Similar Figures

1 Introduction

Similar triangles are fundamentals of geometry. Without them, it would be very difficult in many configurations to convert angle information to length information (e.g. through AA similarity) and vice versa. At the same time, though, similar triangles are often the most difficult things to spot without prior geometric intuition. In this workout, the goal is to strengthen this intuition.

2 Warm-up Problems

The one thing that is suboptimal about teaching a geometry topic to a math club is that, unlike algebra, geometry is not reiterated in the high school mathematics curriculum. As a result, there may be various facets of geometric topics that may have fizzed away over time. To refresh your memories, try the following problems. The first two are more basic in nature, while the third is a tad bit harder (but still quite doable). If there are any gaps in knowledge present from geometric atrophy, let me know and I’ll be glad to help.

1. [AHSME 1995] In $\triangle ABC$, $\angle C = 90^\circ$, $AC = 6$ and $BC = 8$. Points $D$ and $E$ are on $AB$ and $BC$, respectively, and $\angle BDE = 90^\circ$. If $DE = 4$, then $BD = \hfill$
   (A) 5  (B) $\frac{16}{3}$  (C) $\frac{20}{3}$  (D) $\frac{15}{2}$  (E) 8

2. How many equilateral triangles of side length 1 can fit inside an equilateral triangle of side length 3?
   (A) 6  (B) 7  (C) 8  (D) 9  (E) 10

3. $\triangle ABC$ has $AB = 12$, $AC = 13$, and $BC = 15$. Points $X$ and $Y$ are placed on $AB$ and $AC$ respectively such that $\angle AXY = \angle ACB$. If $XY = 6$, what is $AX + AY$?
   (A) 4  (B) 6  (C) 8  (D) 10  (E) 12

3 Utilizing Similar Triangles

In all of the above problems, the similar triangles were fairly easy to spot: an angle condition was given (e.g. parallel or antiparallel lines), this condition was used to find similar triangles, and from there lengths of segments in the diagram could be found. In competitions (or in real life for that matter) the situations are not as simple. In this regard, we will dive straight into a few problems; the hope is that you will pick up on a technique or two throughout the way.

Example 1 (AMC 10A 2002). Points $A$, $B$, $C$, $D$, $E$, and $F$ lie, in that order, on $\overline{AF}$, dividing it into five segments, each of length 1. Point $G$ is not on line $\overline{AF}$. Point $H$ lies on $\overline{GD}$, and point $J$ lies on $\overline{GF}$. The line segments $\overline{HC}$, $\overline{JE}$, and $\overline{AG}$ are parallel. Find $HC/JE$.

Solution. At first glance, this diagram seems quite complicated - the key is to pick it apart. More specifically, we have three sets of parallel lines here: $\overline{HC}$, $\overline{JE}$, and $\overline{AG}$. From $HC \parallel AG$ we see $\triangle HCD \sim \triangle GAD$, so
In a similar fashion, $\triangle JEF \sim \triangle GAF$, so $\frac{EJ}{AG} = \frac{FE}{FA} = \frac{1}{3}$. Now we see that this problem is not so bad after all! Indeed, manipulating the ratios so that the $AG$ terms cancel gives 

$$\frac{HC}{JE} = \left( \frac{HC}{AG} \right) \left( \frac{AG}{JE} \right) = \frac{1}{3} \cdot \frac{5}{3} = \frac{5}{9}.$$ 

**Example 2.** On square $ABCD$, points $E$ and $F$ are constructed on $CD$ and $AB$ respectively such that $DE = EC$ and $AF = 2FB$. Segment $DF$ intersects $AE$ and $AC$ at $P$ and $Q$ respectively. If $AB = 3$, what is $PQ$?

**Solution.** Note that from the condition on point $F$ we have $AF = 2$, so $DF = \sqrt{3^2 + 2^2} = \sqrt{13}$. It now suffices to use similarity to determine $PQ$.

