Gamma}{\gamma \in \Gamma, \gamma'}\ \text{mem}S$$

Figure 12: The inference rules for the context membership judgment.

$$\frac{A \text{ true} \in \Gamma}{\Gamma \rightarrow A \text{ true}} \rightarrow\text{hyp}T \qquad \frac{A \text{ false} \in \Gamma}{\Gamma \rightarrow A \text{ false}} \rightarrow\text{hyp}F$$

$$\frac{}{\Gamma \rightarrow \top \text{ true}} \rightarrow\top T \qquad \frac{}{\Gamma \rightarrow \bot \text{ false}} \rightarrow\bot F$$

$$\frac{A \text{ false} \in \Gamma}{\Gamma \rightarrow \neg A \text{ true}} \rightarrow\neg T \qquad \frac{A \text{ true} \in \Gamma}{\Gamma \rightarrow \neg A \text{ false}} \rightarrow\neg F$$

$$\frac{A \text{ true} \in \Gamma \quad B \text{ true} \in \Gamma}{\Gamma \rightarrow A \wedge B \text{ true}} \rightarrow\wedge T \qquad \frac{A \text{ false} \in \Gamma}{\Gamma \rightarrow A \wedge B \text{ false}} \rightarrow\wedge F_L \qquad \frac{B \text{ false} \in \Gamma}{\Gamma \rightarrow A \wedge B \text{ false}} \rightarrow\wedge F_R$$

$$\frac{A \text{ true} \in \Gamma}{\Gamma \rightarrow A \vee B \text{ true}} \rightarrow\vee T_L \qquad \frac{B \text{ true} \in \Gamma}{\Gamma \rightarrow A \vee B \text{ true}} \rightarrow\vee T_R \qquad \frac{A \text{ false} \in \Gamma \quad B \text{ false} \in \Gamma}{\Gamma \rightarrow A \vee B \text{ false}} \rightarrow\vee F$$

$$\frac{(A \text{ true} \vdash B \text{ true}) \in \Gamma}{\Gamma \rightarrow A \supset B \text{ true}} \rightarrow\supset T \qquad \frac{A \text{ true} \in \Gamma \quad B \text{ false} \in \Gamma}{\Gamma \rightarrow A \supset B \text{ false}} \rightarrow\supset F$$

$$\frac{A \text{ false} \in \Gamma \quad B \text{ true} \in \Gamma}{\Gamma \rightarrow B \setminus A \text{ true}} \rightarrow\setminus T \qquad \frac{(A \text{ false} \vdash B \text{ false}) \in \Gamma}{\Gamma \rightarrow B \setminus A \text{ false}} \rightarrow\setminus F$$

$$\frac{A \text{ true} \in \Gamma \quad A \text{ false} \in \Gamma}{\Gamma \rightarrow \#} \rightarrow\# I \qquad \frac{(A \text{ false} \vdash \#) \in \Gamma}{\Gamma \rightarrow A \text{ true}} \rightarrow\# ET \qquad \frac{(A \text{ true} \vdash \#) \in \Gamma}{\Gamma \rightarrow A \text{ false}} \rightarrow\# EF$$

Figure 13: The inference rules for the single step judgment.

$$\frac{\Gamma \to J}{\Gamma \vdash \texttt{hence}\ J : J}\ \text{step}P \qquad \frac{\Gamma \to H \quad \Gamma, H \vdash P : J}{\Gamma \vdash \texttt{hence}\ H; P : J}\ \text{seq}P \qquad \frac{\Gamma, H \vdash P : J \quad \Gamma, (H \vdash J) \vdash P' : J'}{\Gamma \vdash [H; P]; P' : J'}\ \text{hyp}P$$

Figure 14: The inference rules for the valid proof judgment.

$$\begin{array}{rcl} \overline{A\ \text{true}} & =^{\text{def}} & A \\ \overline{A\ \text{false}} & =^{\text{def}} & \neg A \\ \overline{\#} & =^{\text{def}} & \bot \end{array}$$

$$\begin{array}{rcl} H^* & =^{\text{def}} & \overline{H} \\ (H \vdash J)^* & =^{\text{def}} & \overline{H} \supset \overline{J} \end{array}$$

$$\begin{array}{rcl} .^* & =^{\text{def}} & . \\ (\Gamma, \gamma)^* & =^{\text{def}} & \Gamma^*, \gamma^*\ \text{true} \end{array}$$

Figure 15: A translation from Classical Tutch contexts to classical logic contexts.

analogous rules require knowledge that the same basic judgments hold. In the case of the single step rules, this knowledge is in the form of context membership. In the case of the entailment rules of classical logic, it is in the form of an entailment judgment.

There is a subtle distinction between the single step judgment of Classical Tutch and the entailment judgment of classical logic that is worth noting. While the context Δ in $\Delta \vdash J$ contains only hypotheses, the context Γ in $\Gamma \to J$ contains all previously proven non-hypothetical and hypothetical judgments. This distinction may become clearer after the inference rules for the valid proof judgment are presented.

The inference rules for the valid proof judgment are given in Figure 14. These rules immediately suggest a syntax-directed checking algorithm where the judgment that is proven is taken as the output:

step$P$ Rule: If the proof is a single hence-step, it is correct if the judgment that it claims to prove follows immediately from the context.

seq$P$ Rule: If the proof is a hence-step followed by a proof, it is correct if the step follows immediately from the context and if the proof that follows the step is correct given the context extended with the judgment proved by that step.

hyp$P$ Rule: If the proof is a frame followed by a proof, it is correct if the subproof within the frame is correct, given the context extended with the hypothesis, and if the proof that follows the frame is correct, given the context extended with the hypothetical judgment representing the frame.

5 Soundness

Before we can state the soundness theorem, we must define a translation from Classical Tutch contexts to classical logic contexts, which we will write as Γ^*. Figure 15 presents this translation.

The first translation maps a J to a proposition, and is written as $\overline{J}$. This translation is used to define a second translation that maps a Classical Tutch context entry to a proposition, and is written as γ^*. The final translation maps a Classical Tutch context to a classical logic context by applying the second translation to each entry in the context.

With this translation from Classical Tutch contexts to classical logic contexts, it is now possible to state the soundness theorem as follows:

Theorem (Soundness). *If $\Gamma \vdash P : J$ is derivable, then $\Gamma^* \vdash J$ is derivable in classical logic.*

Before proving the soundness theorem, we state a series of lemmas that will be used in the proof. The proofs of Lemmas 2, 1, and 4 are left to the Appendix.

Lemma 1. *If $\gamma \in \Gamma$ is derivable, then γ^* true $\in \Gamma^*$ is derivable.*

This lemma is needed to show that the context translation "distributes" over the context membership judgment. It is used in the proof of Lemma 3.

Lemma 2. *If $\Delta \vdash \neg A$ true is derivable in classical logic, then $\Delta \vdash A$ false is derivable in classical logic.*

This lemma is needed to invert the effects of the context translation on falsity judgments. It is used in the proof of Lemma 3.

The following is a statement of soundness for the single step judgment. This lemma is used in the first two cases of the proof of the soundness theorem.

Lemma 3. *If $\Gamma \rightarrow J$ is derivable, then $\Gamma^* \vdash J$ is derivable in classical logic.*

Proof. By case analysis on the derivation of $\Gamma \rightarrow J$.

