18-847F: Special Topics in Computer Systems

Foundations of Cloud and Machine Learning Infrastructure

Lecture 4: Basics of Queueing Theory

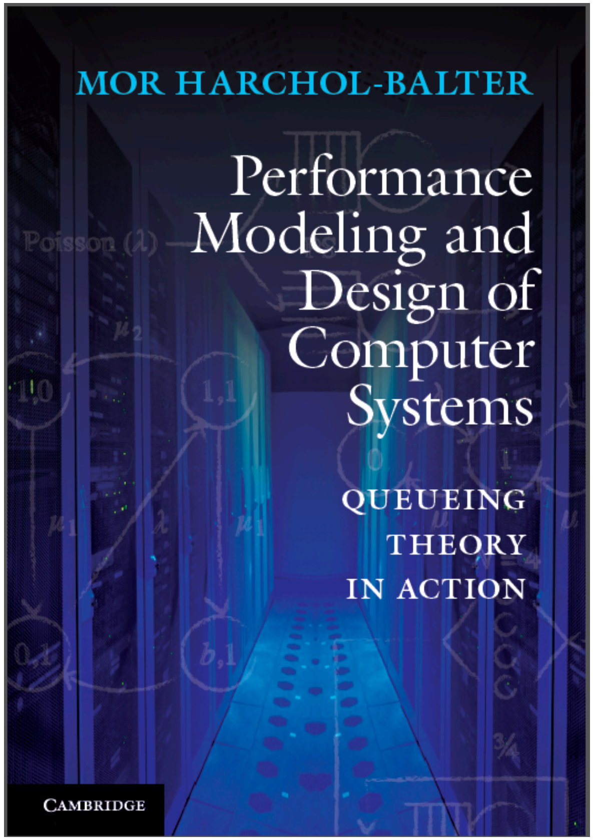
Foundations of Cloud and Machine Learning Infrastructure

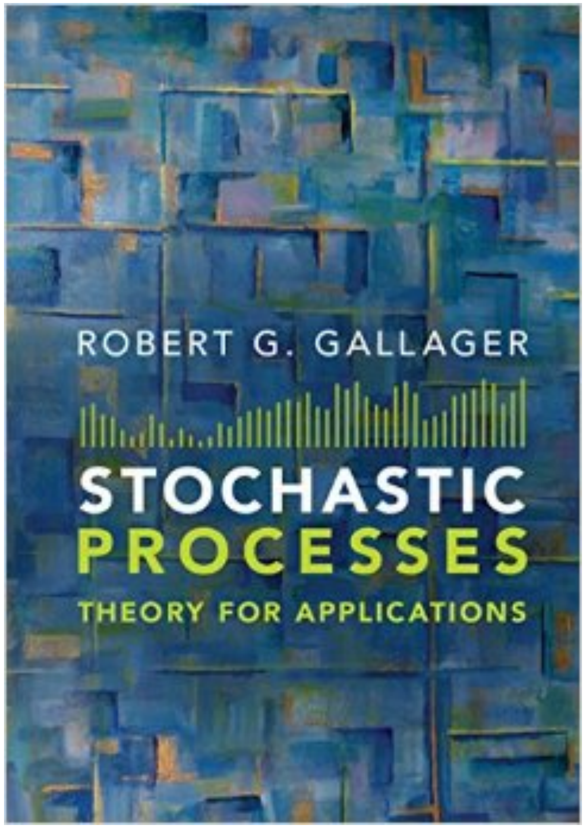
Announcements

- Has everybody submitted their paper reviews?
- Sign-up for Class Presentations
- After your talk, please upload the slides to Canvas
- New TA: Ankur Mallick

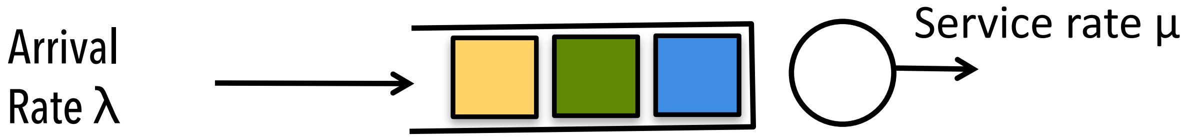
Queueing Theory

Reference Textbooks



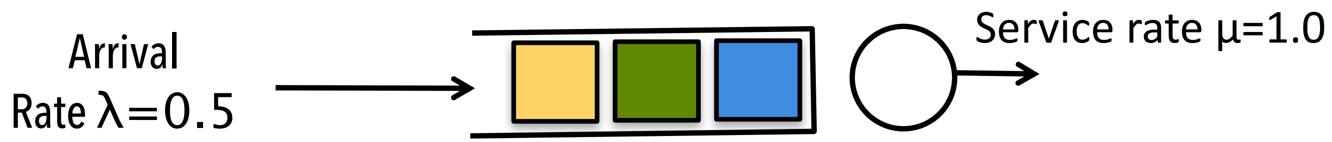


Queueing Terminology



Mean Service Time	$E[S] = 1/\mu$
Mean Waiting Time	$E[W]$
Mean Response Time	$E[T] = E[W] + E[S]$
Mean # Customers in Queue	$E[N]$
Server Utilization or Load	$\rho = \lambda/\mu$

Exercise: First-come first-served Queue



t =0	Yellow job arrives
t = 2.5	Blue job leaves
t= 4	Green job leaves
t = 5	Yellow job leaves at time t = 5

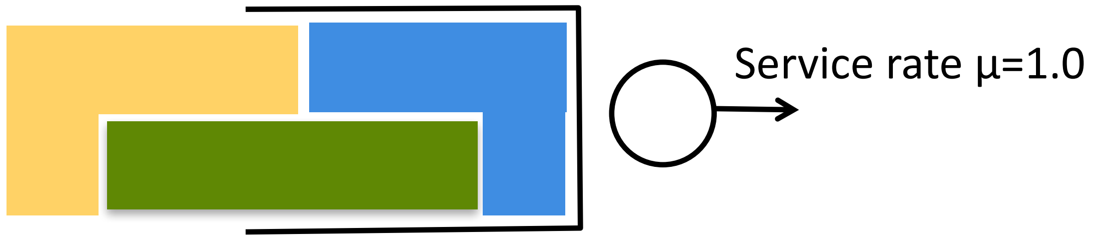
Q1: Waiting Time W of the yellow job?

Q2: Service Time S of the yellow job?

Q3: Response time T of the yellow job?

Q4: Load on the system? What happens if $\lambda=1.1$?

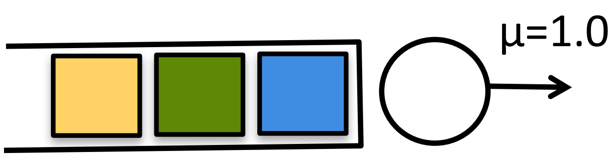
Processor-Sharing Queues

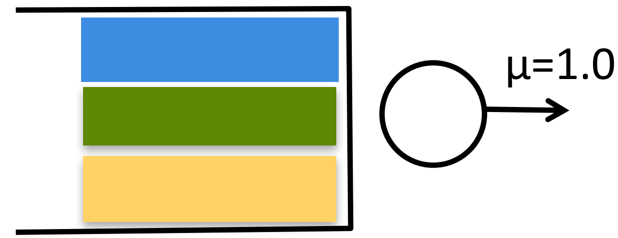


t =0	Blue job arrives
t = 0.5	Green job arrives
t= 1.5	Yellow job arrives
t = 1.5	Blue job leaves
t = 2.5	Green job leaves
t = 3.0	Yellow job leaves

First-come First-served vs. Processor-Sharing

Which is better in terms of E[T]?

