Mathematical Understanding

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Epistemological questions

Since Plato, the philosophy of mathematics has been concerned with:

- the nature of mathematical objects, and
- the appropriate justification for mathematical knowledge.

But we employ other normative judgments as well:

- some theorems are interesting
- some questions are natural
- some concepts are fruitful, or powerful
- some proofs provide better explanations than others
- some historical developments are important
- some observations are insightful

. . . and so on.

The problem of multiple proofs

On the standard account, the value of a mathematical proof is that it warrants the truth of the resulting theorem.

Why, then, do we often value a new proof of a previous established theorem?

For example, Gauss published six proofs of the law of quadratic reciprocity in his lifetime, and left us two unpublished versions as well.

Franz Lemmermeyer has documented 233 proofs (available online, with references).

The problem of multiple proofs

This question not new. For example:

> *It might be said: "—that every proof, even of a proposition which has already been proved, is a contribution to mathematics". But why is it a contribution if its only point was to prove the proposition? Well, one can say: "the new proof shews (or makes) a new connexion". — Wittgenstein, Remarks on the Foundations of Mathematics, III–60*

Indeed, it is *not* a great mystery. There is a lot we can say about what we learn from different proofs.

But the philosophy of mathematics has had relatively little to say about the matter.

The problem of conceptual possibility

It is often said that some mathematical advance was "made possible" by a prior conceptual development.

For example, Riemann's introduction of the complex zeta function and the use of complex analysis made it possible for Hadamard and de la Vallée Poussin to prove the prime number theorem in 1896.

What is the sense of "possibility" here?

Intuition: a certain "understanding" guides us. (But let's focus on the phenomena, not the word.)

Epistemological questions

What the questions have in commmon:

- They have a generally epistemological flavor, involving "knowledge" or "understanding."
- They invoke normative assessments.

This is a starting point for philosophical inquiry.

Mathematical Understanding

Overview:

- General epistemological questions
- Intuitions
- Towards a theory of mathematical understanding
- Strategies
  - Look to mathematical practice
  - Look to interactive theorem proving
  - Look to the history of mathematics

Intuitions

Mathematics is hard.

Mathematical solutions, proofs, and calculations involve long sequences of steps, that have to be chosen and composed in precise ways.

To compound matters, there are too many options; among the many steps we may plausibly take, most will get us absolutely nowhere.

And we have limited cognitive capacities — we can only keep track of so much data, anticipate the result of a few small steps, remember so many background facts.

We rely on our understanding to help us and to guide us.

Intuitions

Does understanding the demonstration of a theorem consist in examining each of the syllogisms of which it is composed in succession, and being convinced that it is correct and conforms to the rules of the game? In the same way, does understanding a definition consist simply in recognizing that the meaning of all the terms employed is already known, and being convinced that it involves no contradiction?

. . . Almost all are more exacting; they want to know not only whether all the syllogisms of a demonstration are correct, but why they are linked together in one order rather than in another. As long as they appear to them engendered by caprice, and not by an intelligence constantly conscious of the end to be attained, they do not think they have understood.

(Poincaré, *Science et méthod*, 1908)

Intuitions

Logic teaches us that on such and such a road we are sure of not meeting an obstacle; it does not tell us which is the road that leads to the desired end. (Ibid.)

Discovery consists precisely in not constructing useless combinations, but in constructing those that are useful, which are an infinitely small minority. Discovery is discernment, selection. (Ibid.)

Intuitions

It seems to me, then, as I repeat an argument I have learned, that I could have discovered it. This is often only an illusion; but even then, even if I am not clever enough to create for myself, I rediscover it myself as I repeat it. (Ibid.)

Intuitions

Now, in calm weather, to swim in the open ocean is as easy to the practised swimmer as to ride in a spring-carriage ashore. But the awful lonesomeness is intolerable. The intense concentration of self in the middle of such a heartless immensity, my God! who can tell it? Mark, how when sailors in a dead calm bathe in the open sea–mark how closely they hug their ship and only coast along her sides.

(Melville, *Moby Dick*, Chapter 93)

Intuitions

But not yet have we solved the incantation of this whiteness, and learned why it appeals with such power to the soul; and more strange and far more portentous...and yet should be as it is, the intensifying agent in things the most appalling to mankind.

Is it that by its indefiniteness it shadows forth the heartless voids and immensities of the universe, and thus stabs us from behind with the thought of annihilation, when beholding the white depths of the milky way? Or is it, that as in essence whiteness is not so much a colour as the visible absence of colour; and at the same time the concrete of all colours; is it for these reasons that there is such a dumb blankness, full of meaning, in a wide landscape of snows — a colourless, all-colour of atheism from which we shrink?