First examine lines $AC$ and $DF$. They form two triangles, $\triangle AEF$ and $\triangle CDQ$, which are similar to each other. Their ratio of similitude is $\frac{AF}{DC} = \frac{2}{3}$, so $DQ = 3x$ and $FQ = 2x$ for some $x$. Now remark $DF = 3x + 2x = 5x = \sqrt{13}$, so $x = \frac{\sqrt{13}}{5}$ and $DQ = \frac{3}{5}\sqrt{13}$. In a similar manner, we examine $AE$ and $DF$. Note that $\triangle AEP \sim \triangle EDP$, so 

$$\frac{DP}{PF} = \frac{DE}{AF} = \frac{3}{2} = \frac{3}{4}.$$ 

Now let $DP = 3y$ and $PF = 4y$; then $DF = 7y$, which implies $y = \frac{\sqrt{13}}{7}$. Thus, $DP = \frac{3}{7}\sqrt{13}$.

Putting everything together, we get 

$$PQ = DQ - DP = \frac{3}{5}\sqrt{13} - \frac{3}{7}\sqrt{13} = \frac{6}{35}\sqrt{13}.$$ 

4 Constructing Auxiliary Lines

Often times, problems will be made easier by constructing additional lines within the diagram. If the right lines are drawn, similar triangles can be exploited. Here is a short example illustrating this idea:

**Example 3** (Math League HS 1981-1982). The area of square $ABCD$ is 1. As illustrated at the right, diagonal $AC$ is extended its own length to point $E$. How long is $BE$?

**Solution.** Let $M$ be the midpoint of $AB$. Then since $BC = 1$ and $BM = \frac{1}{2}$, Pythagorean Theorem gives $CM = \frac{1}{2}\sqrt{5}$. Now remark that $\triangle ACM \sim \triangle AEB$, so 

$$\frac{BE}{MC} = \frac{AB}{AM} = 2 \implies BE = 2 \cdot MC = \frac{\sqrt{5}}{2}.$$ 

This is not the only simple solution to the problem. We could have also dropped a perpendicular from $E$ to $AB$ and scaled upward before utilizing Pythagorean Theorem - that works just as well. In general, if you feel an auxiliary line is necessary, there are three types of lines that work better than others:

- Lines parallel to others in the diagram. As we saw in the previous problem, this creates similar triangles that may be useful.

- Perpendiculars. Right triangles are very powerful tools since they come equipped with their own set of theorems. Drop altitudes when you need information regarding perpendicular distances or when you need to set up similar right triangles. (An altitude from the vertex angle of an isosceles triangle can be especially useful, since its foot also doubles as the midpoint of the base!)

- Radii of a circle. The most recurring case of this occurs with points of tangency: if $\ell$ is a line tangent to a circle with center $O$ at point $A$, then $OA \perp \ell$. (Remember this fact!)
5 A Problem from the AIME

Before we move on to our final example, here is a seemingly-silly problem that demonstrates an important concept with similar triangles:

Example 4. Two triangles, \( \triangle I \) and \( \triangle II \), are known to be similar. Furthermore, it is known that the length of the median to the longest side of \( \triangle I \) is 6 while the length of the median to the longest side of \( \triangle II \) is 9. What is the ratio of the area of \( \triangle I \) to the area of \( \triangle II \)?

Solution. The ratio of similitude between \( \triangle I \) and \( \triangle II \) is \( \frac{2}{3} \). Therefore the ratio of their areas is \( \left( \frac{2}{3} \right)^2 = \frac{4}{9} \). \( \square \)

Notice that despite the fact I was given nothing about the actual side lengths of either triangle, I was still able to determine the ratio of similitude between them. This is because when a triangle is scaled upward or downward, all components of the triangle (not just the sides) are scaled. In particular, I used lengths of medians as corresponding segments between two triangles. Even the fact that the segments are medians does not change the problem; for all I know, I could have been given the distance between the symmedian point of the Fermat point and the orthocenter of the triangle. We’ll end this lecture with a more challenging problem from the 2015 American Invitational Mathematics Exam, which took place exactly one week ago. The question below was placed at the #7 spot, meaning that it is a medium-level problem when compared to the others that appeared on the test.