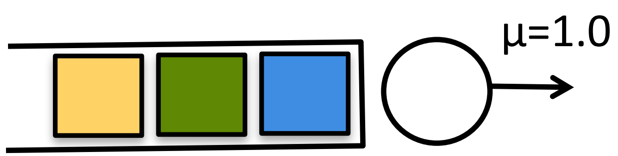


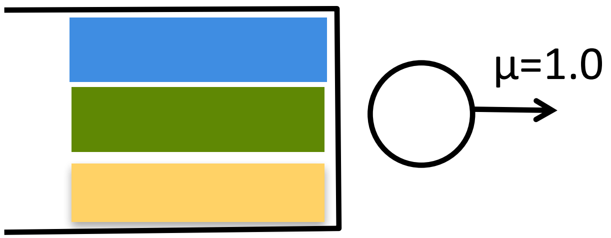


Suppose that all jobs arrive at time t = 0, and service time is deterministic, 1 sec per job

First-come First-served vs. Processor-Sharing

Which is better in terms of E[T]?





Suppose that all jobs arrive at time t = 0, and service time is deterministic, 1 sec per job

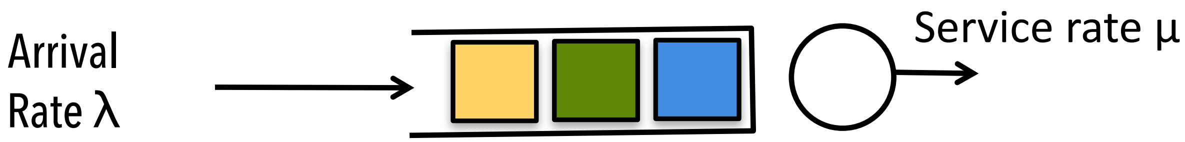
$E[T_{blue}] = 1.0$
$E[T_{green}] = 2.0$
$E[T_{yellow}] = 3.0$

$E[T_{blue}] = 3.0$
$E[T_{green}] = 3.0$
$E[T_{yellow}] = 3.0$

Then why use processor-sharing?

- To avoid starving small jobs that get stuck behind large ones
- For jobs that interact with each other

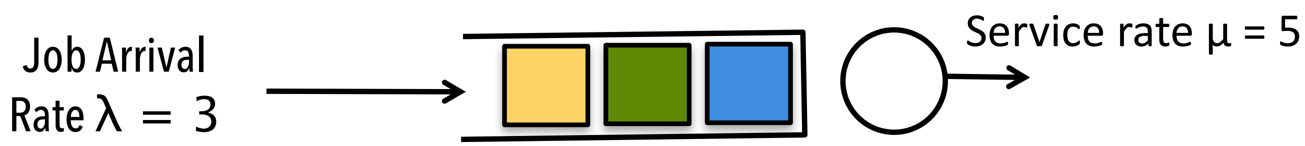
We will focus on FCFS jobs in this lecture



Mean Service Time	$E[S] = 1/\mu$
Mean Waiting Time	$E[W]$
Mean Response Time	$E[T] = E[W] + E[S]$
Mean # Customers in Queue	$E[N]$
Server Utilization or Load	$\rho = \lambda/\mu$

Design Question 1

What if the arrival rate doubles?

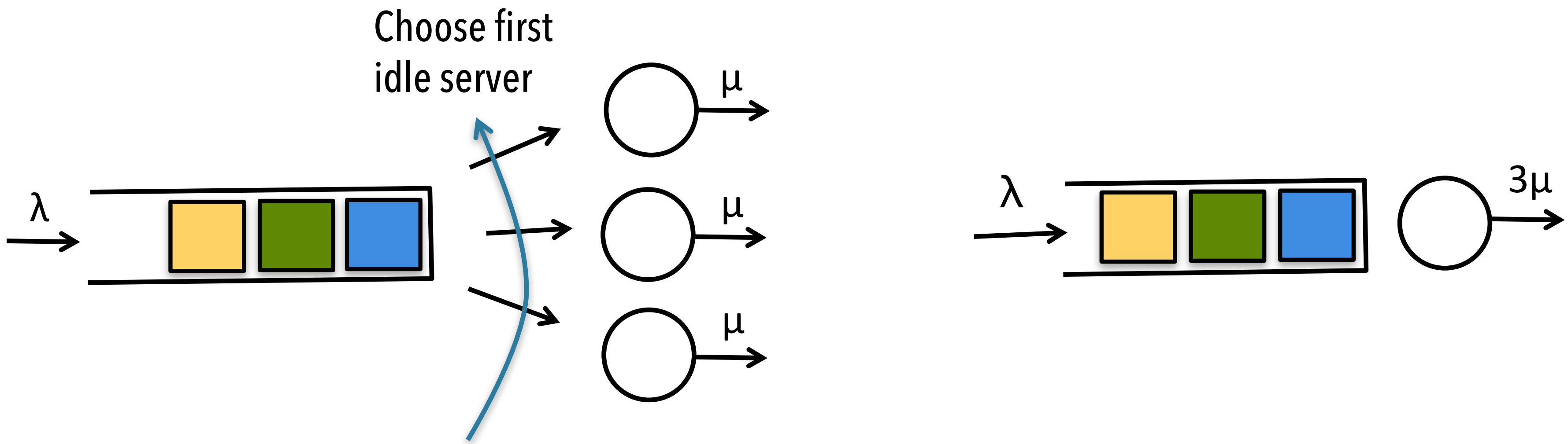


Mean Response Time T = Waiting time in Queue + Service Time

Q: If λ doubles, do you need a server of 2x rate to achieve the same E[T]?

Design Question 2

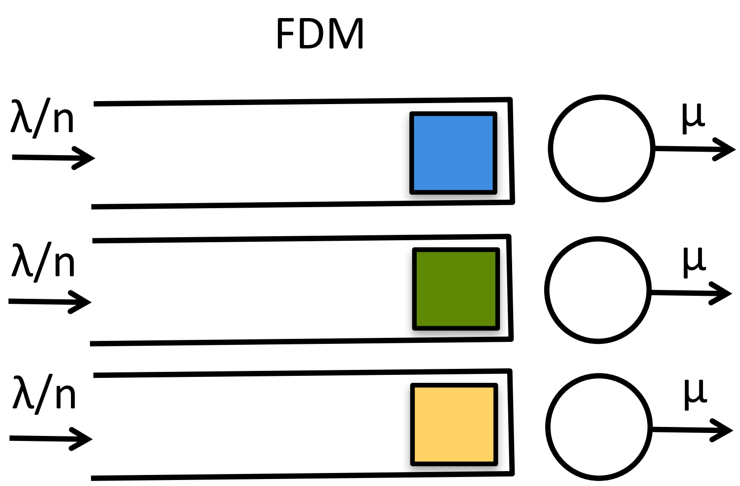
Many slow, or more fast server?