(Meville, *Moby Dick*, Chapter 42).

Intuitions

The sea had jeeringly kept his finite body up, but drowned the infinite of his soul. Not drowned entirely, though. Rather carried down alive to wondrous depths, where strange shapes of the unwarped primal world glided to and fro before his passive eyes; and the miser-merman, Wisdom, revealed his hoarded heaps; and among the joyous, heartless, ever-juvenile eternities, Pip saw the multitudinous, God-omnipresent, coral insects, that out of the firmament of waters heaved the colossal orbs. He saw God's foot upon the treadle of the loom, and spoke it; and therefore his shipmates called him mad. So man's insanity is heaven's sense; and wandering from all mortal reason, man comes at last to that celestial thought, which, to reason, is absurd and frantic; and weal or woe, feels then uncompromised, indifferent as his God.

Towards a theory of understanding

General picture:

- Beyond knowledge, we look to mathematics for modes of *understanding*.
- Understanding involves not just factual knowledge, but something more dynamic: ways of proceeding, modes of analysis, capacities for thought.
- We value mathematical resources for conferring understanding.
- Some mathematical resources are overtly syntactic: definitions, theorems, proofs, questions.
- These give rise to resources that are harder to characterize precisely: concepts, methods, heuristics, intuitions, . . .

A methodological stance

To make progress, we have to pick a methodological framework:

- a way of thinking about mathematics
- a language for talking about the objects of mathematical understanding
- a way of posing questions precisely (or at least trying to)
- precise, disciplined ways of answering them

We just have to do it, and see what happens.

A methodological stance

We want a philosophical theory of mathematical understanding that

- is coherent
- is satisfying
- can inform (and is informed by) other pursuits:
  - history of mathematics
  - interactive theorem proving and automated reasoning
  - psychology and cognitive science
  - mathematics education
  - mathematics itself

I will make some personal recommendations here.

Recommendations

First recommendation: stay grounded in syntax.

What characterizes mathematics with respect to other scholarly disciplines is its level of rigor: there are precise norms that dictate how to make meaningful mathematical claims, and how to establish their truth.

We can (and have) studied these norms in syntactic terms, with great success.

Definitions, theorems, proofs, conjectures, questions, and the like — the "literature" — constitute the starting data.

The more nebulous objects of understanding — concepts, methods, intuitions, etc. — are manifested in the linguistic artifacts.

Recommendations

Second recommendation: think of the philosophy of mathematics as a design science, like automative engineering.

A closer look at the syntactic components of mathematics — definitions, theorems, proofs, theories, and so on — shows them to be highly structured objects.

When one studies the history of mathematics, or tries to model *real* mathematical proofs formally, one has the sense that mathematical language is beautifully *designed* to extend our cognitive reach, make it possible for us to solve increasingly more difficult problems, construct more elaborate proofs.

What are the general principles?

Recommendations

Third recommendation: start with more specific, focused projects.

I will discuss three strategies for making progress:

- look to the everyday practice of mathematics
- look to interactive theorem proving
- look to the history of mathematics

I will also provide illustrative examples.

Strategies

First strategy: look at ordinary mathematical proofs.

- What are the (inferential and communicative) norms that are in play?
- What cognitive capacities that are presupposed by their comprehensibility?

Compare alternative proofs, or textbook presentations, of the same theorem. Explain

- the structuring of information, and
- the understanding or expertise that is conveyed.

We need to rely on what mathematicians *do* rather than their self assessments.

The *Elements*

For more than two thousand years, Euclid's *Elements* was held to be the paradigm for rigorous argumentation.

But the nineteenth century raised concerns:

- Conclusions are drawn from diagrams, using "intuition" rather than precise rules.
- Particular diagrams are used to infer general results (without suitable justification).

Axiomatizations due to Pasch and Hilbert, and Tarksi's formal axiomatization later on, were thought to make Euclid rigorous.

The *Elements*

But in some ways, they are unsatisfactory.

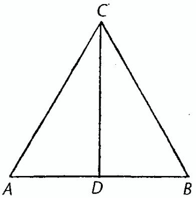
- Proofs in the new systems look very different from Euclid's.
- The initial criticisms belie the fact that Euclidean practice was remarkably stable for more than two thousand years.

What is going on?

What are the norms that govern Euclidean reasoning?

Proposition 10

To bisect a given finite straight line.



Let AB be the given finite straight line.

Thus it is required to bisect the finite straight line AB.

Let the equilateral triangle ABC be constructed on it, [I. 1]
and let the angle ACB be bisected by the straight line CD; [I. 9]

I say that the straight line AB has been bisected at the point D.