Case: $\dfrac{A \text{ true} \in \Gamma \quad B \text{ true} \in \Gamma}{\Gamma \rightarrow A \wedge B \text{ true}} \rightarrow\wedge T$

1. A true $\in \Gamma$	Subderivation
2. B true $\in \Gamma$	Subderivation
3. A true $\in \Gamma^*$	Lemma 1 (1)
4. $\Gamma^* \vdash A$ true	hyp$T$ Rule (3)
5. B true $\in \Gamma^*$	Lemma 1 (2)
6. $\Gamma^* \vdash B$ true	hyp$T$ Rule (5)
7. $\Gamma^* \vdash A \wedge B$ true	$\wedge T$ Rule (4, 6)

Case: $\dfrac{A \text{ false} \in \Gamma}{\Gamma \rightarrow A \wedge B \text{ false}} \rightarrow\wedge F_L$

1. A false $\in \Gamma$	Subderivation
2. $\neg A$ true $\in \Gamma^*$	Lemma 1 (1)
3. $\Gamma^* \vdash \neg A$ true	hyp$T$ Rule (2)
4. $\Gamma^* \vdash A$ false	Lemma 2 (3)
5. $\Gamma^* \vdash A \wedge B$ false	$\wedge F_L$ Rule (4)

Case: $\dfrac{A \text{ false} \in \Gamma \quad B \text{ true} \in \Gamma}{\Gamma \rightarrow B \setminus A \text{ true}} \rightarrow\setminus T$

1. A false $\in \Gamma$	Subderivation
2. B true $\in \Gamma$	Subderivation
3. $\neg A$ true $\in \Gamma^*$	Lemma 1 (1)
4. $\Gamma^* \vdash \neg A$ true	hyp$T$ Rule (3)
5. $\Gamma^* \vdash A$ false	Lemma 2 (4)
6. B true $\in \Gamma^*$	Lemma 1 (2)
7. $\Gamma^* \vdash B$ true	hyp$T$ Rule (6)
8. $\Gamma^* \vdash B \setminus A$ true	$\setminus T$ Rule (5, 7)

Case: $\dfrac{(A \text{ false} \vdash B \text{ false}) \in \Gamma}{\Gamma \to B \setminus A \text{ false}} \to\!\setminus\! F$

1. $(A \text{ false} \vdash B \text{ false}) \in \Gamma$ — Subderivation
2. $\neg A \supset \neg B \text{ true} \in \Gamma^*$ — Lemma 1 (1)
3. $\neg A \supset \neg B \text{ true} \in \Gamma^*, A \text{ false}, B \text{ true}$ — (2)
4. $\Gamma^*, A \text{ false}, B \text{ true} \vdash \neg A \supset \neg B \text{ true}$ — hyp$T$ Rule (3)
5. $A \text{ false} \in \Gamma^*, A \text{ false}, B \text{ true}$
6. $\Gamma^*, A \text{ false}, B \text{ true} \vdash A \text{ false}$ — hyp$F$ Rule (5)
7. $\Gamma^*, A \text{ false}, B \text{ true} \vdash \neg A \text{ true}$ — $\neg T$ Rule (6)
8. $B \text{ true} \in \Gamma^*, A \text{ false}, B \text{ true}$ — mem$F$ Rule
9. $\Gamma^*, A \text{ false}, B \text{ true} \vdash B \text{ true}$ — hyp$T$ Rule (8)
10. $\Gamma^*, A \text{ false}, B \text{ true} \vdash \neg B \text{ false}$ — $\neg F$ Rule (9)
11. $\Gamma^*, A \text{ false}, B \text{ true} \vdash \neg A \supset \neg B \text{ false}$ — $\supset\! F$ Rule (7, 10)
12. $\Gamma^*, A \text{ false}, B \text{ true} \vdash \#$ — $\# I$ Rule (4, 11)
13. $\Gamma^*, A \text{ false} \vdash B \text{ false}$ — $\# EF$ Rule (12)
14. $\Gamma^* \vdash B \setminus A \text{ false}$ — $\setminus F$ Rule (13)

Case: $\dfrac{A \text{ true} \in \Gamma \quad A \text{ false} \in \Gamma}{\Gamma \to \#} \to\!\# I$

1. $A \text{ true} \in \Gamma$ — Subderivation
2. $A \text{ false} \in \Gamma$ — Subderivation
3. $A \text{ true} \in \Gamma^*$ — Lemma 1 (1)
4. $\Gamma^* \vdash A \text{ true}$ — hyp$T$ Rule (3)
5. $\neg A \text{ true} \in \Gamma^*$ — Lemma 1 (2)
6. $\Gamma^* \vdash \neg A \text{ true}$ — hyp$T$ Rule (5)
7. $\Gamma^* \vdash A \text{ false}$ — Lemma 2 (6)
8. $\Gamma^* \vdash \#$ — $\# I$ Rule (4, 7)

Case: $\dfrac{(A \text{ false} \vdash \bot) \in \Gamma}{\Gamma \to A \text{ true}} \to\!\# ET$

1. $(A \text{ false} \vdash \#) \in \Gamma$ — Subderivation
2. $\neg A \supset \bot \text{ true} \in \Gamma^*$ — Lemma 1 (1)
3. $\neg A \supset \bot \text{ true} \in \Gamma^*, A \text{ false}$ — mem$S$ Rule (2)
4. $\Gamma^*, A \text{ false} \vdash \neg A \supset \bot \text{ true}$ — hyp$T$ Rule (3)
5. $A \text{ false} \in \Gamma^*, A \text{ false}$ — mem$F$ Rule

6. Γ^*, A false $\vdash A$ false — hyp$F$ Rule (5)
7. Γ^*, A false $\vdash \neg A$ true — $\neg T$ Rule (6)
8. Γ^*, A false $\vdash \bot$ false — $\bot F$ Rule
9. Γ^*, A false $\vdash \neg A \supset \bot$ false — $\supset F$ Rule (7, 8)
10. Γ^*, A false $\vdash \#$ — $\# I$ Rule (4, 9)
11. $\Gamma^* \vdash A$ true — $\# ET$ Rule (10)

The remaining cases are similar and are left to the Appendix.

□

Lemma 4. *If* $\Delta \vdash J$ *is derivable in classical logic, then* $\Delta \vdash \overline{J}$ true *is derivable in classical logic.*

This lemma is used in the proof of the soundness theorem for the second and third cases.

And now, we repeat the statement of the soundness theorem for convenience and give the proof.

Theorem 1 (Soundness). *If* $\Gamma \vdash P : J$ *is derivable, then* $\Gamma^* \vdash J$ *is derivable in classical logic.*

Proof. By induction on the derivation of $\Gamma \vdash P : J$.

Case: $\dfrac{\Gamma \to J}{\Gamma \vdash \texttt{hence}\ J : J}$

1. $\Gamma \to J$ — Subderivation
2. $\Gamma^* \vdash J$ — Lemma 3 (1)

Case: $\dfrac{\Gamma \to H \quad \Gamma, H \vdash P : J}{\Gamma \vdash \texttt{hence}\ H; P : J}$

1. $\Gamma \to H$ — Subderivation
2. $\Gamma, H \vdash P : J$ — Subderivation
3. $\Gamma^* \vdash H$ — Lemma 3 (1)
4. $\Gamma^* \vdash \overline{H}$ true — Lemma 4 (3)
5. $\Gamma^*, \overline{H}$ true $\vdash J$ — Inductive Hypothesis (2)
6. $\Gamma^* \vdash J$ — Cut Rule (4, 5)

Case: $\dfrac{\Gamma, H \vdash P : J \quad \Gamma, (H \vdash J) \vdash P' : J'}{\Gamma \vdash [H; P]; P' : J'}$

1. $\Gamma, H \vdash P : J$ — Subderivation
2. $\Gamma, (H \vdash J) \vdash P' : J'$ — Subderivation
3. $\Gamma^*, \overline{H}$ true $\vdash J$ — Inductive Hypothesis (1)
4. $\Gamma^*, \overline{H}$ true $\vdash \overline{J}$ true — Lemma 4 (3)
5. $\Gamma^* \vdash \overline{H} \supset \overline{J}$ true — $\supset T$ Rule (4)
6. $\Gamma^*, \overline{H} \supset \overline{J}$ true $\vdash J'$ — Inductive Hypothesis (2)
7. $\Gamma^* \vdash J'$ — Cut Rule (5, 6)

□

This completes the demonstration of soundness for the formal system for Classical Tutch. We now shift our attention to proving completeness for the system.