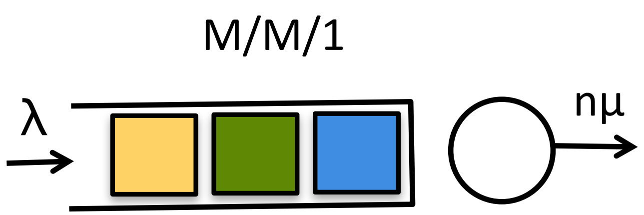


Q: Which of the two systems gives lower E[T]?

Design Question 3

Many slow, or more fast server?





Q: Which of the two systems gives lower E[T]?

Little's Law

Theorem: For any ergodic open system we have

$$E[N] = \lambda E[T]$$

Very general and hence powerful law

- Any # of servers, scheduling policy, queue size limit

Some Variants

$$E[N_w] = \lambda E[W]$$

$$\rho = \lambda E[S]$$

Little’s Law: Exercise

A professor takes 2 new students in even-numbered years, and 1 new student in odd-numbered years.

If avg. graduation time = 6 yrs, how many students will the professor have on average?

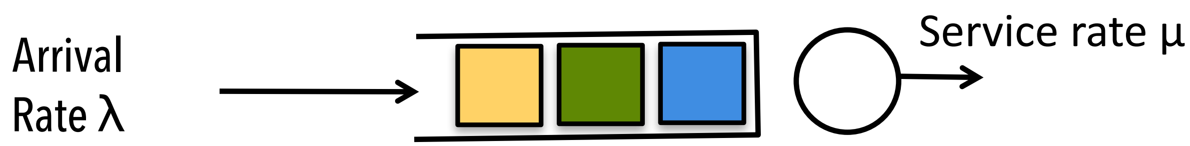
Little's Law: Answer

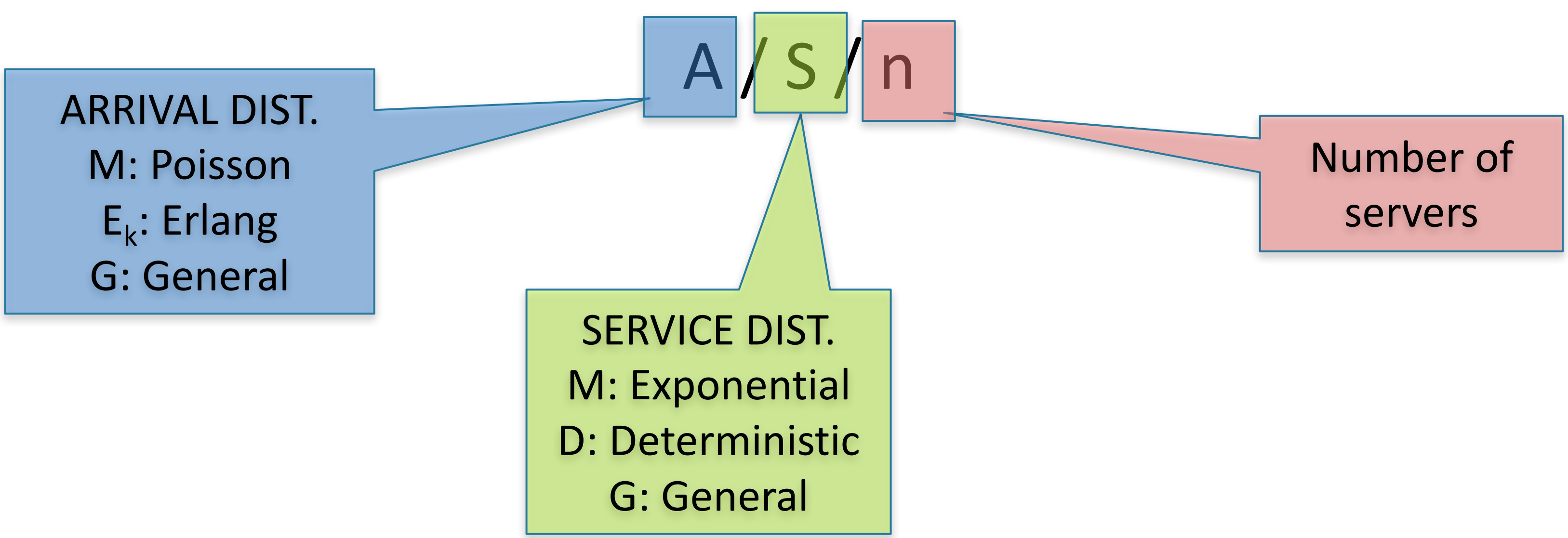
A professor takes 2 new students in even-numbered years, and 1 new student in odd-numbered years.

If avg. graduation time = 6 yrs, how many students will the professor have on average?

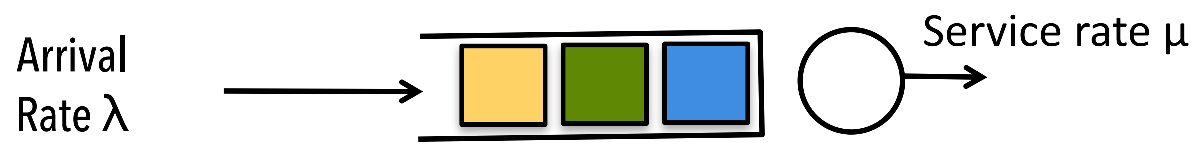
$$
\begin{aligned}
E[N] &= \lambda\, E[T] \\
&= 1.5 * 6 \\
&= 9
\end{aligned}
$$