Example 5 (AIME 2015). In the diagram below, \( ABCD \) is a square. Point \( E \) is the midpoint of \( AD \). Points \( F \) and \( G \) lie on \( CE \), and \( H \) and \( J \) lie on \( AB \) and \( BC \), respectively, so that \( FGHJ \) is a square. Points \( K \) and \( L \) lie on \( GH \), and \( M \) and \( N \) lie on \( AD \) and \( AB \), respectively, so that \( KLMN \) is a square. The area of \( KLMN \) is 99. Find the area of \( FGHJ \).

Solution. Let \( X = HG \cap AD \) and \( Y = AB \cap CE \). Note that \( HG \perp CE \), so \( \angle AHX = \angle YCB \). Combining this with \( \angle XAH = \angle CBY = 90^\circ \) gives \( \triangle CBY \sim \triangle HAX \). Now note that because the two squares \( FGHJ \) and \( KLMN \) are inscribed inside these triangles, their ratio of similitude is the same as the ratio the similitude of the aforementioned two triangles. In other words, \( \frac{[FGHJ]}{[KLMN]} = \left( \frac{BC}{AH} \right)^2 \). (Here \( [X] \) denotes the area of figure \( X \).)

Now let \( s \) be the side length of square \( ABCD \) and \( s_1 \) the side length of square \( FGHJ \). Note that since \( \triangle FJC \sim \triangle GYH \sim \triangle BYC \), we have \( FC = \frac{1}{2} s_1 \) and \( CY = 2s_1 \). Thus

\[
\frac{1}{2} s_1 + s_1 + 2s_1 = \frac{7}{2} s_1 = CY = s\sqrt{5} \implies s_1 = \frac{2\sqrt{5}}{7} s.
\]

This implies \( BH = \left( \frac{2}{\sqrt{5}} \right) s_1 = \frac{2}{7} s \). Finally, we trivially have \( AH = \frac{3}{4} s \), so

\[
[FGHJ] = [KLMN] \left( \frac{BC}{AH} \right)^2 = 99 \left( \frac{s}{\frac{3s}{\sqrt{5}}} \right)^2 = 99 \cdot \frac{49}{9} = 539
\]

\( \square \)

\(^1\)The point where the reflections of the medians across the respective angle bisectors concur

\(^2\)The triangle whose vertices are the tangency points of the incircle with the three sides of the original triangle

\(^3\)The point \( F \) inside \( \triangle ABC \) for which \( AF + BF + CF \) is minimized

\(^4\)The triangle whose vertices are the feet of the three altitudes of the original triangle

\(^5\)The center of the circle passing through the midpoints of the three sides, among other points; also the midpoint of the segment connecting the circumcenter and orthocenter of the triangle
6 Problems

6.1 Problem Set A

1. Let points $A, B, C, D, E, F$ lie in the plane such that $\triangle ABC \sim \triangle DEF$. Let $H_a$ denote the unique point on $BC$ for which $AH_a \perp BC$, and define $H_d$ similarly. If $AH_a = 5$, $BC = 6$, and $DH_d = 10$, what is $EF$?

2. [AHSME 1990] Let $ABCD$ be a parallelogram with $\angle ABC = 120^\circ$, $AB = 16$ and $BC = 10$. Extend $CD$ through $D$ to $E$ so that $DE = 4$. If $BE$ intersects $AD$ at $F$, then $FD$ is closest to

3. [Adapted from HMMT 2007] We are given four similar triangles whose areas are $1^2$, $3^2$, $5^2$, and $7^2$. If the smallest triangle has a perimeter of 4, what is the sum of all the triangles’ perimeters?

4. [AMC 10A 2012] Externally tangent circles with centers at points $A$ and $B$ have radii of lengths 5 and 3, respectively. A line externally tangent to both circles intersects ray $AB$ at point $C$. What is $BC$?

5. [AMC 10A 2009] Rectangle $ABCD$ has $AB = 4$ and $BC = 3$. Segment $EF$ is constructed through $B$ so that $EF \perp DB$, and $A$ and $C$ lie on $DE$ and $DF$, respectively. What is $EF$?

