Mathematical Understanding

Jeremy Avigad

Department of Philosophy and
Department of Mathematical Sciences
Carnegie Mellon University

May 2016

Sequence of lectures

1. Mathematical Understanding
2. The History of Dirichlet's Theorem
3. Formalization and Interactive Theorem Proving
4. The Role of the Diagram in Euclid's *Elements*
5. Modularity in Mathematics

Mathematical Understanding

Overview:

- General epistemological questions
- An example: sums of squares
- Talking about mathematical understanding
- Towards a general theory

Epistemological questions

Since Plato, the philosophy of mathematics has been concerned with:

- the nature of mathematical objects, and
- the appropriate justification for mathematical knowledge.

But we employ other normative judgments as well:

- some theorems are interesting
- some questions are natural
- some concepts are fruitful, or powerful
- some proofs provide better explanations than others
- some historical developments are important
- some observations are insightful

. . . and so on.

The problem of multiple proofs

On the standard account, the value of a mathematical proof is that it warrants the truth of the resulting theorem.

Why, then, do we often value a new proof of a previous established theorem?

For example, Gauss published six proofs of the law of quadratic reciprocity in his lifetime, and left us two unpublished versions as well.

Franz Lemmermeyer has documented 233 proofs (available online, with references).

The problem of multiple proofs

This question not new. For example:

> *It might be said: "—that every proof, even of a proposition which has already been proved, is a contribution to mathematics". But why is it a contribution if its only point was to prove the proposition? Well, one can say: "the new proof shews (or makes) a new connexion". — Wittgenstein, Remarks on the Foundations of Mathematics, III–60*

Indeed, it is *not* a great mystery. There is a lot we can say about what we learn from different proofs.

But the philosophy of mathematics has had relatively little to say about the matter.

The problem of conceptual possibility

It is often said that some mathematical advance was "made possible" by a prior conceptual development.

For example, Riemann's introduction of the complex zeta function and the use of complex analysis made it possible for Hadamard and de la Vallée Poussin to prove the prime number theorem in 1896.

What is the sense of "possibility" here?

Intuition: a certain "understanding" guides us. (But let's focus on the phenomena, not the word.)

Epistemological questions

What the questions have in commmon:

- They have a generally epistemological flavor, involving "knowledge" or "understanding."
- They invoke normative considerations.

This is a starting point for philosophical inquiry.

Sums of squares

To prod our intuitions, let's consider an example.

In the *Arithmetic*, Diophantus notes that

- $5 = 2^2 + 1^2$
- $13 = 3^2 + 2^2$
- $5 \times 13 = 65 = 8^2 + 1^2 = 7^2 + 4^2$.

Theorem. If x and y can each be written as a sum of two integer squares, then so can xy.

Sums of squares

Proof #1. Suppose $x = a^2 + b^2$, and $y = c^2 + d^2$. Then

$$xy = (ac - bd)^2 + (ad + bc)^2,$$

a sum of two squares. □

In more detail:

$$\begin{aligned}
&(ac - bd)^2 + (ad + bc)^2 \\
&\qquad = a^2c^2 - 2abcd + b^2d^2 + a^2d^2 + 2abcd + b^2d^2 \\
&\qquad = a^2c^2 + b^2d^2 + a^2d^2 + b^2d^2 \\
&\qquad = (a^2 + b^2)(c^2 + d^2)
\end{aligned}$$

Note: $(ac + bd)^2 + (ad - bc)^2$ works just as well.

Sums of squares

Pros: elementary, general.

Cons: mysterious, "unmotivated," hard to remember, have to calculate.

Sums of squares

Define the *Gaussian integers*:

$$\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$$

If $\alpha = u + vi$, define the *conjugate*:

$$\overline{\alpha} = u - vi.$$

We have $\overline{\alpha\beta} = \overline{\alpha} \cdot \overline{\beta}$.