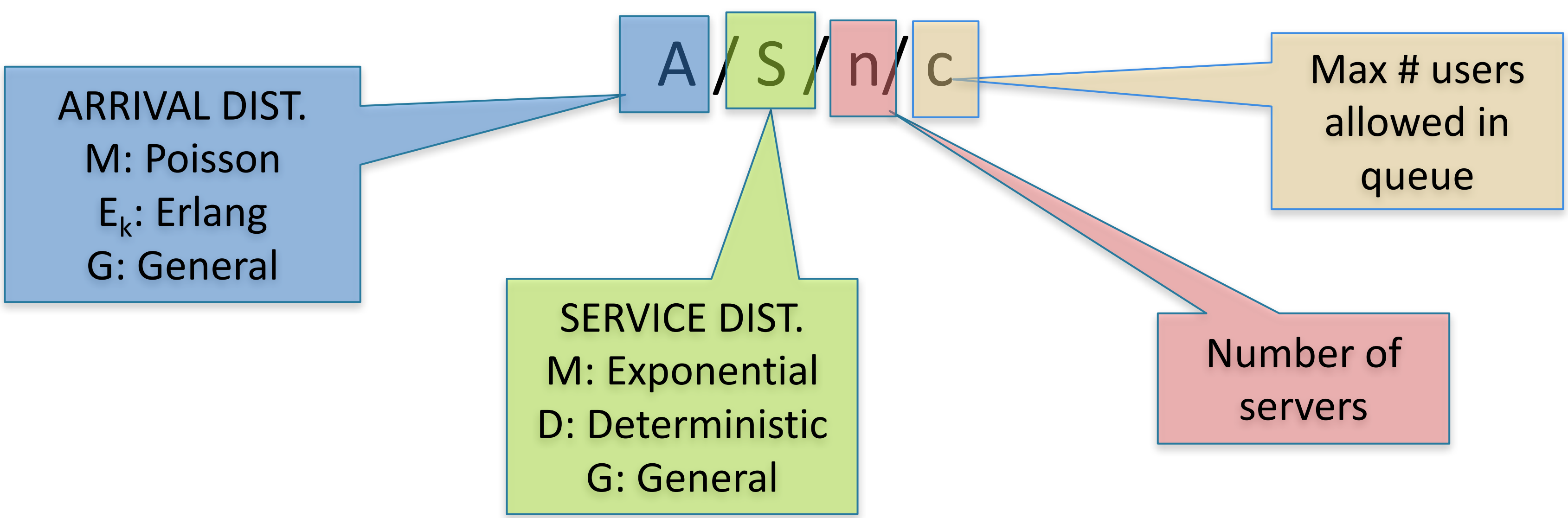
Kendall's Notation



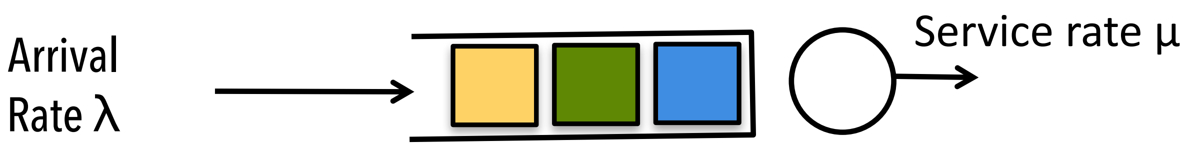


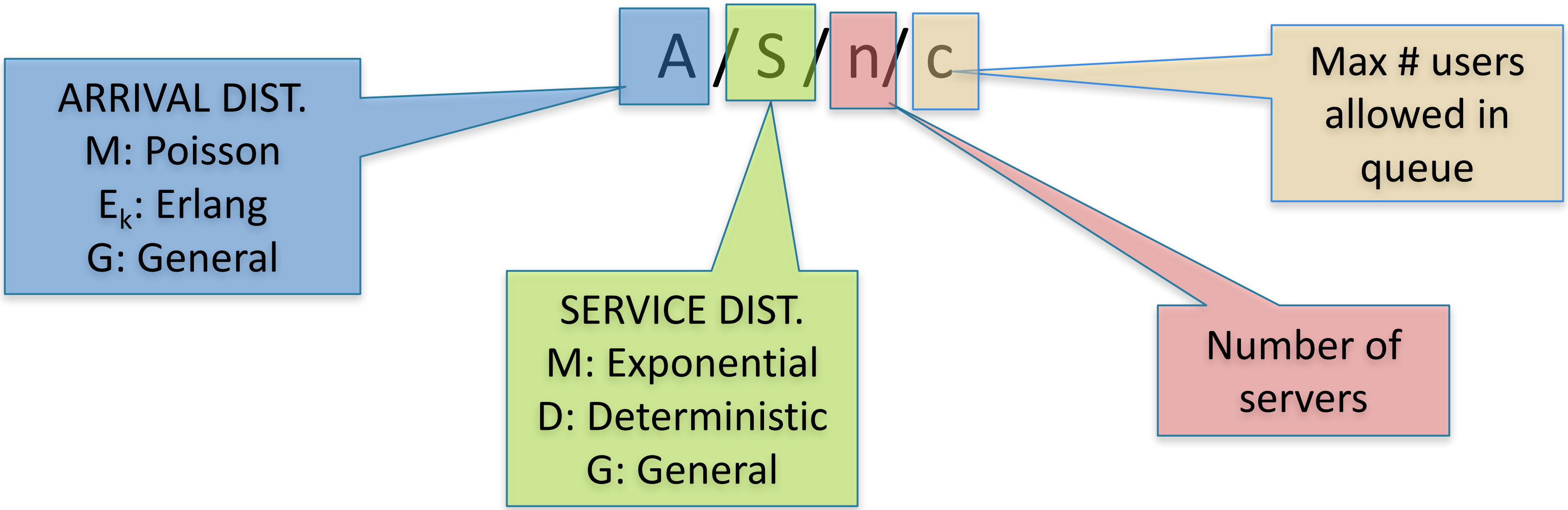
Kendall's Notation



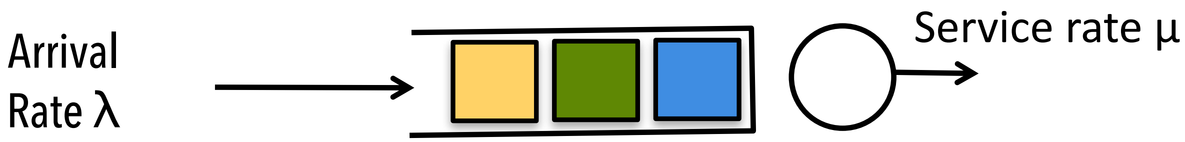


Exercise: What are the distributions of Poisson and Exponential random variables?





