

# A Quick Introduction to Representation Theory with Applications

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# Outline

- 1 Introduction
- 2 Characters
- 3 Applications

# Group Representations: Definition

- Let  $G$  be a finite group. A **representation** of  $G$  is a pair  $(V, \rho)$ , where
  - $V$  is a finite dimensional vector space over  $\mathbb{C}$
  - $\rho : G \rightarrow \mathbf{GL}(V)$  is a group homomorphism
- Every group homomorphism  $G \rightarrow \mathbb{C}^*$  is a one dimensional representation. (Think of scalars as  $1 \times 1$  matrices.)

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# Examples

- If  $X$  is a finite  $G$ -set then let  $V$  be the vector space with basis  $\{e_x \mid x \in X\}$ . Then  $\rho : G \rightarrow \mathbf{GL}(V)$  defined by

$$\rho_s(e_x) = e_{s \cdot x}$$

is called the **permutation** representation associated with  $X$ .

- The **regular** representation is a special case of this.
- Let  $(V, \rho)$  and  $(W, \zeta)$  be representations. The **direct sum** representation is

$$\rho \oplus \zeta : G \rightarrow \mathbf{GL}(V \oplus W) : s \mapsto \begin{bmatrix} \rho_s & 0 \\ 0 & \zeta_s \end{bmatrix}$$

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# Subrepresentations and Irreducible Representations

- If  $(V, \rho)$  is a representation and  $W \subseteq V$  is a linear subspace such that

$$\rho_s W \subseteq W \quad \text{for all } s \in G$$

then  $W$  is a **subrepresentation** of  $V$ .

- If  $V$  has no subrepresentations other than  $\{0\}$  and  $V$  then it is said to be **irreducible**.
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# Schur's Lemma

## Lemma

*Suppose that  $(V, \rho)$  and  $(W, \zeta)$  are irreducible representations and  $f : V \rightarrow W$  is linear such that*

$$\rho_s f(v) = f(\zeta_s v) \quad \text{for all } s \in G \text{ and } v \in V$$

*Then  $f$  is either the zero map or an isomorphism.*

# Proof of Schur's Lemma

- Suppose that  $f$  is non-zero.
- $\ker f \subseteq V$  is a subspace. If  $v \in \ker f$  then for any  $s \in G$ ,

$$f(\rho_s v) = \zeta_s f(v) = \zeta_s 0 = 0$$

Therefore  $\ker f$  is a subrepresentation of  $V$ .

But  $V$  is irreducible and  $f$  is non-zero, so  $\ker f = \{0\}$ .

- $f(V) \subseteq W$  is a subspace. If  $f(v) \in f(V)$  then for any  $s \in G$ ,

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# Characters: Definition

- Let  $\rho : G \rightarrow \mathbf{GL}(V)$  be a representation of  $G$ .  
The **character** of  $\rho$  is  $\chi_V$ , defined by

$$\chi_V(s) = \text{Tr}(\rho_s)$$

- Notice that  $\chi_V$  is a class function since

$$\chi_V(hsh^{-1}) = \text{Tr}(\rho_{hsh^{-1}}) = \text{Tr}(\rho_h \rho_s \rho_h^{-1}) = \text{Tr}(\rho_s) = \chi_V(s)$$

- For any  $s \in G$ ,  $\rho_s^{|G|} = \rho_{s^{|G|}} = \rho_e = I$  so the eigenvalues of  $\rho_s$  are roots of unity. Hence  $\chi_V(s)$  is a sum of roots of unity.

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# Facts

## Proposition

Let  $(V, \rho)$  and  $(W, \zeta)$  be representations of  $G$ .

- $\chi_V(e) = \dim V$
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# Characters of Irreducible Representations

The collection of all class functions  $f : G \rightarrow \mathbb{C}$  is a vector space. It has an inner product

$$\langle f, g \rangle = \frac{1}{|G|} \sum_{s \in G} f(s) \overline{g(s)}$$

## Theorem

*The characters of the irreducible representations of  $G$  form an orthonormal basis of this vector space.*

Proof of this fact uses Schur's Lemma.