6 Completeness

We state the completeness theorem for Classical Tutch as follows:

Theorem. *If $\Delta \vdash J$ is derivable in classical logic, then there exists a proof P such that $\Delta \vdash P : J$ is derivable.*

Before proving the completeness theorem, we state a series of lemmas that will be used in the proof. The proof of Lemma 5 is left to the Appendix.

Lemma 5. *If $\Gamma, \Gamma' \rightarrow J$ is derivable, then $\Gamma, \gamma, \Gamma' \rightarrow J$ is derivable.*

This lemma is a statement of weakening for the single step judgment. It is used in the first two cases of the proof of weakening for the valid proof judgment, Lemma 6.

The following is a statement of weakening for the valid proof judgment. This lemma is used in the proof of Lemma 7 and a number of cases in the proof of the completeness theorem.

Lemma 6 (Weakening). *If $\Gamma, \Gamma' \vdash P : J$ is derivable, then $\Gamma, \gamma, \Gamma' \vdash P : J$ is derivable.*

Proof. By induction on the derivation of $\Gamma, \Gamma' \vdash P : J$.

Case: $\dfrac{\Gamma, \Gamma' \rightarrow J}{\Gamma, \Gamma' \vdash \texttt{hence}\ J : J}\ \text{step}P$

1. $\Gamma, \Gamma' \rightarrow J$ — Subderivation
2. $\Gamma, \gamma, \Gamma' \rightarrow J$ — Lemma 5 (1)
3. $\Gamma, \gamma, \Gamma' \vdash \texttt{hence}\ J : J$ — step$P$ Rule (2)

Case: $\dfrac{\Gamma, \Gamma' \rightarrow H \quad \Gamma, \Gamma', H \vdash P : J}{\Gamma, \Gamma' \vdash \texttt{hence}\ H; P : J}\ \text{seq}P$

1. $\Gamma, \Gamma' \rightarrow H$ — Subderivation
2. $\Gamma, \Gamma', H \vdash P : J$ — Subderivation
3. $\Gamma, \gamma, \Gamma' \rightarrow H$ — Lemma 5 (1)
4. $\Gamma, \gamma, \Gamma', H \vdash P : J$ — Inductive Hypothesis (2)
5. $\Gamma, \gamma, \Gamma' \vdash \texttt{hence}\ H; P : J$ — seq$P$ Rule (3, 4)

Case: $\dfrac{\Gamma, \Gamma', H \vdash P : J \quad \Gamma, \Gamma', (H \vdash J) \vdash P' : J'}{\Gamma, \Gamma' \vdash [H; P]; P' : J'}\ \text{hyp}P$

1. $\Gamma, \Gamma', H \vdash P : J$ — Subderivation
2. $\Gamma, \Gamma', (H \vdash J) \vdash P' : J'$ — Subderivation
3. $\Gamma, \gamma, \Gamma', H \vdash P : J$ — Inductive Hypothesis (1)
4. $\Gamma, \gamma, \Gamma', (H \vdash J) \vdash P' : J'$ — Inductive Hypothesis (2)
5. $\Gamma, \gamma, \Gamma' \vdash [H; P]; P' : J'$ — hyp$P$ Rule (3, 4)

□

$$
\begin{array}{rcl}
(\texttt{hence}\ H) \cdot P' & =^{\text{def}} & \texttt{hence}\ H; P' \\
(\texttt{hence}\ \#) \cdot P' & =^{\text{def}} & \text{Undefined} \\
(\texttt{hence}\ H; P) \cdot P' & =^{\text{def}} & \texttt{hence}\ H; (P \cdot P') \\
([H; P_1]; P_2) \cdot P' & =^{\text{def}} & [H; P_1]; (P_2 \cdot P')
\end{array}
$$

Figure 16: A concatenation operator for Classical Tutch proofs.

Before we can state and prove a concatenation lemma for Classical Tutch proofs, we must define an appropriate notion of concatenation, which we will write as $P \cdot P'$ (Fig. 16). The concatenation of a hence-step that proves an H and a proof is defined to be that step followed by the proof. The concatenation of a hence-step that proves a contradiction judgment and a proof is left undefined. The concatenation of a hence-step followed by a proof and another proof is that hence-step followed by the concatenation of the proofs. The concatenation of a frame followed by a proof and another proof is that frame followed by the concatenation of the proofs.

We now state and prove the concatenation lemma for the valid proof judgment. This lemma is used in nearly every case of the proof of completeness.

Lemma 7 (Concatenation). *If* $\Gamma \vdash P : H$ *and* $\Gamma, H \vdash P' : J$ *are derivable, then* $\Gamma \vdash P \cdot P' : J$ *is derivable.*

Proof. By induction on the derivation of $\Gamma \vdash P : H$.

Case: $\dfrac{\Gamma \rightarrow H}{\Gamma \vdash \texttt{hence}\ H : H}\ \text{step}P$

1. $\Gamma \rightarrow H$ — Subderivation
2. $\Gamma, H \vdash P' : J$ — Assumption
3. $\Gamma \vdash \texttt{hence}\ H; P' : J$ — seq$P$ Rule (1, 2)

Case: $\dfrac{\Gamma \rightarrow H' \quad \Gamma, H' \vdash P : H}{\Gamma \vdash \texttt{hence}\ H'; P : H}\ \text{seq}P$

1. $\Gamma \rightarrow H'$ — Subderivation
2. $\Gamma, H' \vdash P : H$ — Subderivation
3. $\Gamma, H \vdash P' : J$ — Assumption
4. $\Gamma, H', H \vdash P' : J$ — Lemma 6 (3)
5. $\Gamma, H' \vdash P \cdot P' : J$ — Inductive Hypothesis (2, 4)
6. $\Gamma \vdash \texttt{hence}\ H'; (P \cdot P') : J$ — seq$P$ Rule (1, 5)

Case: $\dfrac{\Gamma, H_1 \vdash P_1 : J_1 \quad \Gamma, (H_1 \vdash J_1) \vdash P_2 : H}{\Gamma \vdash [H_1; P_1]; P_2 : H}\ \text{hyp}P$

1. $\Gamma, H_1 \vdash P_1 : J_1$ — Subderivation
2. $\Gamma, (H_1 \vdash J_1) \vdash P_2 : H$ — Subderivation
3. $\Gamma, H \vdash P' : J$ — Assumption

4. $\Gamma, (H_1 \vdash J_1), H \vdash P' : J$ Lemma 6 (3)
5. $\Gamma, (H_1 \vdash J_1) \vdash P_2 \cdot P' : J$ Inductive Hypothesis (2, 4)
6. $\Gamma \vdash [H_1; P_1]; (P_2 \cdot P') : J$ hyp$P$ Rule (1, 5)

□

We now restate the completeness theorem for convenience and then give the proof.

Theorem 2 (Completeness). *If $\Delta \vdash J$ is derivable in classical logic, then there exists a proof P such that $\Delta \vdash P : J$ is derivable.*

Proof. By induction on the derivation of $\Delta \vdash J$.

