Neighborhood Semantics for Modal Logic

Propositional and First-Order Systems

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Modality

- Going back to Aristotle, and then with the beginnings of modern modal logic, the modalities of interest were ***alethic***: $\Box$ as necessity, $\Diamond$ as contingency.

- Thus we find, say, Carnap in *Meaning and Necessity* struggling to give a semantic account of $\Box$ as analyticity, or logical truth, or necessity.

- Adequate working semantics were eventually devloped, of course, by Kripke and others. This semantics is based on possible worlds, or states, linked together by an ***accessibility relation***.

Propositional Kripke Semantics

- A ***Kripke frame*** is a pair $\mathcal{F} = \langle W, R \rangle$, where W is a set of worlds (states), and $R \subseteq W^2$ is the accessibility relation, which is intended to capture the notion of one world being possible relative to another.
- A ***Kripke model*** is then a pair $\mathcal{M} = \langle \mathcal{F}, V \rangle$, where $\mathcal{F}$ is a Kripke frame and V is a ***valuation***

$$V : \text{At} \longrightarrow 2^W$$

assigning each atomic proposition letter to a set of worlds.

Propositional Kripke Semantics

Truth is defined inductively in the usual way:

1. $\mathcal{M}, w \models p \Leftrightarrow w \in V(p)$ (atomic),
2. obvious boolean clauses,
3. $\mathcal{M}, w \models \Box p \Leftrightarrow \mathcal{M}, u \models p$ for all u such that wRu.

- This semantics only works to characterize ***normal*** modal logics, which can be defined as those with the scheme $\Diamond p \leftrightarrow \neg\Box\neg p$, and closed under the inference rule

$$\frac{(p_1 \wedge \cdots \wedge p_n) \to q}{(\Box p_1 \wedge \cdots \wedge \Box p_n) \to \Box q}\ \mathrm{RK}$$

Other Modalities

- Of course nowadays people use modal logics for a panoply of situations in which non-alethic modalities are the notions of interest.
- Using $\Box$ as a knowledge operator or belief operator, for instance.
- Even,say, modelling distributed computation, with possible worlds as nodes of a network; values of type $\Box p$ can represent mobile code which is executable at any node, while values of type $\Diamond p$ represent addresses remote values of type p. (Pfenning, et al.)

Other Modalities

- A potentially objectionable property for a knowledge operator to have is ***logical omniscience***:

$$\Box p \wedge (p \rightarrow q) \rightarrow \Box q.$$

- For a likelihood operator (based on some fixed probability threshhold), we would want to avoid ***adjunction***: $\Box p \wedge \Box q \rightarrow \Box (p \wedge q)$.
- But these schemes will hold in any normal modal logic, and so the usual Kripke semantics do not work for such modal operators.

Propositional Neighborhood Semantics

- A ***neighborhood frame*** is a pair $\mathcal{F} = \langle W, N \rangle$, where W is a set of worlds (states), and N is a ***neighborhood function***

  $$N : W \to 2^{2^W}$$

  assigning a collection of sets of worlds to each world in W.

- A ***neighborhood model*** is then a pair $M = \langle \mathcal{F}, V \rangle$, where $\mathcal{F}$ is a neighborhood frame and V is a valuation.

Propositional Neighborhood Semantics

- The inductive clauses for truth in a model are exactly as for Kripke semantics, with the exception of the modal one:

  $$\mathcal{M}, w \models \Box p \Leftrightarrow ||p||^{\mathcal{M}} \in N(w),$$

  where $||p||^{\mathcal{M}} = \{u \in W \mid \mathcal{M}, u \models p\}$ is called the ***truth set***.

- Kripke: to decide whether $\Box p$ holds at a world w, we check whether p holds at all worlds u accessible from w (wRu).

- Neighborhood: to decide whether $\Box p$ holds at w, we check whether the truth set for p is in $N(w)$.

Propositional Neighborhood Semantics

- Note that we enforce no structure on the neighborhood function. Neighborhoods can consist of any $A \subseteq 2^{2^W}$.
- Thus we can invalidate schemes like adjunction, say.

$$\Box p \wedge \Box q \rightarrow \Box(p \wedge q)$$

will be valid in a neighborhood frame iff $N(p)$ is closed under intersections for every p.

- We can determine all classical modal logics, which are those satisfying the scheme $\Diamond p \leftrightarrow \neg\Box\neg p$, and closed under the rule

$$\frac{p \leftrightarrow q}{\Box p \leftrightarrow \Box q}\,\mathrm{RE}$$

A Bridge to Kripke

- A neighborhood frame $\mathcal{F} = \langle W, N \rangle$ is ***augmented*** if, for all $w \in W$,
  1. $X \cap Y \in N(w) \;\Rightarrow\; X, Y \in N(w)$,
  2. $\bigcap N(w) \in N(w)$.