For, since AC is equal to CB, and CD is common,
the two sides AC, CD are equal to the two sides BC, CD respectively;
and the angle ACD is equal to the angle BCD;
therefore the base AD is equal to the base BD. [I. 4]

Therefore the given finite straight line AB has been bisected at D.

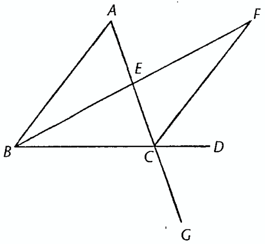
Q.E.F.

Proposition 16

In any triangle, if one of the sides be produced, the exterior angle is greater than either of the interior and opposite angles.

Let *ABC* be a triangle, and let one side of it *BC* be produced to *D*;

I say that the exterior angle *ACD* is greater than either of the interior and opposite angles *CBA*, *BAC*.



Let *AC* be bisected at *E*, **[I. 10]**
and let *BE* be joined and produced in a straight line to *F*;
let *EF* be made equal to *BE*, **[I. 3]**
let *FC* be joined, **[Post. 1]**
and let *AC* be drawn through to *G*. **[Post. 2]**

Then, since *AE* is equal to *EC*, and *BE* to *EF*,
the two sides *AE*, *EB* are equal to the two sides *CE*, *EF* respectively;
and the angle *AEB* is equal to the angle *FEC*, for they are vertical angles. **[I. 15]**

Therefore the base *AB* is equal to the base *FC*, and the triangle *ABE* is equal to the triangle *CFE*,
and the remaining angles are equal to the remaining angles respectively, namely those which the equal sides subtend; **[I. 4]**

therefore the angle *BAE* is equal to the angle *ECF*.

First salient feature: the use of diagrams

Observation (Manders): In a Euclidean proof, diagrams are only used to infer "co-exact" (regional / topological) information, such as incidence, intersection, containment, etc.

Exact (metric) information, like congruence, is always made explicit in the text.

Poincaré: "Geometry is the art of precise reasoning from badly constructed diagrams."

Abstraction: take the "diagram" to be a representation of the relevant data.

Second salient feature: generality

Not every feature found in a particular diagram is generally valid.

Euclid manages to avoid drawing invalid conclusions. We need an explanation as to what secures the generality.

The *Elements*

Edward Dean, John Mumma, and I did the following:

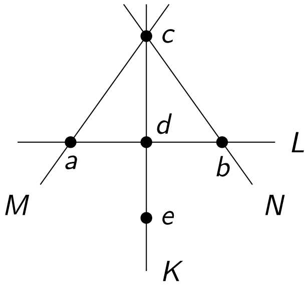
- Described a formal system that is much more faithful to Euclid.
- Argued that the system is sound and complete (for the theorems it can express) relative to Euclidean fields.
- Showed that the system can easily be implemented using contemporary automated reasoning technology.

In particular, we gave an account of "diagrammatic inference" that (we argue) explains what we see in Euclid.

Proposition I.10. Assume a and b are distinct points on L. Construct a point d such that d is between a and b, and $\overline{ad} = \overline{db}$.

By Proposition I.1 applied to a and b, let c be a point such that $\overline{ab} = \overline{bc}$ and $\overline{bc} = \overline{ca}$ and c is not on L.

Let M be the line through c and a.
Let N be the line through c and b.
By Proposition I.9 applied to a, c, b, M, N, let e be a point such that $\angle ace = \angle bce$, b and e are on the same side of M, and a and e are on the same side of N.
Let K be the line through c and e.
Let d be the intersection of K and L.
Hence $\angle ace = \angle acd$.
Hence $\angle bce = \angle bcd$.



By Proposition I.4 applied to a, c, d, b, c, d have $\overline{ad} = \overline{bd}$.
Q.E.F.

Lessons

What we learned:

- What makes something *mathematics* is that there are effective norms and mechanisms that secure agreement from practitioners. These are often implicit, and not obvious.
- There is nothing very mysterious about diagrams. They carry discrete information, and their use is governed by rules, just as words and symbols do.
- One can disentangle the mathematics from history and psychology.

Lessons

Philosophy of mathematics *should* interact with, and provide conceptual foundations for, fields that rely on some understanding of what it means to do mathematics:

- mathematics itself
- computer science
- history of mathematics
- psychology and cognitive science
- education, pedagogy

But, to be make progress on core issues, we have to be clear about the questions we are asking.

Lessons

Sample questions:

- Logical: what role does visualization and diagrammatic reasoning play in mathematics?
- Cognitive: how do we do it?
- Computational: how can we support it or emulate it?
- Historical: how did these uses arise and evolve?
- Pedagogical: how should we use visualization in teaching?
- ...