Define the norm:

$$N(\alpha) = \alpha\overline{\alpha} = (u + iv)(u - iv) = u^2 - i^2 v^2 = u^2 + v^2.$$

Then

$$N(\alpha\beta) = \alpha\beta \cdot \overline{\alpha\beta} = \alpha \cdot \beta \cdot \overline{\alpha} \cdot \overline{\beta} = \alpha\overline{\alpha} \cdot \beta\overline{\beta} = N(\alpha)N(\beta).$$

Sums of squares

Proof #2. Suppose $x = N(\alpha)$ and $y = N(\beta)$ are sums of two squares. Then $xy = N(\alpha\beta)$, a sum of two squares. $\square$

Sums of squares

Features:

- The norm (and its square root, the modulus, or absolute value) are generally useful. For example, the Gaussian integers are a Euclidean domain.
- The proof is easy to remember and reconstruct.
- It avoids calculation.
- Generalizations to the quaternions and octonians give product rules for sums of 4 and 8 squares — and there are no others.
- Conjugates and norms lie at the heart of algebraic number theory.
- They provide, for example, a general theory of *quadratic forms* (expressions $ax^2 + bxy + cy^2$).

Sums of squares

...the complex numbers of Gauss, Jacobi, and M. Kummer force themselves upon our consideration, not because their properties are generalizations of the properties of ordinary integers, but because certain of the properties of integral numbers can only be explained by a reference to them. (H. J. S. Smith, Report on the Theory of Numbers, 1860.)

Sums of squares

In *Sur la théorie des nombres entiers algébrique* (1877), Dedekind emphasized concepts over calculation:

> *Even if there were such a theory, based on calculation, it still would not be of the highest degree of perfection, in my opinion. It is preferable, as in the modern theory of functions, to seek proofs based immediately on fundamental characteristics, rather than on calculation, and indeed to construct the theory in such a way that it is able to predict the results of calculation...*

Sums of squares

Proof #3. Suppose $x = a^2 + b^2$ and $y = c^2 + d^2$. Then

$$\begin{aligned}
xy &= (a^2 + b^2)(c^2 + d^2) \\
&= (a + bi)(a - bi)(c + di)(c - di) \\
&= (a + bi)(c + di)(a - bi)(c - di) \\
&= ((ac - bd) + (ad + bc)i)((ac - bd) - (ad + bc)i) \\
&= (ac - bd)^2 + (ad + bc)^2,
\end{aligned}$$

a sum of two squares. □

This was given by Euler in 1770.

Morals

- There's a lot one can say about what we like about different proofs.
- Only some of it has to do with correctness.
- Judgments can depend on context, including:
  - background
  - goals
  - personal preference
- Proofs generally convey things (knowledge? understanding?) that are useful for other purposes.

Instructions in proofs

Even everyday proof language has procedural elements:

- "…the first law may be proved by induction on n."
- "…by successive applications of the definition, the associative law, the induction assumption, and the definition again."
- "By choice of m, $P(k)$ will be true for all $k < m$."
- "Hence, by the well-ordering postulate…"
- "From this formula it is clear that…"
- "This reduction can be repeated on b and r_1…"
- "This can be done by expressing the successive remainders r_i in terms of a and b…"

Instructions in proofs

- "By the definition of a prime..."
- "On multiplying through by b..."
- "...by the second induction principle, we can assume $P(b)$ and $P(c)$ to be true..."
- "Continue this process until no primes are left on one side of the resulting equation..."
- "Collecting these occurrences, ..."
- "By definition, the hypothesis states that..."
- "...Theorem 10 allows us to conclude ..."

Morals

It is hard to characterize everything we get from a proof in terms of propositional knowledge.

We seem to value various forms of *procedural* knowledge: methods, heuristics, perspectives, ways of proceeding, means of analysis.

Reading a good proof makes me a better mathematician.

What we really want is to *understand*.

Talking about understanding

Mathematics is hard.

Mathematical solutions, proofs, and calculations involve long sequences of steps, that have to be chosen and composed in precise ways.

To compound matters, there are too many options; among the many steps we may plausibly take, most will get us absolutely nowhere.

And we have limited cognitive capacities — we can only keep track of so much data, anticipate the result of a few small steps, remember so many background facts.

We rely on our understanding to help us and to guide us.

