Stochastic volatility model calibration

via likelihood approximations

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Outline

1. Financial models
   - What is done in mathematical finance?
   - Black-Scholes-Merton model
   - Stochastic volatility models

2. Model calibration
   - How are the models calibrated?
   - Transition density function approximation

Derivative contracts

- Call option: gives the holder the right (but not the obligation) to buy the asset for price K at time T.
- Put option: gives the holder the right (but not the obligation) to sell the asset for price K at time T.
- Barrier options, forward start options, Asian options, variance swaps, cliquets, options on baskets, etc.

Mathematical set-up

- Specify models in the language of stochastic calculus

$$dX_t = \mu(X_t)dt + \sigma(X_t)dB_t$$

  where $X^{(1)}$ represents the spot price of a traded security.
- Pricing/risk-neutral measure, under which the discounted price processes of all traded securities are martingales.
- Under any risk-neutral measure the drift of a traded asset is the interest rate.
- The price of a derivative security is the discounted expected value under the risk-neutral measure. E.g. price of a call is

$$\mathbb{E}[e^{-rT}\max(X_T^{(1)} - K, 0)]$$

Black-Scholes-Merton

- Model (risk-neutral world):

$$dS_t = S_t(rdt + \sigma dW_t)$$

- The price of a call can be found explicitly as

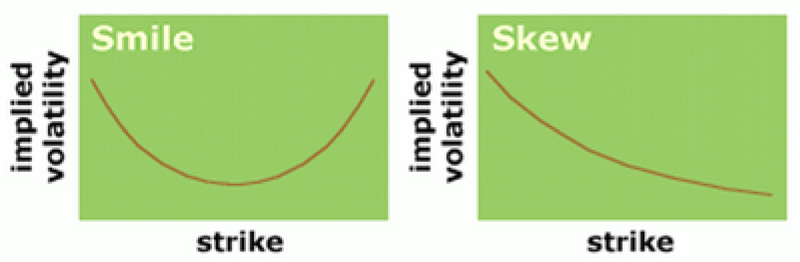
$$C(S_0, K, T, r, \sigma) = S_0 N(d_1) - e^{rT} K N(d_2)$$

$$\text{where } d_1 = \frac{\log \frac{S_0}{K} + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \text{ and } d_2 = d_1 - \sigma\sqrt{T}$$

  and N is the standard normal cumulative distribution function.
- Includes a hedging recipe…

Implied volatility

- For a fixed contract, $\sigma \mapsto C(S_0, K, T, r, \sigma)$ is increasing.
- Nowadays call and put options are very liquid, so their prices are set by the market. For a fixed maturity T, inverting the price data we get a graph (σ vs. K)



Stochastic volatility models

- General stochastic volatility model (risk-neutral world)

$$dS_t = S_t(r dt + \sigma_t dW_t)$$
$$d\sigma_t = \alpha(\sigma_t)dt + \beta(\sigma_t)dZ_t$$

  where W and Z are correlated Brownian motions.

- α and β are chosen so that the volatility process has certain intuitive properties, e.g.
  - positivity
  - mean-reversion
  - auto-correlation

Example: Heston model

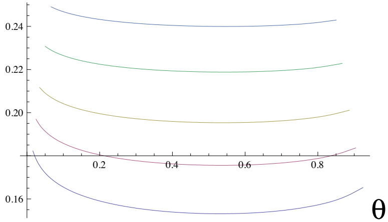
- Define $\sigma_t = \sqrt{V_t}$ and

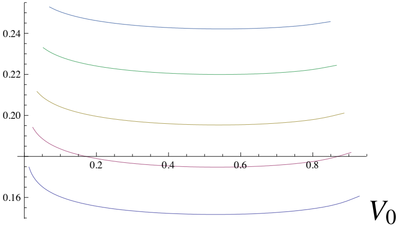
$$dS_t = S_t(\mu dt + \sqrt{V_t} dW_t)$$
$$dV_t = \kappa(\theta - V_t)dt + \eta\sqrt{V_t} dZ_t$$

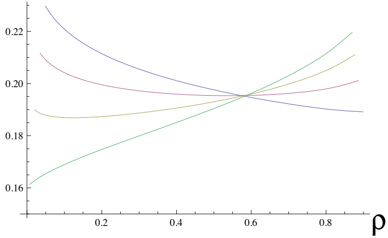
- Volatility is mean-reverting, non-negative, and auto-correlated.
- Parameters:
  - κ – mean-reversion rate
  - θ – long-term variance
  - η – volatility of variance
  - ρ – instantaneous correlation between W and Z
  - V_0 – initial variance

and of course S_0 and r.