Case: $\dfrac{\Delta \vdash A \text{ true} \quad \Delta \vdash B \text{ true}}{\Delta \vdash A \wedge B \text{ true}} \wedge T$

1. $\Delta \vdash A$ true Subderivation
2. $\Delta \vdash B$ true Subderivation
3. $A \text{ true} \in \Delta, A \text{ true}, B \text{ true}$
4. $B \text{ true} \in \Delta, A \text{ true}, B \text{ true}$ mem$F$ Rule
5. $\Delta, A \text{ true}, B \text{ true} \rightarrow A \wedge B \text{ true}$ $\rightarrow\wedge T$ Rule (3, 4)
6. $\Delta, A \text{ true}, B \text{ true} \vdash \texttt{hence}\ A \wedge B \text{ true} : A \wedge B \text{ true}$ step$P$ Rule (5)
7. $\Delta \vdash P_2 : B$ true Inductive Hypothesis (2)
8. $\Delta, A \text{ true} \vdash P_2 : B$ true Lemma 6 (7)
9. $\Delta, A \text{ true} \vdash P_2 \cdot \texttt{hence}\ A \wedge B \text{ true} : A \wedge B \text{ true}$ Lemma 7 (8, 6)
10. $\Delta \vdash P_1 : A$ true Inductive Hypothesis (1)
11. $\Delta \vdash P_1 \cdot (P_2 \cdot \texttt{hence}\ A \wedge B \text{ true}) : A \wedge B \text{ true}$ Lemma 7 (10, 9)

Case: $\dfrac{\Delta \vdash A \text{ false}}{\Delta \vdash A \wedge B \text{ false}} \wedge F_L$

1. $\Delta \vdash A$ false Subderivation
2. $A \text{ false} \in \Delta, A \text{ false}$ mem$F$ Rule
3. $\Delta, A \text{ false} \rightarrow A \wedge B \text{ false}$ $\rightarrow\wedge F_L$ Rule (2)
4. $\Delta, A \text{ false} \vdash \texttt{hence}\ A \wedge B \text{ false} : A \wedge B \text{ false}$ step$P$ Rule (3)
5. $\Delta \vdash P : A$ false Inductive Hypothesis (1)
6. $\Delta \vdash P \cdot \texttt{hence}\ A \wedge B \text{ false} : A \wedge B \text{ false}$ Lemma 7 (5, 4)

Case: $\dfrac{\Delta \vdash A \text{ false} \quad \Delta \vdash B \text{ true}}{\Delta \vdash B \setminus A \text{ true}} \setminus T$

1. $\Delta \vdash A$ false — Subderivation
2. $\Delta \vdash B$ true — Subderivation
3. A false $\in \Delta, A$ false$, B$ true
4. B true $\in \Delta, A$ false$, B$ true — mem$F$ Rule
5. Δ, A false$, B$ true $\rightarrow B \setminus A$ true — $\rightarrow\setminus T$ Rule (3, 4)
6. Δ, A false$, B$ true $\vdash$ `hence` $B \setminus A$ true : $B \setminus A$ true — step$P$ Rule (5)
7. $\Delta \vdash P_2 : B$ true — Inductive Hypothesis (2)
8. Δ, A false $\vdash P_2 : B$ true — Lemma 6 (7)
9. Δ, A false $\vdash P_2 \cdot$ `hence` $B \setminus A$ true : $B \setminus A$ true — Lemma 7 (8, 6)
10. $\Delta \vdash P_1 : A$ false — Inductive Hypothesis (1)
11. $\Delta \vdash P_1 \cdot (P_2 \cdot$ `hence` $B \setminus A$ true$) : B \setminus A$ true — Lemma 7 (10, 9)

Case: $\dfrac{\Delta, A \text{ false} \vdash B \text{ false}}{\Delta \vdash B \setminus A \text{ false}} \setminus F$

1. Δ, A false $\vdash B$ false — Subderivation
2. $(A$ false $\vdash B$ false$) \in \Delta, (A$ false $\vdash B$ false$)$ — mem$F$ Rule
3. $\Delta, (A$ false $\vdash B$ false$) \rightarrow B \setminus A$ false — $\rightarrow\setminus F$ Rule (2)
4. $\Delta, (A$ false $\vdash B$ false$) \vdash$ `hence` $B \setminus A$ false : $B \setminus A$ false — step$P$ Rule (3)
5. Δ, A false $\vdash P : B$ false — Inductive Hypothesis (1)
6. $\Delta \vdash [A$ false$; P];$ `hence` $B \setminus A$ false : $B \setminus A$ false — hyp$P$ Rule (5, 4)

Case: $\dfrac{\Delta \vdash A \text{ true} \quad \Delta \vdash A \text{ false}}{\Delta \vdash \#} \#I$

1. $\Delta \vdash A$ true — Subderivation
2. $\Delta \vdash A$ false — Subderivation
3. A true $\in \Delta, A$ true$, A$ false
4. A false $\in \Delta, A$ true$, A$ false — mem$F$ Rule
5. Δ, A true$, A$ false $\rightarrow \#$ — $\rightarrow\#I$ Rule (3, 4)
6. Δ, A true$, A$ false $\vdash$ `hence` $\# : \#$ — step$P$ Rule (5)
7. $\Delta \vdash P_2 : A$ false — Inductive Hypothesis (2)
8. Δ, A true $\vdash P_2 : A$ false — Lemma 6 (7)
9. Δ, A true $\vdash P_2 \cdot$ `hence` $\# : \#$ — Lemma 7 (8, 6)
10. $\Delta \vdash P_1 : A$ true — Inductive Hypothesis (1)
11. $\Delta \vdash P_1 \cdot (P_2 \cdot$ `hence` $\#) : \#$ — Lemma 7 (10, 9)

Case: $\dfrac{\Delta, A \text{ false} \vdash \#}{\Delta \vdash A \text{ true}} \ \#ET$

1. $\Delta, A \text{ false} \vdash \#$ — Subderivation
2. $(A \text{ false} \vdash \#) \in \Delta, (A \text{ false} \vdash \#)$ — mem$F$ Rule
3. $\Delta, (A \text{ false} \vdash \#) \rightarrow A \text{ true}$ — $\rightarrow\#ET$ Rule (2)
4. $\Delta, (A \text{ false} \vdash \#) \vdash \texttt{hence}\ A \text{ true} : A \text{ true}$ — step$P$ Rule (3)
5. $\Delta, A \text{ false} \vdash P : \#$ — Inductive Hypothesis (1)
6. $\Delta \vdash [A \text{ false}; P]; \texttt{hence}\ A \text{ true} : A \text{ true}$ — hyp$P$ Rule (5, 4)

The remaining cases are similar and are left to the Appendix.

□

This completes the demonstration of the completeness proof.

7 Related Work

As noted in [2], the Block Calculus given by Dahn and Wolf in [5] is similar to the linear syntax used in Tutch, and consequently similar to the linear syntax used in Classical Tutch. In addition, [2] references the seminal work in human-readable machine-verifiable proofs done in the *Mizar* project [8].

In addition to their work on the original Tutch system, Abel, Chang, and Pfenning have investigated a proof checker with step sizes of increased granularity. Abel and Pfenning were able to increase the proof step size while maintaining simplicity. Only four constructs were used: `assume`, `triv`, `lemma`, and `case`. At the time of publication of their paper, this system had not been tested in a classroom setting. [2]

Sridhar Ramesh and Karl Crary constructed Linear Tutch, a proof checker for dual intuitionistic linear logic based on Abel, Chang, and Pfenning's Tutch. Adapting the Tutch system to the resource-sensitive dual intuitionistic linear logic necessitated the addition of search and backtracking to avoid requiring the user to explicitly state resource use. Linear Tutch was used in the Spring 2006 offering of the Constructive Logic course at Carnegie Mellon University. Anecdotal evidence suggests that it was very effective in its role as a pedagogic tool. [7]

8 Future Work

Classical Tutch and, more generally, proof checkers provide several possibilities for future work. The addition of proof terms to Classical Tutch is an immediate opportunity for extension. The Constructive Logic course at Carnegie Mellon University includes proof terms in the discussion of classical logic, and it is likely that students in this course would find a proof term checker both beneficial and complementary to Classical Tutch.

It is also reasonable to consider developing versions of Tutch for other logics, for example, modal logic. Modal logic was briefly covered in the Spring 2006 offering of the Constructive Logic course. If modal logic is covered in greater depth in future offerings of the course, we assert that a version of Tutch for modal logic would be equally valuable as a pedagogic tool as we believe Classical Tutch will be.

9 Conclusion

This completes our discussion of a formal system for Classical Tutch that was introduced in Section 4 and adheres to the formal system for classical logic presented in Section 3. We have shown through the proofs given in Sections 5 and 6 that the described system is both sound and complete. We concluded with an overview of related work, which includes a higher-level version of Tutch and Linear Tutch, and possible future work, which includes proof terms for Classical Tutch and the possibility of "Modal Tutch," in Sections 7 and 8.