Intuitions

Does understanding the demonstration of a theorem consist in examining each of the syllogisms of which it is composed in succession, and being convinced that it is correct and conforms to the rules of the game? In the same way, does understanding a definition consist simply in recognizing that the meaning of all the terms employed is already known, and being convinced that it involves no contradiction?

. . . Almost all are more exacting; they want to know not only whether all the syllogisms of a demonstration are correct, but why they are linked together in one order rather than in another. As long as they appear to them engendered by caprice, and not by an intelligence constantly conscious of the end to be attained, they do not think they have understood.

(Poincaré, *Science et méthod*)

Intuitions

Logic teaches us that on such and such a road we are sure of not meeting an obstacle; it does not tell us which is the road that leads to the desired end. (Ibid.)

Discovery consists precisely in not constructing useless combinations, but in constructing those that are useful, which are an infinitely small minority. Discovery is discernment, selection. (Ibid.)

Intuitions

It seems to me, then, as I repeat an argument I have learned, that I could have discovered it. This is often only an illusion; but even then, even if I am not clever enough to create for myself, I rediscover it myself as I repeat it. (Ibid.)

Intuitions

Now, in calm weather, to swim in the open ocean is as easy to the practised swimmer as to ride in a spring-carriage ashore. But the awful lonesomeness is intolerable. The intense concentration of self in the middle of such a heartless immensity, my God! who can tell it? Mark, how when sailors in a dead calm bathe in the open sea–mark how closely they hug their ship and only coast along her sides.

(Melville, *Moby Dick*, Chapter 93)

Intuitions

The sea had jeeringly kept his finite body up, but drowned the infinite of his soul. Not drowned entirely, though. Rather carried down alive to wondrous depths, where strange shapes of the unwarped primal world glided to and fro before his passive eyes; and the miser-merman, Wisdom, revealed his hoarded heaps; and among the joyous, heartless, ever-juvenile eternities, Pip saw the multitudinous, God-omnipresent, coral insects, that out of the firmament of waters heaved the colossal orbs. He saw God's foot upon the treadle of the loom, and spoke it; and therefore his shipmates called him mad. So man's insanity is heaven's sense; and wandering from all mortal reason, man comes at last to that celestial thought, which, to reason, is absurd and frantic; and weal or woe, feels then uncompromised, indifferent as his God.

Towards a theory of understanding

General outlook:

- Beyond knowledge, we look to mathematics for modes of *understanding*.
- Understanding involves not just factual knowledge, but something more dynamic: ways of proceeding, modes of analysis, capacities for thought.
- We value mathematical resources for conferring understanding.
- Some mathematical resources are overtly syntactic: definitions, theorems, proofs, questions.
- These give rise to resources that are harder to characterize precisely: concepts, methods, heuristics, guiding intuitions, . . .

Towards a theory of understanding

Philosophical goal: to develop a robust theory of mathematical understanding

General questions:

- How can we / should we talk about things like concepts, methods, heuristics, and guiding intuitions?
- To what extent do we need to model the cognizing agent?
- Is this really philosophy, or cognitive science?
- Do we need a theory of syntax, or semantics?
- What is the relationship between mathematics and language?

A methodological stance

To make progress, we have to pick a "scientific framework":

- a way of thinking about mathematics
- a language for talking about the objects of mathematical understanding
- a way of posing questions precisely (or at least trying to)
- precise, disciplined ways of answering them

We just have to do it, and see what happens.

A methodological stance

We are looking for:

- a coherent theory
- satisfying answers
- a theory that can inform (and are informed by) other pursuits:
  - history of mathematics
  - interactive theorem proving and automated reasoning
  - psychology and cognitive science
  - mathematics education
  - mathematics itself

I'll describe one perspective, and (over the coming lectures) try to convince you that it is fruitful.

A methodological stance

What is the relationship between mathematics and language?

What characterizes mathematics with respect to other scholarly disciplines is its level of rigor: there are precise norms that govern how to make meaningful mathematical claims, and how to establish their truth.

We can (and have) studied these norms in syntactic terms, with great success.

A closer look at the syntactic components of mathematics — definitions, theorems, proofs, theories, and so on — shows them to be highly structured objects.