M/M/1 Queue

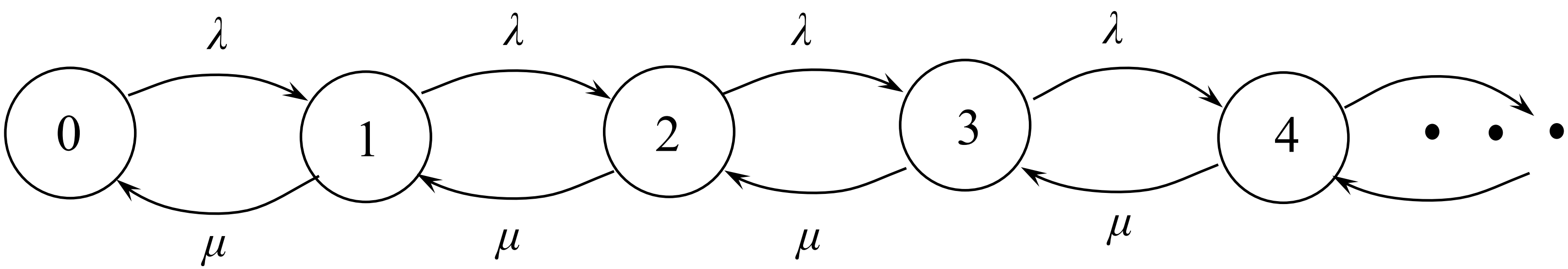


WANT TO FIND

1. Mean Response Time E[T]

2. Mean Waiting Time E[W]

M/M/1: Markov Model

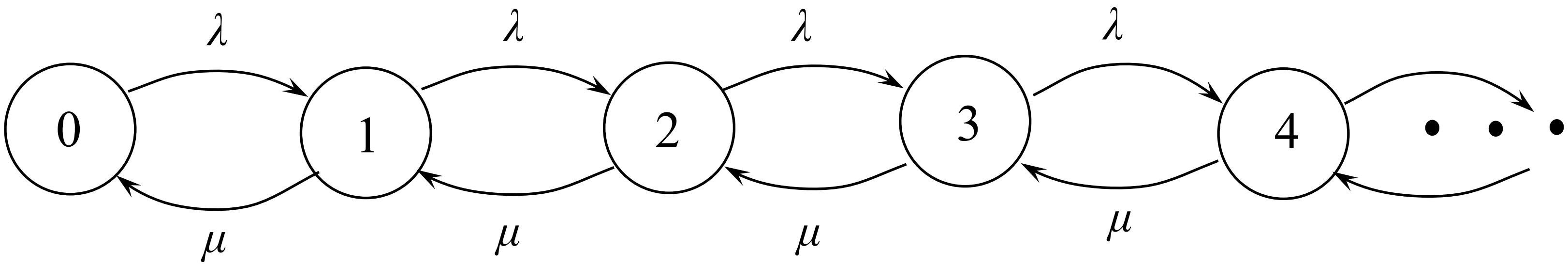


$$\pi_i = \rho^i(1-\rho)$$
$$\pi_0 = (1-\rho)$$

where $\rho = \frac{\lambda}{\mu}$

$$\mathbb{E}[N] = \sum_{i=0}^{\infty} i\pi_i = \rho(1-\rho)\sum_{i=1}^{\infty} i\rho^{i-1} = \frac{\rho}{1-\rho}$$

M/M/1: Mean Response Time



$$\mathbb{E}[N] = \sum_{i=0}^{\infty} i\pi_i = \rho(1-\rho)\sum_{i=1}^{\infty} i\rho^{i-1} = \frac{\rho}{1-\rho}$$

$$\mathbb{E}[T] = \frac{\mathbb{E}[N]}{\lambda} = \frac{\rho}{\lambda(1-\rho)} = \boxed{\frac{1}{\mu-\lambda}}$$

$$\mathbb{E}[W] = \frac{1}{\mu-\lambda} - \frac{1}{\mu} = \boxed{\frac{\rho}{\mu-\lambda}}$$

Exercise: Design Question 1

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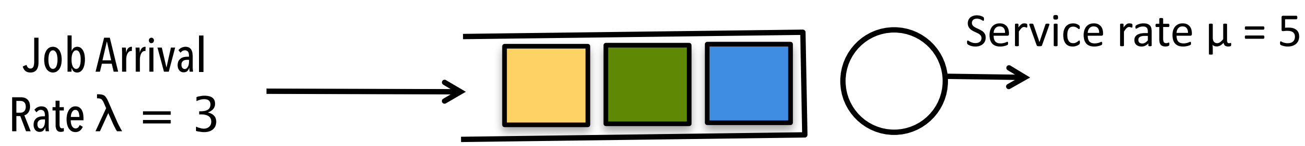
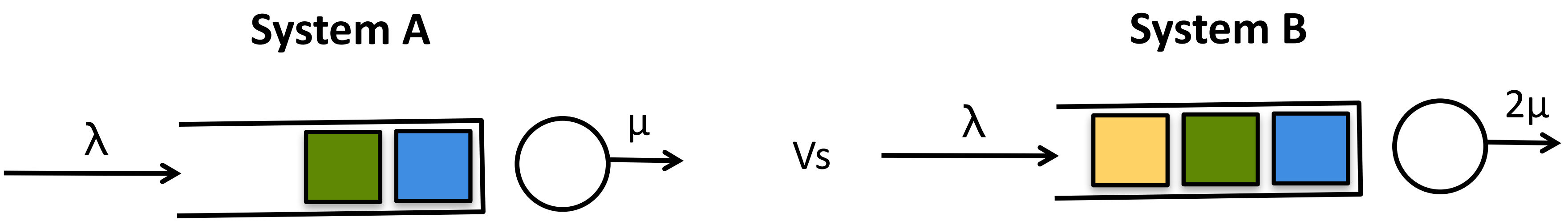


Mean Response Time T = Waiting time in Queue + Service Time

Q: If λ doubles, do you need a server of 2x rate to achieve the same E[T]?

A: Service rate 6+2 = 8 is sufficient

Exercise: M/M/1 Queue
What if the service rate doubles?

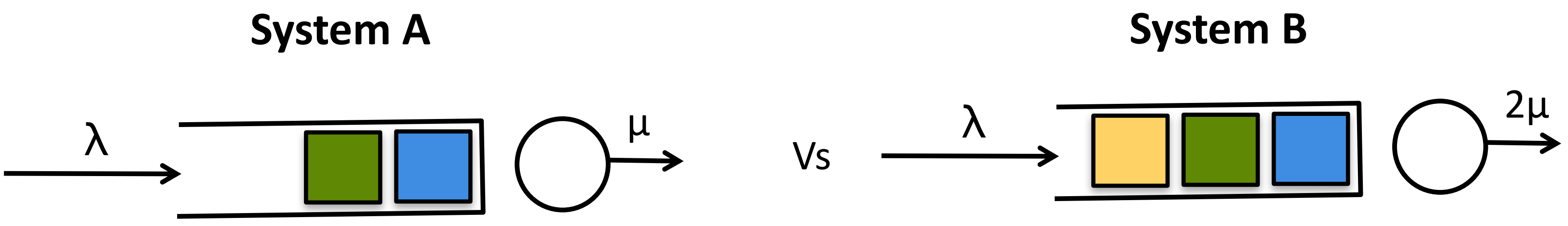


Q: Is the first queue twice (or more) longer than the second?

What is $E[W^{(A)}] / E[W^{(B)}]$ as a function of $\rho = \lambda/\mu$?

Exercise: M/M/1 Queue

What if the service rate doubles?

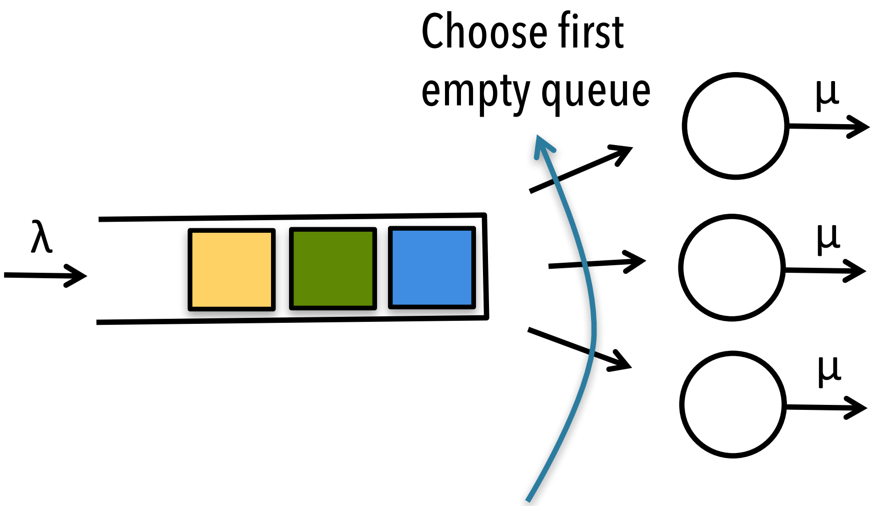


Q: Is the first queue twice (or more) longer than the second?