# Decomposing Representations

- The dimension of the space of class functions is equal to the number of conjugacy classes, so there are that many irreducible representations.
- Suppose that  $(V_1, \psi_1), \dots, (V_h, \psi_h)$  are all of the irreducible representations of  $G$ .
- If  $\rho : G \rightarrow \mathbf{GL}(V)$  is any representation then there are integers  $a_1, \dots, a_h \geq 0$  such that

$$V \cong a_1 V_1 \oplus \cdots \oplus a_h V_h$$

- Hence  $\chi_V = a_1 \chi_1 + \cdots + a_h \chi_h$ , and so

$$a_i = \langle \chi_V, \chi_i \rangle$$

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# Character Tables

Notice that  $\langle \chi_V, \chi_V \rangle = a_1^2 + \cdots + a_h^2$ , so  $\chi_V$  is irreducible if and only if  $\langle \chi_V, \chi_V \rangle = 1$ .

## Theorem

*It is enough to know the character of a representation to know it up to isomorphism (once you already know all of the irreducible representations).*

- It is convenient to record this information in a table.
- Here is the character table for  $\mathfrak{S}_3$ :

	$id$	$(1\ 2)$	$(1\ 2\ 3)$
$U$	1	1	1
$U'$	1	-1	1
$W$	2	0	-1

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# Calculation of the Character Table for $\mathfrak{S}_3$

- $U$  is the trivial representation, every group has it.
- $U'$  is the alternating representation. It is obtained by considering the sign of the permutation.
- $\mathfrak{S}_3$  acts on the set  $\{1, 2, 3\}$ , so consider the associated permutation representation  $(V, \rho)$ . We have

$$\rho_{id} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \rho_{(1\ 2)} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \rho_{(1\ 2\ 3)} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

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## Calculation, continued

- $(V, \rho)$  is not irreducible, since the vector  $e_1 + e_2 + e_3$  is fixed by all elements of the group. The span of this vector forms a copy of  $U$  sitting inside of  $V$ .
- Suppose we write  $V = W \oplus U$ . Then  $\chi_V = \chi_W + 1$ , so

$$\chi_W(id) = 2 \quad \chi_W(1\ 2) = 0 \quad \chi_W(1\ 2\ 3) = -1$$

- $W$  is irreducible since

$$\langle \chi_W, \chi_W \rangle = \frac{1}{|\mathfrak{S}_3|} \sum_{s \in \mathfrak{S}_3} |\chi_W(s)|^2 = \frac{1}{6}(2 + 4 \cdot 0 + 2 \cdot 1) = 1$$

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# Burnside's Theorem

Using representation theory we can show

## Lemma

*If  $G$  has a conjugacy class of order a power of a prime then  $G$  is not simple.*

## Theorem

*If  $|G|$  is divisible by at most two primes then  $G$  is solvable.*

# Burnside's Theorem

Using representation theory we can show

## Lemma

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# Proof of Burnside's Theorem

- Suppose that  $G$  is a minimal counterexample. Then  $G$  is simple since a normal subgroup  $N$  would give solvable groups  $N$  and  $G/N$ , which would show  $G$  is solvable.
- Say  $|G| = p^a q^b$ , where  $b > 0$ . Let  $H$  be the Sylow subgroup of size  $q^b$ . Then  $H$  has non-trivial center.
- Let  $h$  be a non-identity element in the center of  $H$  and let  $K$  be the centralizer of  $h$  in  $G$ .
- $|K| = p^c q^b$ , and  $c < a$  since otherwise  $h$  would be in the center of  $G$ . The size of the conjugacy class of  $h$  is  $|G|/|K| = p^{a-c}$ .
- This is a contradiction by the Lemma. □

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# For Further Reading



Jean-Pierre Serre

*Linear Representations of Finite Groups.*

GTM 42, Springer-Verlag, 1977.