We hope that this work will allow an implementation of Classical Tutch to prove useful to students learning classical logic.

A Appendix

A.1 Proof of Lemma 1

The proof of Lemma 1 that was omitted from Section 5 follows. The statement of the lemma is reproduced for convenience.

Lemma 1. *If* $\gamma \in \Gamma$ *is derivable, then* γ^* true $\in \Gamma^*$ *is derivable.*

Proof. By induction on the derivation of $\gamma \in \Gamma$.

Case: $\dfrac{}{\gamma \in \Gamma, \gamma}\ \text{mem}F$

1. $(\Gamma, \gamma)^* = \Gamma^*, \gamma^*$ true — Definition of Translation
2. γ^* true $\in \Gamma^*, \gamma^*$ true — mem$F$ Rule

Case: $\dfrac{\gamma_1 \in \Gamma}{\gamma_1 \in \Gamma, \gamma_2}\ \text{mem}S$

1. $\gamma_1 \in \Gamma$ — Subderivation
2. γ_1^* true $\in \Gamma^*$ — Inductive Hypothesis (1)
3. γ_1^* true $\in \Gamma^*, \gamma_2^*$ true — mem$S$ Rule (2)

□

A.2 Proof of Lemma 2

The proof of Lemma 2 follows. This proof was omitted from Section 5. We restate the lemma for convenience.

Lemma 2. *If* $\Delta \vdash \neg A$ true *is derivable in classical logic, then* $\Delta \vdash A$ false *is derivable in classical logic.*

Proof.
1. $\Delta \vdash \neg A$ true — Assumption
2. Δ, A true $\vdash \neg A$ true — Weakening (1)
3. A true $\in \Delta, A$ true
4. Δ, A true $\vdash A$ true — hyp$T$ Rule (3)
5. Δ, A true $\vdash \neg A$ false — $\neg F$ Rule (4)
6. Δ, A true $\vdash \#$ — $\#I$ Rule (2, 5)
7. $\Delta \vdash A$ false — $\#EF$ Rule (6)

□

A.3 Proof of Lemma 3

The cases of the proof of Lemma 3 that were omitted from Section 5 are given here. The lemma is restated for convenience.

Lemma 3. *If* $\Gamma \rightarrow J$ *is derivable, then* $\Gamma^* \vdash J$ *is derivable in classical logic.*

Proof. By case analysis on the derivation of $\Gamma \rightarrow J$.

Case: $\dfrac{A \text{ true} \in \Gamma}{\Gamma \to A \text{ true}} \to\text{hyp}T$

1. A true $\in \Gamma$ — Subderivation
2. A true $\in \Gamma^*$ — Lemma 1 (1)
3. $\Gamma^* \vdash A$ true — hyp$T$ Rule (2)

Case: $\dfrac{A \text{ false} \in \Gamma}{\Gamma \to A \text{ false}} \to\text{hyp}F$

1. A false $\in \Gamma$ — Subderivation
2. $\neg A$ true $\in \Gamma^*$ — Lemma 1 (1)
3. $\Gamma^* \vdash \neg A$ true — hyp$T$ Rule (2)
4. $\Gamma^* \vdash A$ false — Lemma 2 (3)

Case: $\dfrac{}{\Gamma \to \top \text{ true}} \to\top T$

1. $\Gamma^* \vdash \top$ true — $\top T$ Rule

Case: $\dfrac{}{\Gamma \to \bot \text{ false}} \to\bot F$

1. $\Gamma^* \vdash \bot$ false — $\bot F$ Rule

Case: $\dfrac{A \text{ false} \in \Gamma}{\Gamma \to \neg A \text{ true}} \to\neg T$

1. A false $\in \Gamma$ — Subderivation
2. $\neg A$ true $\in \Gamma^*$ — Lemma 1 (1)
3. $\Gamma^* \vdash \neg A$ true — hyp$T$ Rule (2)

Case: $\dfrac{A \text{ true} \in \Gamma}{\Gamma \to \neg A \text{ false}} \to\neg F$

1. A true $\in \Gamma$ — Subderivation
2. A true $\in \Gamma^*$ — Lemma 1 (1)
3. $\Gamma^* \vdash A$ true — hyp$T$ Rule (2)
4. $\Gamma^* \vdash \neg A$ false — $\neg F$ Rule (3)

Case: $\dfrac{B \text{ false} \in \Gamma}{\Gamma \to A \wedge B \text{ false}} \to\wedge F_R$

1. B false $\in \Gamma$ — Subderivation
2. $\neg B$ true $\in \Gamma^*$ — Lemma 1 (1)
3. $\Gamma^* \vdash \neg B$ true — hyp$T$ Rule (2)
4. $\Gamma^* \vdash B$ false — Lemma 2 (3)
5. $\Gamma^* \vdash A \wedge B$ false — $\wedge F_R$ Rule (4)

Case: $\dfrac{A \text{ true} \in \Gamma}{\Gamma \rightarrow A \vee B \text{ true}} \rightarrow\!\vee T_L$

1. A true $\in \Gamma$ — Subderivation
2. A true $\in \Gamma^*$ — Lemma 1 (1)
3. $\Gamma^* \vdash A$ true — hyp$T$ Rule (2)
4. $\Gamma^* \vdash A \vee B$ true — $\vee T_L$ Rule (3)

Case: $\dfrac{B \text{ true} \in \Gamma}{\Gamma \rightarrow A \vee B \text{ true}} \rightarrow\!\vee T_R$

1. B true $\in \Gamma$ — Subderivation
2. B true $\in \Gamma^*$ — Lemma 1 (1)
3. $\Gamma^* \vdash B$ true — hyp$T$ Rule (2)
4. $\Gamma^* \vdash A \vee B$ true — $\vee T_R$ Rule (3)

Case: $\dfrac{A \text{ false} \in \Gamma \quad B \text{ false} \in \Gamma}{\Gamma \rightarrow A \vee B \text{ false}} \rightarrow\!\vee F$

1. A false $\in \Gamma$ — Subderivation
2. B false $\in \Gamma$ — Subderivation
3. $\neg A$ true $\in \Gamma^*$ — Lemma 1 (1)
4. $\Gamma^* \vdash \neg A$ true — hyp$T$ Rule (3)
5. $\Gamma^* \vdash A$ false — Lemma 2 (4)
6. $\neg B$ true $\in \Gamma^*$ — Lemma 1 (2)
7. $\Gamma^* \vdash \neg B$ true — hyp$T$ Rule (6)
8. $\Gamma^* \vdash B$ false — Lemma 2 (7)
9. $\Gamma^* \vdash A \vee B$ false — $\vee F$ Rule (5, 8)

Case: $\dfrac{(A \text{ true} \vdash B \text{ true}) \in \Gamma}{\Gamma \rightarrow A \supset B \text{ true}} \rightarrow\!\supset T$

1. $(A$ true $\vdash B$ true$) \in \Gamma$ — Subderivation
2. $A \supset B$ true $\in \Gamma^*$ — Lemma 1 (1)
3. $\Gamma^* \vdash A \supset B$ true — hyp$T$ Rule (2)

Case: $\dfrac{A \text{ true} \in \Gamma \quad B \text{ false} \in \Gamma}{\Gamma \rightarrow A \supset B \text{ false}} \rightarrow\supset F$

1. A true $\in \Gamma$ — Subderivation
2. B false $\in \Gamma$ — Subderivation
3. A true $\in \Gamma^*$ — Lemma 1 (1)
4. $\Gamma^* \vdash A$ true — hyp$T$ Rule (3)
5. $\neg B$ true $\in \Gamma^*$ — Lemma 1 (2)
6. $\Gamma^* \vdash \neg B$ true — hyp$T$ Rule (5)
7. $\Gamma^* \vdash B$ false — Lemma 2 (6)
8. $\Gamma^* \vdash A \supset B$ false — $\supset F$ Rule (4, 7)

Case: $\dfrac{}{\Gamma, (A \text{ true} \vdash \#) \rightarrow A \text{ false}} \rightarrow \# EF$

1. $(A$ true $\vdash \#) \in \Gamma$ — Subderivation
2. $A \supset \bot$ true $\in \Gamma^*$ — Lemma 1 (1)
3. $A \supset \bot$ true $\in \Gamma^*, A$ true — mem$S$ Rule (2)
4. Γ^*, A true $\vdash A \supset \bot$ true — hyp$T$ Rule (3)
5. A true $\in \Gamma^*, A$ true — mem$F$ Rule
6. Γ^*, A true $\vdash A$ true — hyp$T$ Rule (5)
7. Γ^*, A true $\vdash \bot$ false — $\bot F$ Rule
8. Γ^*, A true $\vdash A \supset \bot$ false — $\supset F$ Rule (6, 7)
9. Γ^*, A true $\vdash \#$ — $\# I$ Rule (4, 8)
10. $\Gamma^* \vdash A$ false — $\# EF$ Rule (9)

The remaining cases were given in Section 5.