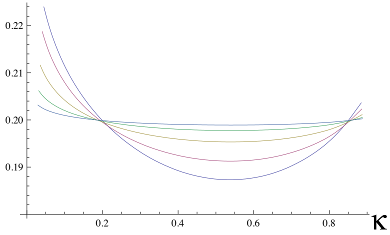
Example: parameters affect implied volatility

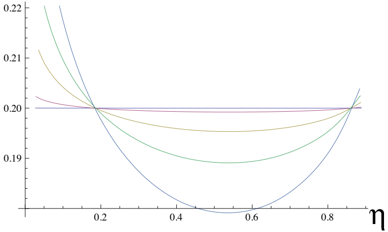






Example: parameters affect implied volatility





- Only ρ affects the skew. For stock indices we tend to see a skew corresponding to ρ between -0.5 and -0.9.
- θ and V_0 affect the level of the implied volatility curve, but not the shape, while κ and η affect the convexity.

Determining model parameters

- Need to determine reasonable values for the model parameters.
- Have data on the spot price and (less data) on the prices of derivative securities (put and call options).
- What methods do we have available?
  - Time-series methods (about which I know nothing)
  - Least-squares fit
  - Maximum likelihood estimate

Transition density function (TDF)

- Transition density function is $p_t(x \mid x_0)$, the conditional probability density for $X_t = x$ given that $X_0 = x_0$.
- Note that p_t depends on the model parameters.
- $p_t(x \mid x_0)$ satisfies the Kolmogorov forward PDE

$$\frac{\partial}{\partial t} p_t = -\sum_{i=1}^{m} \frac{\partial}{\partial x_i}[\mu_i(x) p_t(x)] + \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} \frac{\partial^2}{\partial x_i \partial x_j}[\nu_{ij}(x) p_t(x)]$$

  with initial condition $p_0 = \delta_{x_0}$ (in the sense of distributions).
- E.g. the KFPDE for Brownian Motion is the heat equation.

Maximum likelihood estimate (MLE)

- Data is equally spaced, consecutive observations

$$x_0 = X_0,\ x_1 = X_{\Delta t},\ x_2 = X_{2\Delta t},\ \ldots,\ x_n = X_{n\Delta t}.$$

- Log-likelihood function for this data is

$$\ell = \sum_{i=1}^{n} \log p_{\Delta t}(x_i \mid x_{i-1})$$

- Note that ℓ is a function of the model parameters only.
- Maximizing ℓ over the parameter space gives the parameters for which the observed data has the greatest probability of occurring (this is the maximum likelihood estimate).

Problem with MLE via TDF

- In general it is impossible to get a closed-form expression for the TDF (and hence for the LLF).
- One approach is to use an approximation to the LLF. But what do we mean by approximation? Which one do we take?
- E.g. if $X_t = \mu t + \sigma B_t$ is m-dimensional BM with drift then

$$\log p_t(x \mid x_0) = -\frac{m}{2}\log(2\pi t) - \frac{1}{2}\log|\sigma\sigma^T| - \frac{1}{2t}(x - x_0 - \mu t)^T(\sigma\sigma^T)^{-1}(x - x_0 - \mu t)$$

Forward and backward PDE

- It might be reasonable to ask that

$$\log p_t(x \mid x_0) \approx -\frac{m}{2}\log(2\pi t) - \frac{1}{2}\log|\sigma\sigma^T(x_0)| + \sum_{k=-1}^{K} c_k(x - x_0)\frac{t^k}{k!}$$

  for some polynomials c_k of low degree.
- If we require the RHS to satisfy the Kolmogorov forward PDE up to order K, then we can recursively solve for the polynomials c_k.
- Aït-Sahalia was able to prove that the approximations converge uniformly in probability as $t \to 0$, independently of K and n. Under reasonable conditions the maxima converge, so (for smallish t) we have approximations for the MLE.

One more thing

- Of course, instantaneous volatility data is not available.
- Variance swap: On one hand, the price of a variance swap is determined (in a model independent way) by the implied volatility curve, while on the other, in the Heston model the price of a variance swap can be computed exactly to be

$$\frac{(e^{-\kappa T}-1)(\theta - V_0)}{\kappa T} + \theta.$$

- TDF for the actual data is obtained from the TDF for the model by multiplying by the Jacobian of the map that takes model values to data values, namely

$$(S,V) \mapsto \left(S, \frac{(e^{-\kappa T}-1)(\theta - V)}{\kappa T} + \theta\right)$$

Conclusions and further work

- Applied to the EUROSTOXX50 index, obtained parameters

$$\{\kappa = 2, \theta = 0.05, \eta = 0.45, \rho = -0.65\}$$

- Apply this method to other stochastic volatility models.
- Careful analysis of convergence of the approximation.

For Further Reading

- Steven E. Shreve
  Stochastic Calculus for Finance II: Continuous-Time Models.
  Springer Finance, 2004.

- Yacine Aït-Sahalia
  Closed-Form Likelihood Expansions for Multivariate Diffusions.
  Annals of Statistics, 2008, 36, 906-937.