What is $E[W^{(A)}] / E[W^{(B)}]$ as a function of $\rho = \lambda/\mu$?

ANSWER: $$\frac{2(2-\rho)}{1-\rho}$$

M/M/n Queue



WANT TO FIND

1. Mean Response Time E[T]
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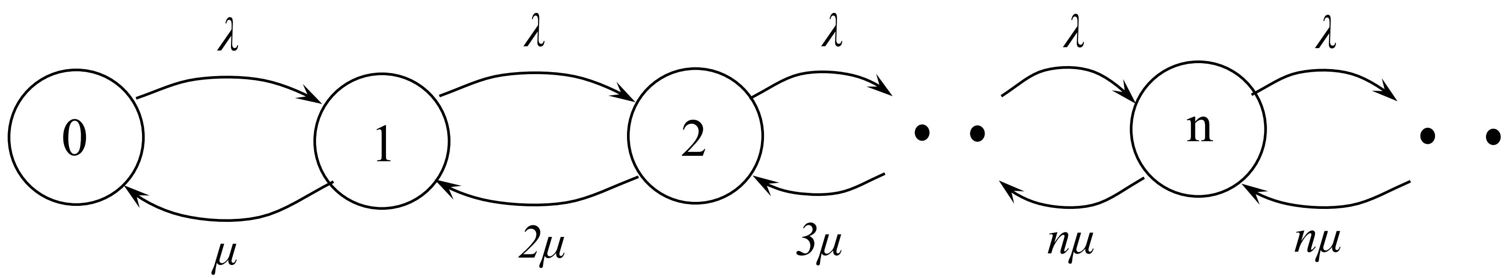
M/M/n Queue



$$P_Q = \sum_{i=n}^{\infty} \pi_i$$

$$\rho = \frac{\lambda}{n\mu}$$

$$= \pi_0 \frac{n^n}{n!} \sum_{i=n}^{\infty} \rho^i$$

where $\pi_0 = \left[\sum_{i=0}^{n-1} \frac{(n\rho)^i}{i!} + \frac{(n\rho)^n}{n!(1-\rho)} \right]^{-1}$

$$= \frac{n^n \pi_0}{n!(1-\rho)}$$

Erlang-C Formula

Used in call centers to determine number of agents required

M/M/n Queue

$$\mathbb{E}[N_w] = \sum_{i=n}^{\infty} \pi_i (i - n)$$

$$= \pi_0 \sum_{i=n}^{\infty} \frac{\rho^i n^n}{n!} (i - n)$$

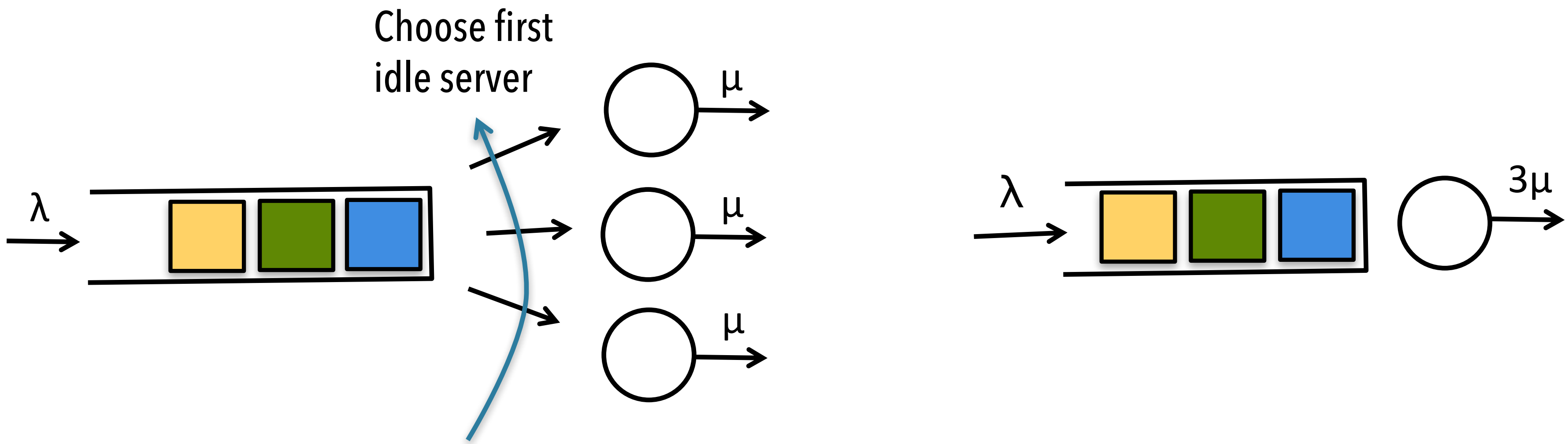
$$= P_Q \frac{\rho}{1 - \rho}$$

$$\mathbb{E}[W] = \frac{\mathbb{E}[N_w]}{\lambda} = P_Q \frac{\rho}{\lambda(1 - \rho)}$$

$$\mathbb{E}[T] = P_Q \frac{\rho}{\lambda(1 - \rho)} + \frac{1}{\mu}$$

Design Question 2

Many slow, or more fast server?



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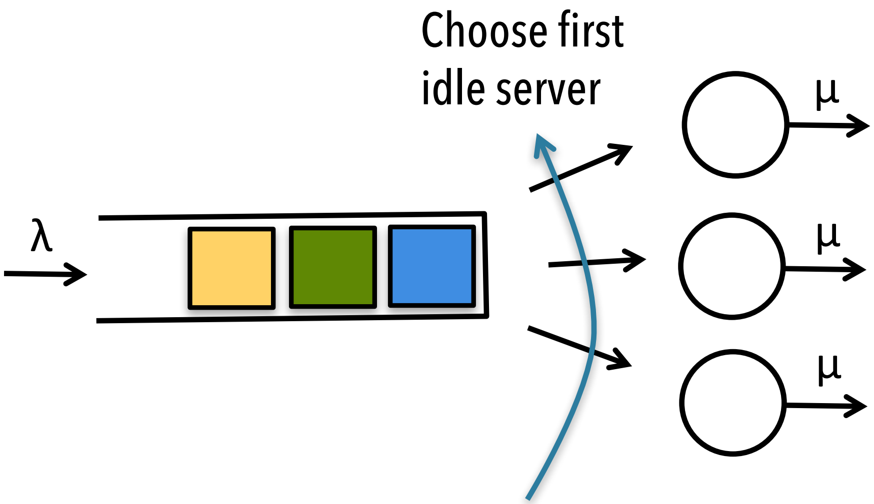
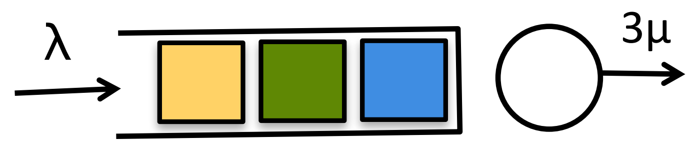