□

A.4 Proof of Lemma 4

The proof of Lemma 4, which was omitted from Section 5, follows. The lemma is restated for convenience.

Lemma 4. *If* $\Delta \vdash J$ *is derivable in classical logic, then* $\Delta \vdash \overline{J}$ true *is derivable in classical logic.*

Proof. By case analysis on J.

Case: $J = H$

Subcase: $H = A$ true

1. $\Delta \vdash J$ — Assumption
2. $\Delta \vdash A$ true — $J = H$ and $H = A$ true (1)
3. $\Delta \vdash \overline{A \text{ true}}$ true — (2)

Subcase: $H = A$ false

1. $\Delta \vdash J$ — Assumption
2. $\Delta \vdash A$ false — $J = H$ and $H = A$ false (1)
3. $\Delta \vdash \neg A$ true — $\neg T$ Rule (2)
4. $\Delta \vdash \overline{A \text{ false}}$ true — (3)

Case: $J = \#$

1. $\Delta \vdash J$ — Assumption
2. $\Delta \vdash \#$ — $J = \#$ (1)
3. $\Delta, \bot$ false $\vdash \#$ — Weakening (2)
4. $\Delta \vdash \bot$ true — $\#ET$ Rule (3)
5. $\Delta \vdash \overline{\#}$ true — (4)

□

A.5 Statement and Proof of Weakening for Context Membership

The following is a statement and proof of weakening for the context membership judgment. It is used in the proof of Lemma 5. The statement and proof were omitted from Section 6.

Lemma. *If $\gamma \in \Gamma, \Gamma'$ is derivable, then $\gamma \in \Gamma, \gamma', \Gamma'$ is derivable.*

Proof. By induction on the structure of Γ'.

Case: $\Gamma' = \cdot$

1. $\gamma \in \Gamma$ — Assumption
2. $\gamma \in \Gamma, \gamma'$ — mem$S$ Rule (1)

Case:

Subcase: $\Gamma' = \Gamma'', \gamma$

1. $\gamma \in \Gamma, \gamma', \Gamma'', \gamma$ — mem$F$ Rule

Subcase: $\Gamma' = \Gamma'', \gamma''$

1. $\gamma \in \Gamma, \Gamma'', \gamma''$ — Assumption
2. $\gamma \in \Gamma, \Gamma''$ — Inversion (1)
3. $\gamma \in \Gamma, \gamma', \Gamma''$ — Inductive Hypothesis (2)
4. $\gamma \in \Gamma, \gamma', \Gamma'', \gamma''$ — mem$S$ Rule (3)

□

A.6 Proof of Lemma 5

The following is a proof of Lemma 5, which was omitted from Section 6. We restate the lemma for convenience.

Lemma 5. *If* $\Gamma, \Gamma' \to J$ *is derivable, then* $\Gamma, \gamma, \Gamma' \to J$ *is derivable.*

Proof. By case analysis on the derivation of $\Gamma, \Gamma' \to J$.

Case: $\dfrac{A \text{ true} \in \Gamma, \Gamma'}{\Gamma, \Gamma' \to A \text{ true}} \to\!\text{hyp}T$

1. A true $\in \Gamma, \Gamma'$	Subderivation
2. A true $\in \Gamma, \gamma, \Gamma'$	Lemma A.5 (1)
3. $\Gamma, \gamma, \Gamma' \to A$ true	$\to$hyp$T$ Rule (2)

Case: $\dfrac{A \text{ false} \in \Gamma, \Gamma'}{\Gamma, \Gamma' \to A \text{ false}} \to\!\text{hyp}F$

1. A false $\in \Gamma, \Gamma'$	Subderivation
2. A false $\in \Gamma, \gamma, \Gamma'$	Lemma A.5 (1)
3. $\Gamma, \gamma, \Gamma' \to A$ false	$\to$hyp$F$ Rule (2)

Case: $\dfrac{}{\Gamma, \Gamma' \to \top \text{ true}} \to\!\top T$

1. $\Gamma, \gamma, \Gamma' \to \top$ true	$\to\top T$ Rule

Case: $\dfrac{}{\Gamma, \Gamma' \to \bot \text{ false}} \to\!\bot F$

1. $\Gamma, \gamma, \Gamma' \to \bot$ false	$\to\bot F$ Rule

Case: $\dfrac{A \text{ false} \in \Gamma, \Gamma'}{\Gamma, \Gamma' \to \neg A \text{ true}} \to\!\neg T$

1. A false $\in \Gamma, \Gamma'$	Subderivation
2. A false $\in \Gamma, \gamma, \Gamma'$	Lemma A.5 (1)
3. $\Gamma, \gamma, \Gamma' \to \neg A$ true	$\to\neg T$ Rule (2)

Case: $\dfrac{A \text{ true} \in \Gamma, \Gamma'}{\Gamma, \Gamma' \to \neg A \text{ false}} \to\!\neg F$

1. A true $\in \Gamma, \Gamma'$	Subderivation
2. A true $\in \Gamma, \gamma, \Gamma'$	Lemma A.5 (1)
3. $\Gamma, \gamma, \Gamma' \to \neg A$ false	$\to\neg F$ Rule (2)

Case: $\dfrac{A \text{ true} \in \Gamma, \Gamma' \quad B \text{ true} \in \Gamma, \Gamma'}{\Gamma, \Gamma' \to A \wedge B \text{ true}} \to\wedge T$

1. A true $\in \Gamma, \Gamma'$ — Subderivation
2. B true $\in \Gamma, \Gamma'$ — Subderivation
3. A true $\in \Gamma, \gamma, \Gamma'$ — Lemma A.5 (1)
4. B true $\in \Gamma, \gamma, \Gamma'$ — Lemma A.5 (2)
5. $\Gamma, \gamma, \Gamma' \to A \wedge B$ true — $\to\wedge T$ Rule (3, 4)

Case: $\dfrac{A \text{ false} \in \Gamma, \Gamma'}{\Gamma, \Gamma' \to A \wedge B \text{ false}} \to\wedge F_L$

1. A false $\in \Gamma, \Gamma'$ — Subderivation
2. A false $\in \Gamma, \gamma, \Gamma'$ — Lemma A.5 (1)
3. $\Gamma, \gamma, \Gamma' \to A \wedge B$ false — $\to\wedge F_L$ Rule (2)

Case: $\dfrac{B \text{ false} \in \Gamma, \Gamma'}{\Gamma, \Gamma' \to A \wedge B \text{ false}} \to\wedge F_R$

1. B false $\in \Gamma, \Gamma'$ — Subderivation
2. B false $\in \Gamma, \gamma, \Gamma'$ — Lemma A.5 (1)
3. $\Gamma, \gamma, \Gamma' \to A \wedge B$ false — $\to\wedge F_R$ Rule (2)