Theorem. *For every Kripke model* $\mathcal{M} = \langle W, R, V \rangle$*, there is a pointwise equivalent augmented neighborhood model* $\mathcal{N} = \langle W, N, V \rangle$*, and vice versa.*

- It is in this sense that neighborhood semantics can be called a generalization of Kripke semantics.

A Bridge to Kripke

Proof Sketch (see Chellas). To get $\mathcal{N}$ from $\mathcal{M}$ we set

$$X \in N(w) \;\Leftrightarrow\; \{u \in W \mid wRu\} \subseteq X.$$

That a world satisfies the same sentences in each model is proved by induction on sentence complexity, including the step

$$\begin{aligned}\mathcal{M}, w \models \Box\varphi \quad &\Leftrightarrow \forall u(wRu \to \mathcal{M}, u \models \varphi)\\ &\Leftrightarrow \{u \in W \mid wRu\} \subseteq ||\varphi||^{\mathcal{M}}\\ &\Leftrightarrow ||\varphi||^{\mathcal{N}} \in N(w) \Leftrightarrow \mathcal{N}, w \models \Box\varphi\end{aligned}$$

For the other direction define $wRu \;\Leftrightarrow\; u \in \bigcap N(w)$, and the proof is similar. ☐

First-Order Kripke Semantics

- We saw in the propositional setting that n’hood semantics allows us to do things that Kripke does not; there is more to be had in the first-order setting.
- For first-order Kripke frames, we have ***varying domains***: each world w has a domain D_w which need bear no relation to other worlds’ domains.
- If we want to have ***constant domains***, it is at the cost of validating both the Barcan and Converse Barcan schemes:

$$\text{(BF)} \quad \forall x \Box \varphi \rightarrow \Box \forall x \varphi,$$

$$\text{(CBF)} \quad \Box \forall x \varphi \rightarrow \forall x \Box \varphi.$$

First-Order Kripke Semantics

- First-order Kripke frame conditions equivalent to the schemes:

$$\begin{array}{llcl} \mathrm{BF:} & wRu & \Rightarrow & D_w \supseteq D_u \\ \mathrm{CBF:} & wRu & \Rightarrow & D_w \subseteq D_u \end{array}$$

- Thus any constant domain Kripke frame necessarily validates both BF and CBF.
- Now varying domains are quite natural for modelling necessity. It seems we should be able to talk about things coming into, or passing out of, existence from alternative to alternative.

Modalities Again

- But again there are modalities for which we would like constant domains, but which should not satsify both BF and CBF.
- For instance, a belief operator $\Box$ satisfying the Barcan scheme

  $$\forall x \Box \varphi \rightarrow \Box \forall x \varphi$$

  would run afoul of Kyburg's lottery paradox.
- First-order neighborhood semantics allows for the use of constant domains without validating either BF or CBF.

First-Order Neighborhood Semantics

- A ***constant domain neighborhood frame*** is a tuple $\langle W, N, D\rangle$, where W is a set of worlds, N is a neighborhood function, and D is a non-empty set.

- A ***constant domain neighborhood model*** is then a pair $\langle \mathcal{F}, I\rangle$, where $\mathcal{F}$ is such a frame and I is an interpretation function: for any n-ary predicate F and world w, $I(F, w) \subseteq D^n$.

- An ***assignment*** is a function $\sigma : \mathcal{V} \rightarrow D$, mapping variables to elements of the domain. Truth at a world in the model will be defined relative to an assignment ...

First-Order Neighborhood Semantics

An assignment ρ is an x***-variant*** of σ, written $\rho \sim_x \sigma$, if $\rho(y) = \sigma(y)$ for all variables y except (possibly) x.

1. $\mathcal{M}, w \models_\sigma F(\vec{x}) \;\Leftrightarrow\; \langle \sigma(x_0), \ldots, \sigma(x_k) \rangle \in I(F, w)$
2. $\mathcal{M}, w \models_\sigma \neg\varphi \;\Leftrightarrow\; \mathcal{M}, w \not\models_\sigma \varphi$
3. $\mathcal{M}, w \models_\sigma (\varphi \wedge \psi) \;\Leftrightarrow\; \mathcal{M}, w \models_\sigma \varphi$ and $\mathcal{M}, w \models_\sigma \psi$
4. $\mathcal{M}, w \models_\sigma \forall x \varphi(x) \;\Leftrightarrow\; \mathcal{M}, w \models_\rho \varphi(x)$ for all $\rho \sim_x \sigma$
5. $\mathcal{M}, w \models_\sigma \Box\varphi \;\Leftrightarrow\; ||\varphi||^{\mathcal{M},\sigma} \in N(w)$

   where again $||\varphi||^{\mathcal{M},\sigma} = \{u \in W \mid \mathcal{M}, u \models_\sigma \varphi\}$.