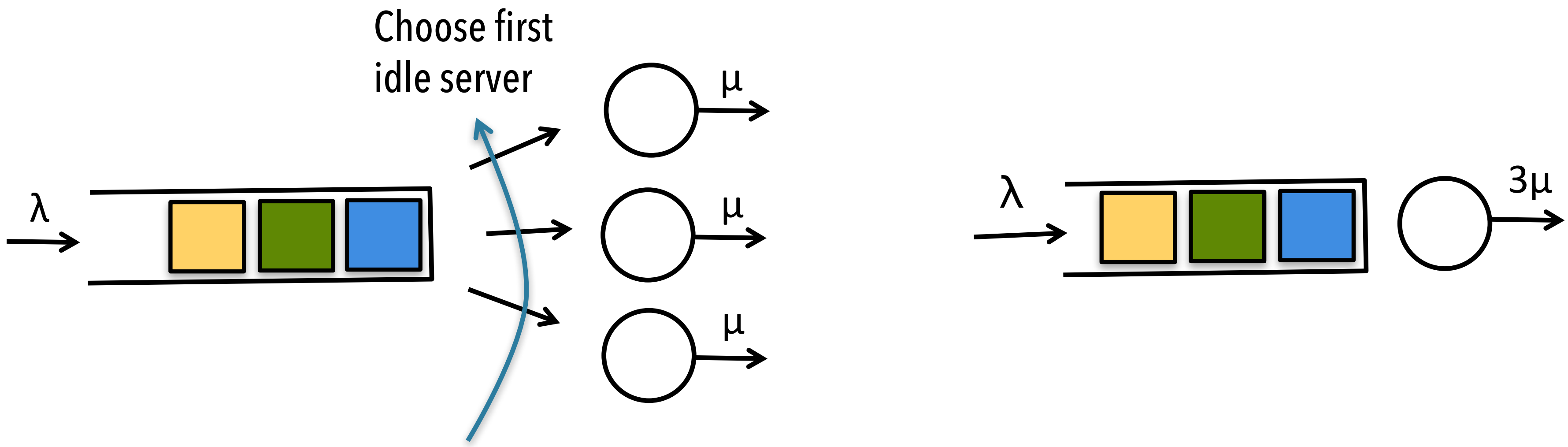
$$\mathbb{E}[T]^{M/M/n} = P_Q \frac{\rho}{\lambda(1-\rho)} + \frac{1}{\mu}$$

$$\mathbb{E}[T]^{M/M/1} = \frac{\rho}{\lambda(1-\rho)}$$

$$\text{System Load } \rho = \frac{\lambda}{3\mu}$$

Design Question 3

Many slow, or more fast server?



M/M/n is n times slower when ρ→0

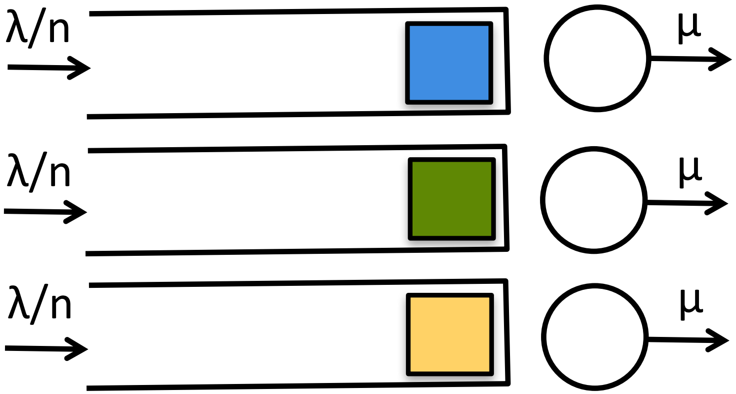
$$\frac{\mathbb{E}[T]^{M/M/n}}{\mathbb{E}[T]^{M/M/1}} = P_Q + n(1-\rho)$$

M/M/n and M/M/1 are almost equal when ρ→ 1

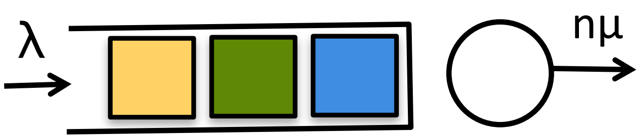
Design Question 3

Many slow, or more fast server?

Freq. Division Multiplexing (FDM)



M/M/1



Design Question 3

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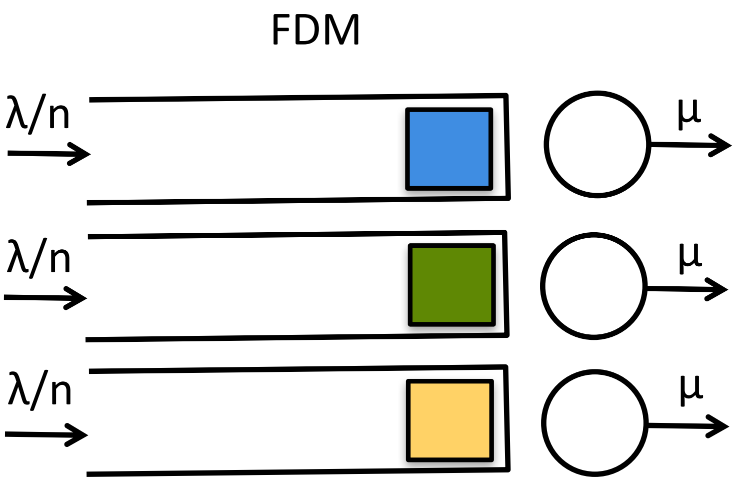
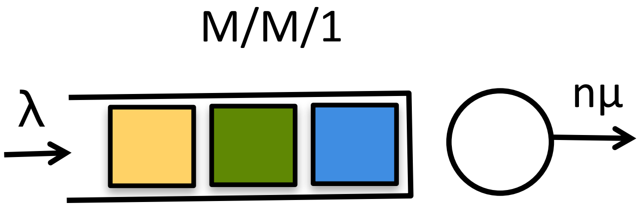