A methodological stance

When one studies the history of mathematics, or tries to model *real* mathematical proofs formally, one has the sense that mathematical language is beautifully *designed* to extend our cognitive reach, make it possible for us to solve increasingly more difficult problems, construct more elaborate proofs.

The more abstracts objects of understanding — concepts, methods, intuitions, etc. — are manifested in the linguistic artifacts.

(I have borrowed the term “artifact” from Ken Manders.)

A methodological stance

Do we need a theory of syntax, or semantic objects?

At least, let's start with the syntax: that is the raw data, and there is a lot going on there.

I expect that the more abstract objects of understanding — concepts, methods, intuitions, and so on — will be best described in terms of quasi-syntactic representations and quasi-algorithmic ways of acting on them.

(At heart, I am a proof theorist in the Hilbert tradition, and skeptical of anything that is not ultimately grounded in syntax.)

A methodological stance

Is this really philosophy, or cognitive science?

We need a healthy interaction between these two disciplines that is mindful of the differences between them:

- data: psychology studies human behavior, phil math studies *mathematics* (a shared practice)
- normativity: psychology describes what people do, phil math explains what people *should* do
- method: psychologists conduct experiments, philosophers (should) study the mathematics

A methodological stance

To what extent do we need to model the cognizing agent?

I am less sure of this.

One can get pretty far with a crude characterization: we are cognitively bounded agents, trying to carry out complex tasks efficiently.

As needed, we can develop better measures of complexity, simplicity, difficulty, . . .

A methodological stance

We can look for a suitable division of labor:

- philosophy explains why certain methodological strategies are advantageous, for agents with certain (idealized) cognitive capacities and constraints.
- psychology tells more about the specifically human capacities and constraints.

Although computers face difference constraints, to some extent, we are up against the same thing, combinatorial explosion.

A methodological stance

A priori, a mathematical knowledge is a relationship between the knowing agent and the object known.

Conventional theories of mathematical knowledge boil down to theories of mathematical inference and proof.

- We start with a notion of *proposition*.
- We characterize relations between propositions, like entailment.

The role of the agent disappears.

By analogy, a theory of mathematical understanding might take the notion of a mathematical *ability* or *capacity* as basic.

A theory of understanding could then amount to a theory of mathematical capacities, and how they interact.

A methodological stance

How can we / should we talk about things like capacities, abilities, concepts, methods, heuristics, or intuitions?

I don't know.

But I know a few ways to get started.

Strategies

One strategy: look to the history of mathematics.

Find an important historical development (what Ken Manders calls a "big deal difference").

This suggests that we were in

- a certain epistemological state beforehand, and
- a certain epistemological state after,

and that they are different in some important way.

Explain the difference.

Strategies

One can do the same with contemporary mathematics, and smaller differences.

For example, one compare alternative proofs, or textbook presentations, with an eye towards explaining

- the structuring of information, and
- the understanding or expertise that is conveyed.

We need to rely on what mathematicians *do* rather than their self assessments.

Strategies

Look to interactive theorem proving and automated reasoning.

"Proof languages" provide expressive models of ordinary mathematical language, designed to convey knowledge (and expertise) efficiently.

Understanding what is needed to develop mathematics formally provides insight into how the informal languages work as well.

Mathematical Understanding

Overview:

- General epistemological questions
- An example: sums of squares
- Talking about mathematical understanding
- Towards a general theory

Sequence of lectures

1. Mathematical Understanding
2. The History of Dirichlet's Theorem
3. Formalization and Interactive Theorem Proving
4. The Role of the Diagram in Euclid's *Elements*
5. Modularity in Mathematics

References

Main sources for this talk:

- *Mathematical method and proof*
- *Understanding proof*
- *Understanding, formal verification, and the philosophy of mathematics*
- *Mathematics and language*

See also:

- *Number theory and elementary arithmetic*
- *Computers in mathematical inquiry*
- Review of Marcus Giaquinto, *Visual Thinking in Mathematics*
- Review of Bonnie A. Gold and Roger Simons, editors, *Proof and Other Dilemmas*