6. [Wikipedia, et. al.] A closed planar shape is said to be *equiable* if the numerical values of its perimeter and area are the same. For example, a square with side length 4 is equiable since its perimeter and area are both 16. Show that any closed shape in the plane can be stretched or shrunk to become equiable.

7. [AHSME 1986] In $\triangle ABC$, $AB = 8$, $BC = 7$, $CA = 6$ and side $BC$ is extended, as shown in the figure, to a point $P$ so that $\triangle PAB$ is similar to $\triangle PCA$. The length of $PC$ is

8. Let $\triangle ABC$ be a triangle with $AB = 13$, $BC = 14$, and $AC = 15$. Square $BCYX$ is erected outside $\triangle ABC$. Segment $AX$ intersects $BC$ at point $P$, while $AY$ intersects it at point $Q$. Determine the length of $PQ$.

6.2 Problem Set B

9. [Mandelbrot 2006-2007] Suppose that $ABCD$ is a trapezoid in which $AD \parallel BC$. Given $AC \perp CD$, $AC$ bisects angle $\angle BAD$, and $\text{area}(ABCD) = 42$, then compute $\text{area}(ACD)$.

10. [ITAMO 2012] On the sides of a triangle $ABC$ right angled at $A$ three points $D, E$ and $F$ (respectively $BC, AC,$ and $AB$) are chosen so that the quadrilateral $AFDE$ is a square. If $x$ is the length of the side of the square, show that

$$\frac{1}{x} = \frac{1}{AB} + \frac{1}{AC}.$$
11. [AHSME 1981] In $\triangle ABC$, $M$ is the midpoint of side $BC$, $AN$ bisects $\angle BAC$, and $BN \perp AN$. If sides $AB$ and $AC$ have lengths 14 and 19, respectively, then find $MN$.

(Hint: Extend $BN$ past $N$ to intersect $AC$ at a point $Q$.)

12. [Mandelbrot 2008-2009] A pyramid has a square base, triangular sides, and eight edges that are each 80 meters long. A straight path begins at one corner of the square base, slanting upwards to meet the next edge at a point 30 meters along that edge from the corner, as shown. The path continues around the pyramid, always slanting upward at the same angle, making infinitely many turns. What is the total length of the path?

13. [OMO 2014] The points $A$, $B$, $C$, $D$, $E$ lie on a line $\ell$ in this order. Suppose $T$ is a point not on $\ell$ such that $\angle BTC = \angle DTE$, and $AT$ is tangent to the circumcircle of triangle $BTE$. If $AB = 2$, $BC = 36$, and $CD = 15$, compute $DE$.

14. [Math League HS 1984-1985] What is the area of a trapezoid whose altitude has a length of 12 and one of whose perpendicular diagonals has a length of 15?

6.3 Problem Set C

15. [AIME 1998] Let $ABCD$ be a parallelogram. Extend $DA$ through $A$ to a point $P$, and let $PC$ meet $AB$ at $Q$ and $DB$ at $R$. Given that $PQ = 735$ and $QR = 112$, find $RC$.

16. [Thomas Mildorf] $ABC$ is an isosceles triangle with base $AB$. $D$ is a point on $AC$ and $E$ is the point on the extension of $BD$ past $D$ such that $\angle BAE$ is right. If $BD = 15$, $DE = 2$, and $BC = 16$, then $CD$ can be expressed as $\frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers. Determine $m + n$.

17. [Math League HS 1977-1978] In $\triangle ABC$, $AC = 18$, and $D$ is the point on $AC$ for which $AD = 5$. Perpendiculars drawn from $D$ to $AB$ and $AC$ have lengths 4 and 5 respectively. What is the area of $\triangle ABC$?

18. [AIME 1986] In $\triangle ABC$, $AB = 425$, $BC = 450$, and $AC = 510$. An interior point $P$ is then drawn, and segments are drawn through $P$ parallel to the sides of the triangle. If these three segments are of an equal length $d$, find $d$. 