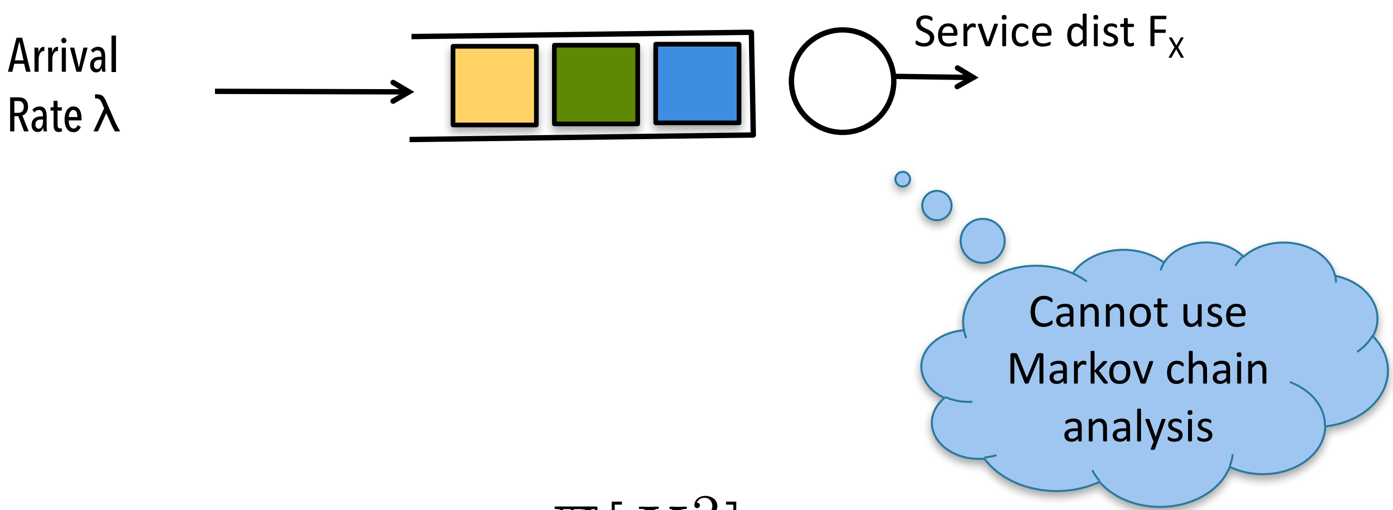
$$\mathbb{E}[T]^{FDM} = \frac{n}{n\mu - \lambda}$$

$$\mathbb{E}[T]^{M/M/1} = \frac{1}{n\mu - \lambda}$$

FDM is n times slower than M/M/1

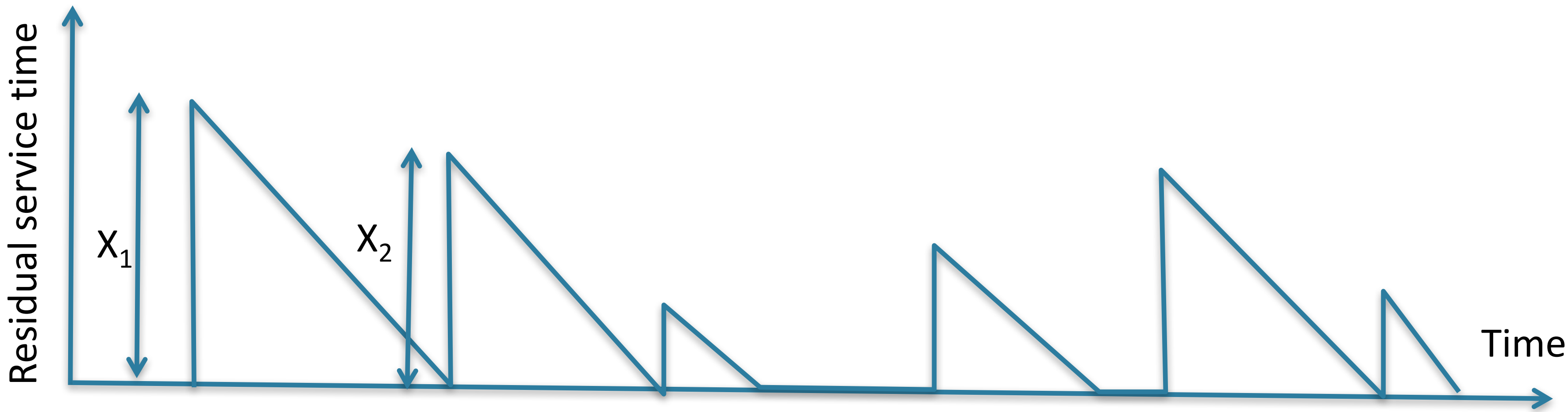
M/G/1 Queue

Pollaczek-Khinchine Formula



$$\mathbb{E}[T] = \mathbb{E}[X] + \frac{\mathbb{E}[X^2]}{2(1-\lambda\mathbb{E}[X])}$$

Proof of PK formula

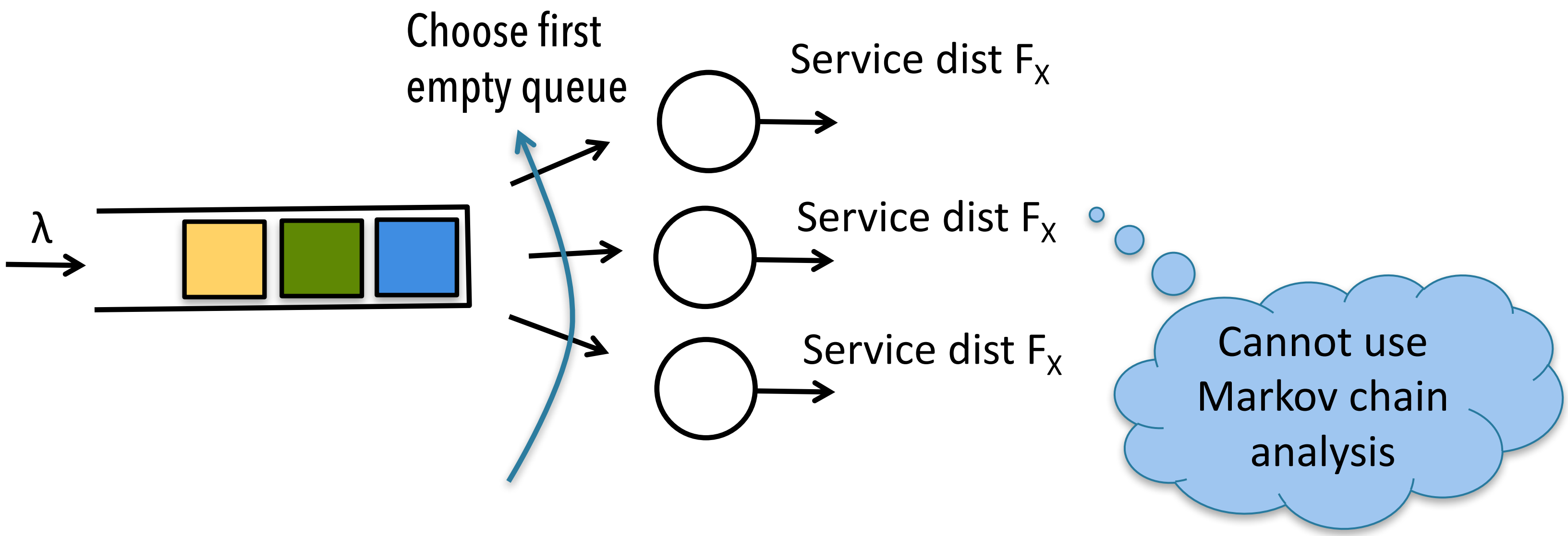


$$\mathbb{E}[T_w] = \mathbb{E}[N_w] \cdot \mathbb{E}[X] + E[R]$$

$$= \lambda \mathbb{E}[T_w] \cdot \mathbb{E}[X] + \frac{\mathbb{E}[X^2]}{2}$$

$$= \frac{\mathbb{E}[X^2]}{2(1 - \lambda \mathbb{E}[X])}$$

M/G/n Queue



$$\mathbb{E}[T] \approx \mathbb{E}[X] + \frac{\mathbb{E}[X^2]}{2\mathbb{E}[X]} \cdot \mathbb{E}[W^{M/M/n}]$$