Case: $\dfrac{A \text{ true} \in \Gamma, \Gamma'}{\Gamma, \Gamma' \to A \vee B \text{ true}} \to\vee T_L$

1. A true $\in \Gamma, \Gamma'$ — Subderivation
2. A true $\in \Gamma, \gamma, \Gamma'$ — Lemma A.5 (1)
3. $\Gamma, \gamma, \Gamma' \to A \vee B$ true — $\to\vee T_L$ Rule (2)

Case: $\dfrac{B \text{ true} \in \Gamma, \Gamma'}{\Gamma, \Gamma' \to A \vee B \text{ true}} \to\vee T_R$

1. B true $\in \Gamma, \Gamma'$ — Subderivation
2. B true $\in \Gamma, \gamma, \Gamma'$ — Lemma A.5 (1)
3. $\Gamma, \gamma, \Gamma' \to A \vee B$ true — $\to\vee T_R$ Rule (2)

Case: $\dfrac{A \text{ false} \in \Gamma, \Gamma' \quad B \text{ false} \in \Gamma, \Gamma'}{\Gamma, \Gamma' \to A \vee B \text{ false}} \to\vee F$

1. A false $\in \Gamma, \Gamma'$ — Subderivation
2. B false $\in \Gamma, \Gamma'$ — Subderivation
3. A false $\in \Gamma, \gamma, \Gamma'$ — Lemma A.5 (1)
4. B false $\in \Gamma, \gamma, \Gamma'$ — Lemma A.5 (2)
5. $\Gamma, \gamma, \Gamma' \to A \vee B$ false — $\to\vee F$ Rule (3, 4)

Case: $$\frac{(A \text{ true} \vdash B \text{ true}) \in \Gamma, \Gamma'}{\Gamma, \Gamma' \to A \supset B \text{ true}} \to\!\supset\! T$$

1. $(A \text{ true} \vdash B \text{ true}) \in \Gamma, \Gamma'$ — Subderivation
2. $(A \text{ true} \vdash B \text{ true}) \in \Gamma, \gamma, \Gamma'$ — Lemma A.5 (1)
3. $\Gamma, \gamma, \Gamma' \to A \supset B \text{ true}$ — $\to\!\supset\! T$ Rule (2)

Case: $$\frac{A \text{ true} \in \Gamma, \Gamma' \quad B \text{ false} \in \Gamma, \Gamma'}{\Gamma, \Gamma' \to A \supset B \text{ false}} \to\!\supset\! F$$

1. $A \text{ true} \in \Gamma, \Gamma'$ — Subderivation
2. $B \text{ false} \in \Gamma, \Gamma'$ — Subderivation
3. $A \text{ true} \in \Gamma, \gamma, \Gamma'$ — Lemma A.5 (1)
4. $B \text{ false} \in \Gamma, \gamma, \Gamma'$ — Lemma A.5 (2)
5. $\Gamma, \gamma, \Gamma' \to A \supset B \text{ false}$ — $\to\!\supset\! F$ Rule (3, 4)

Case: $$\frac{A \text{ false} \in \Gamma, \Gamma' \quad B \text{ true} \in \Gamma, \Gamma'}{\Gamma, \Gamma' \to B \setminus A \text{ true}} \to\!\setminus\! T$$

1. $A \text{ false} \in \Gamma, \Gamma'$ — Subderivation
2. $B \text{ true} \in \Gamma, \Gamma'$ — Subderivation
3. $A \text{ false} \in \Gamma, \gamma, \Gamma'$ — Lemma A.5 (1)
4. $B \text{ true} \in \Gamma, \gamma, \Gamma'$ — Lemma A.5 (2)
5. $\Gamma, \gamma, \Gamma' \to B \setminus A \text{ true}$ — $\to\!\setminus\! T$ Rule (3, 4)

Case: $$\frac{(A \text{ false} \vdash B \text{ false}) \in \Gamma, \Gamma'}{\Gamma, \Gamma' \to B \setminus A \text{ false}} \to\!\setminus\! F$$

1. $(A \text{ false} \vdash B \text{ false}) \in \Gamma, \Gamma'$ — Subderivation
2. $(A \text{ false} \vdash B \text{ false}) \in \Gamma, \gamma, \Gamma'$ — Lemma A.5 (1)
3. $\Gamma, \gamma, \Gamma' \to B \setminus A \text{ false}$ — $\to\!\setminus\! F$ Rule (2)

Case: $$\frac{A \text{ true} \in \Gamma, \Gamma' \quad A \text{ false} \in \Gamma, \Gamma'}{\Gamma, \Gamma' \to \#} \to\!\# I$$

1. $A \text{ true} \in \Gamma, \Gamma'$ — Subderivation
2. $A \text{ false} \in \Gamma, \Gamma'$ — Subderivation
3. $A \text{ true} \in \Gamma, \gamma, \Gamma'$ — Lemma A.5 (1)
4. $A \text{ false} \in \Gamma, \gamma, \Gamma'$ — Lemma A.5 (2)
5. $\Gamma, \gamma, \Gamma' \to \#$ — $\to\!\# I$ Rule (3, 4)

Case: $\dfrac{(A \text{ false} \vdash \#) \in \Gamma, \Gamma'}{\Gamma, \Gamma' \rightarrow A \text{ true}} \rightarrow\#ET$

1. $(A \text{ false} \vdash \#) \in \Gamma, \Gamma'$ — Subderivation
2. $(A \text{ false} \vdash \#) \in \Gamma, \gamma, \Gamma'$ — Lemma A.5 (1)
3. $\Gamma, \gamma, \Gamma' \rightarrow A \text{ true}$ — $\rightarrow\#ET$ Rule (2)

Case: $\dfrac{(A \text{ true} \vdash \#) \in \Gamma, \Gamma'}{\Gamma, \Gamma' \rightarrow A \text{ false}} \rightarrow\#EF$

1. $(A \text{ true} \vdash \#) \in \Gamma, \Gamma'$ — Subderivation
2. $(A \text{ true} \vdash \#) \in \Gamma, \gamma, \Gamma'$ — Lemma A.5 (1)
3. $\Gamma, \gamma, \Gamma' \rightarrow A \text{ false}$ — $\rightarrow\#EF$ Rule (2)

□

A.7 Proof of Theorem 2

The cases of the proof of the completeness theorem, Theorem 2, that were omitted from Section 6 are given here. The theorem is restated for convenience.

Theorem 2. *If $\Delta \vdash J$ is derivable in classical logic, then there exists a proof P such that $\Delta \vdash P : J$ is derivable.*

Proof. By induction on the derivation of $\Delta \vdash J$.

Case: $\dfrac{A \text{ true} \in \Delta}{\Delta \vdash A \text{ true}}\ \text{hyp}T$

1. $A \text{ true} \in \Delta$ — Subderivation
2. $\Delta \rightarrow A \text{ true}$ — $\rightarrow$hyp$T$ Rule (1)
3. $\Delta \vdash \texttt{hence}\ A \text{ true} : A \text{ true}$ — step$P$ Rule (2)

Case: $\dfrac{A \text{ false} \in \Delta}{\Delta \vdash A \text{ false}}\ \text{hyp}F$

1. $A \text{ false} \in \Delta$ — Subderivation
2. $\Delta \rightarrow A \text{ false}$ — $\rightarrow$hyp$F$ Rule (1)
3. $\Delta \vdash \texttt{hence}\ A \text{ false} : A \text{ false}$ — step$P$ Rule (2)

Case: $\dfrac{}{\Delta \vdash \top \text{ true}}\ \top T$

1. $\Delta \rightarrow \top \text{ true}$ — $\rightarrow\top T$ Rule
2. $\Delta \vdash \texttt{hence}\ \top \text{ true} : \top \text{ true}$ — step$P$ Rule (1)

Case: $\dfrac{}{\Delta \vdash \bot \text{ false}}\ \bot F$

1. $\Delta \rightarrow \bot \text{ false}$ — $\rightarrow\bot F$ Rule
2. $\Delta \vdash \texttt{hence}\ \bot \text{ false} : \bot \text{ false}$ — step$P$ Rule (1)