Canonical Models

A set Γ of formulas has the ***Henkin property*** if for each formula $\varphi \in \Gamma$ and each variable x, there is a variable y such that

$$\varphi[y/x] \rightarrow \forall x \varphi(x)$$

is in Γ.

Lemma (Lindenbaum-y)**.** *Let* Γ *be a consistent set of formulas of* $\mathscr{L}$*, and let* $\mathscr{L}^+$ *have countably many new variables. There is then a consistent* $\Delta \supseteq \Gamma$ *of* $\mathscr{L}^+$*-formulas that has the Henkin property.*

Proof. See, e.g., Hughes & Cresswell. ☐

Canonical Models

For any classical first-order modal logic Σ, the ***smallest canonical model*** for Σ is $\mathcal{M}_\Sigma = \langle W_\Sigma, N_\Sigma, D_\Sigma, I_\Sigma \rangle$, where

- $W_\Sigma = \{\Gamma \mid \mathrm{Max}_\Sigma(\Gamma) \wedge \mathrm{Henkin}(\Gamma)\}$
- $X \in N_\Sigma(\Gamma) \Leftrightarrow \exists\varphi(\Box\varphi \in \Gamma \wedge X = \{\Delta \mid \varphi \in \Delta \in W_\Sigma\})$
  Note that this basically says that a proposition is necessary at world Γ precisely when Γ says it should be.
- $D_\Sigma = \mathcal{V}^+$
- $\langle v_{i_0}, \ldots, v_{i_n} \rangle \in I_\Sigma(\varphi, \Gamma) \Leftrightarrow \varphi(v_{i_1}, \ldots, v_{i_n}) \in \Gamma$

In what follows, we will use σ_Σ to denote the identity assignment: $\sigma_\Sigma(x) = x$.

Canonical Models

Lemma (Truth Lemma). *For each* $\Gamma \in W_\Sigma$ *and* $\varphi \in \mathscr{L}$*,*

$$\varphi \in \Gamma \Leftrightarrow \mathcal{M}_\Sigma, \Gamma \models_{\sigma_\Sigma} \varphi.$$

Proof. Induction on φ complexity. Atomic and boolean cases are straightforward. The modal case is immediate from the construction:

$$\Box\psi \in \Gamma \Leftrightarrow |\psi|_\Sigma \in N_\Sigma(\Gamma) \Leftrightarrow \mathcal{M}_\Sigma, \Gamma \models_\sigma \Box\varphi.$$

Let $\varphi = \forall x\psi(x)$. If $\forall x\psi(x) \notin \Gamma$ then $\neg\forall x\psi(x) \in \Gamma$ (maximal), so there's a $y \in \mathcal{V}^+$ s.t. $\neg\varphi(y) \in \Gamma$ (Henkin), so $\varphi(y) \notin \Gamma$. By inductive hypothesis, $\mathcal{M}_\Sigma, \Gamma \not\models_\rho \varphi(y)$, where $\rho \sim_x \sigma_\Sigma$ with $\rho(x) = y$. Thus

$$\mathcal{M}_\Sigma, \Gamma \not\models_{\sigma_\Sigma} \forall x\psi(x).$$

$\Box$

Completeness

- Using the Truth lemma one can show

Theorem. *For any classical modal logic* Σ*,* φ *is valid in a canonical model iff* $\vdash_\Sigma \varphi$*.*

- Noting that we call the minimal classical modal logic $\mathbf{E}$, and

$$\mathbf{M} = \mathbf{E} + (\Box(\varphi \wedge \psi) \to \Box\varphi \wedge \Box\psi),$$

$$\mathbf{K} = \mathbf{E} + (\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)),$$

we remark that Arló-Costa and Pacuit have shown, for instance, ...

Assorted Frame Completeness Results

Corollary. $\mathrm{FOL} + \mathrm{E}$ *is sound and complete for the class of all constant domain neighborhood frames.*

Corollary. $\mathrm{FOL} + \mathrm{M}$ *is sound and complete for the class of all supplemented constant domain neighborhood frames.*

Corollary. $\mathrm{FOL} + \mathrm{K}$ *is sound and complete for the class of all constant domain neighborhood frames that are filters.*

- Here a frame is ***supplemented*** if its N is closed under supersets, and it is a ***filter*** if N additionally always contains the set of all worlds, and is closed under finite intersections.