Case: $\dfrac{\Delta \vdash A \text{ false}}{\Delta \vdash \neg A \text{ true}} \neg T$

1. $\Delta \vdash A$ false — Subderivation
2. A false $\in \Delta, A$ false — mem$F$ Rule
3. Δ, A false $\rightarrow \neg A$ true — $\rightarrow\neg T$ Rule (2)
4. Δ, A false $\vdash$ `hence` $\neg A$ true : $\neg A$ true — step$P$ Rule (3)
5. $\Delta \vdash P : A$ false — Inductive Hypothesis (1)
6. $\Delta \vdash P \cdot$ `hence` $\neg A$ true : $\neg A$ true — Lemma 7 (5, 4)

Case: $\dfrac{\Delta \vdash A \text{ true}}{\Delta \vdash \neg A \text{ false}} \neg F$

1. $\Delta \vdash A$ true — Subderivation
2. A true $\in \Delta, A$ true — mem$F$ Rule
3. Δ, A true $\rightarrow \neg A$ false — $\rightarrow\neg F$ Rule (2)
4. Δ, A true $\vdash$ `hence` $\neg A$ false : $\neg A$ false — step$P$ Rule (3)
5. $\Delta \vdash P : A$ true — Inductive Hypothesis (1)
6. $\Delta \vdash P \cdot$ `hence` $\neg A$ false : $\neg A$ false — Lemma 7 (5, 4)

Case: $\dfrac{\Delta \vdash B \text{ false}}{\Delta \vdash A \wedge B \text{ false}} \wedge F_R$

1. $\Delta \vdash B$ false — Subderivation
2. B false $\in \Delta, B$ false — mem$F$ Rule
3. Δ, B false $\rightarrow A \wedge B$ false — $\rightarrow\wedge F_R$ Rule (2)
4. Δ, B false $\vdash$ `hence` $A \wedge B$ false : $A \wedge B$ false — step$P$ Rule (3)
5. $\Delta \vdash P : B$ false — Inductive Hypothesis (1)
6. $\Delta \vdash P \cdot$ `hence` $A \wedge B$ false : $A \wedge B$ false — Lemma 7 (5, 4)

Case: $\dfrac{\Delta \vdash A \text{ true}}{\Delta \vdash A \vee B \text{ true}} \vee T_L$

1. $\Delta \vdash A$ true — Subderivation
2. A true $\in \Delta, A$ true — mem$F$ Rule
3. Δ, A true $\rightarrow A \vee B$ true — $\rightarrow\vee T_L$ Rule (2)
4. Δ, A true $\vdash$ `hence` $A \vee B$ true : $A \vee B$ true — step$P$ Rule (3)
5. $\Delta \vdash P : A$ true — Inductive Hypothesis (1)
6. $\Delta \vdash P \cdot$ `hence` $A \vee B$ true : $A \vee B$ true — Lemma 7 (5, 4)

Case: $\dfrac{\Delta \vdash B \text{ true}}{\Delta \vdash A \vee B \text{ true}} \vee T_R$

1. $\Delta \vdash B$ true — Subderivation
2. B true $\in \Delta, B$ true — mem$F$ Rule
3. Δ, B true $\rightarrow A \vee B$ true — $\rightarrow\vee T_R$ Rule (2)
4. Δ, B true $\vdash$ `hence` $A \vee B$ true : $A \vee B$ true — step$P$ Rule (3)
5. $\Delta \vdash P : B$ true — Inductive Hypothesis (1)
6. $\Delta \vdash P \cdot$ `hence` $A \vee B$ true : $A \vee B$ true — Lemma 7 (5, 4)

Case: $\dfrac{\Delta \vdash A \text{ false} \quad \Delta \vdash B \text{ false}}{\Delta \vdash A \vee B \text{ false}} \vee F$

1. $\Delta \vdash A$ false — Subderivation
2. $\Delta \vdash B$ false — Subderivation
3. A false $\in \Delta, A$ false$, B$ false
4. B false $\in \Delta, A$ false$, B$ false — mem$F$ Rule
5. Δ, A false$, B$ false $\rightarrow A \vee B$ false — $\rightarrow\vee F$ Rule (3, 4)
6. Δ, A false$, B$ false $\vdash$ `hence` $A \vee B$ false : $A \vee B$ false — step$P$ Rule (5)
7. $\Delta \vdash P_2 : B$ false — Inductive Hypothesis (2)
8. Δ, A false $\vdash P_2 : B$ false — Lemma 6 (7)
9. Δ, A false $\vdash P_2 \cdot$ `hence` $A \vee B$ false : $A \vee B$ false — Lemma 7 (8, 6)
10. $\Delta \vdash P_1 : A$ false — Inductive Hypothesis (1)
11. $\Delta \vdash P_1 \cdot (P_2 \cdot$ `hence` $A \vee B$ false$) : A \vee B$ false — Lemma 7 (10, 9)

Case: $\dfrac{\Delta, A \text{ true} \vdash B \text{ true}}{\Delta \vdash A \supset B \text{ true}} \supset T$

1. Δ, A true $\vdash B$ true — Subderivation
2. $(A$ true $\vdash B$ true$) \in \Delta, (A$ true $\vdash B$ true$)$ — mem$F$ Rule
3. $\Delta, (A$ true $\vdash B$ true$) \rightarrow A \supset B$ true — $\rightarrow\supset T$ Rule (2)
4. $\Delta, (A$ true $\vdash B$ true$) \vdash$ `hence` $A \supset B$ true : $A \supset B$ true — step$P$ Rule (3)
5. Δ, A true $\vdash P : B$ true — Inductive Hypothesis (1)
6. $\Delta \vdash [A$ true$; P];$ `hence` $A \supset B$ true : $A \supset B$ true — hyp$P$ Rule (5, 4)

Case: $\dfrac{\Delta \vdash A \text{ true} \quad \Delta \vdash B \text{ false}}{\Delta \vdash A \supset B \text{ false}} \supset F$

1. $\Delta \vdash A$ true — Subderivation
2. $\Delta \vdash B$ false — Subderivation
3. A true $\in \Delta, A$ true$, B$ false
4. B false $\in \Delta, A$ true$, B$ false — mem$F$ Rule
5. Δ, A true$, B$ false $\rightarrow A \supset B$ false — $\rightarrow\supset F$ Rule (3, 4)
6. Δ, A true$, B$ false $\vdash$ `hence` $A \supset B$ false : $A \supset B$ false — step$P$ Rule (5)
7. $\Delta \vdash P_2 : B$ false — Inductive Hypothesis (2)
8. Δ, A true $\vdash P_2 : B$ false — Lemma 6 (7)
9. Δ, A true $\vdash P_2 \cdot$ `hence` $A \supset B$ false : $A \supset B$ false — Lemma 7 (8, 6)
10. $\Delta \vdash P_1 : A$ true — Inductive Hypothesis (1)
11. $\Delta \vdash P_1 \cdot (P_2 \cdot$ `hence` $A \supset B$ false$) : A \supset B$ false — Lemma 7 (10, 9)

Case: $\dfrac{\Delta, A \text{ true} \vdash \#}{\Delta \vdash A \text{ false}} \#EF$

1. Δ, A true $\vdash \#$ — Subderivation
2. $(A$ true $\vdash \#) \in \Delta, (A$ true $\vdash \#)$ — mem$F$ Rule
3. $\Delta, (A$ true $\vdash \#) \rightarrow A$ false — $\rightarrow\#EF$ Rule (2)
4. $\Delta, (A$ true $\vdash \#) \vdash$ `hence` A false : A false — step$P$ Rule (3)
5. Δ, A true $\vdash P : \#$ — Inductive Hypothesis (1)
6. $\Delta \vdash [A$ true$; P];$ `hence` A false : A false — hyp$P$ Rule (5, 4)

The remaining cases were given in Section 6.

□

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