The first question informs, and is informed by, the others.

Strategies

Second strategy: look to interactive theorem proving and automated reasoning.

Formal verification involves the use of formal methods to verify correctness, for example:

- verifying that a circuit description, an algorithm, or a network or security protocol meets its specification; or
- verifying that a proof of a mathematical theorem is correct.

"Interactive theorem proving" is one important approach.

Strategies

Working with a proof assistant involves conveying enough information to the system to confirm that there is a formal axiomatic proof.

In fact, most proof systems actually construct a formal proof object, a complex piece of data that can be verified independently.

"Proof languages" provide expressive models of ordinary mathematical language, designed to convey knowledge (and expertise) efficiently.

Understanding what is needed to develop mathematics formally provides insight into how the informal languages work as well.

Interactive theorem proving

```
theorem PrimeNumberTheorem:
  "(%n. pi n * ln (real n) / (real n)) ----> 1"

!C. simple_closed_curve top2 C ==>
  (?A B. top2 A /\ top2 B /\
    connected top2 A /\ connected top2 B /\
  ~(A = EMPTY) /\ ~(B = EMPTY) /\
   (A INTER B = EMPTY) /\ (A INTER C = EMPTY) /\
       (B INTER C = EMPTY) /\
      (A UNION B UNION C = euclid 2)

!d k. 1 <= d /\ coprime(k,d)
  ==> INFINITE { p | prime p /\ (p == k) (mod d) }
```

Interactive theorem proving

```
Theorem Sylow's_theorem :
  [/\ forall P,
      [max P | p.-subgroup(G) P] = p.-Sylow(G) P,
      [transitive G, on 'Syl_p(G) | 'JG],
      forall P, p.-Sylow(G) P ->
        #|'Syl_p(G)| = #|G : 'N_G(P)|
   &  prime p -> #|'Syl_p(G)| %% p = 1%N].

Theorem Feit_Thompson (gT : finGroupType)
   (G : {group gT}) :
  odd #|G| → solvable G.

Theorem simple_odd_group_prime (gT : finGroupType)
    (G : {group gT}) :
  odd #|G| → simple G → prime #|G|.
```

Interactive theorem proving

```
theorem (in prob_space) central_limit_theorem:
  fixes X :: "nat ⇒ 'a ⇒ real"
    and μ :: "real measure"
    and σ c :: real
    and S :: "nat ⇒ 'a ⇒ real"
  assumes X_indep: "indep_vars (λi. borel) X UNIV"
    and X_integrable: "⋀n. integrable M (X n)"
    and X_mean: "⋀n. expectation (X n) = c"
    and σ_pos: "σ > 0"
    and X_square_integrable:
        "⋀n. integrable M (λx. (X n x)²)"
    and X_variance: "⋀n. variance (X n) = σ²"
    and X_distrib: "⋀n. distr M borel (X n) = μ"
  defines "S n x ≡ ∑ i<n. X i x"
  shows "weak_conv_m (λn. distr M borel
       (λx. (S n x - n * c) / sqrt (n*σ²)))
    std_normal_distribution"
```

Interactive theorem proving

Challenges:

- Modeling mathematical assertions in a natural way.
- Modeling mathematical proof in a natural way.
- Modeling mathematical expertise, and filling in "straightforward" inferences automatically.
- Managing large libraries of information.
- Verifying long computations.

Assertion languages

In ordinary mathematics, an expression may denote:

- a natural number: 3, $n^2 + 1$
- an integer: -5, $2j$
- an ordered triple of natural numbers: $(1, 2, 3)$
- a function from natural numbers to reals: $(s_n)_{n \in \mathbb{N}}$
- a set of reals: $[0, 1]$
- a function which takes a measurable function from the reals to the reals and a set of reals and returns a real: $\int_A f \; d\lambda$
- an additive group: $\mathbb{Z}/m\mathbb{Z}$
- a ring: $\mathbb{Z}/m\mathbb{Z}$
- a module over some ring: $\mathbb{Z}/m\mathbb{Z}$ as a $\mathbb{Z}$-module
- an element of a group: $g \in G$
- a function which takes an element of a group and a natural number and returns another element of the group: g^n
- a homomorphism between groups: $f : G \to G$
- a function which takes a sequence of groups and returns a group: $\prod_i G_i$
- a function which takes a sequence of groups indexed by some diagram and homomorphisms between them and returns a group: $\lim_{i \in D} G_i$

Assertion languages

In ordinary mathematical language, a lot is left implicit.

In an interactive theorem prover, everything has to be made explicit.

The infrastructure needed to model ordinary mathematical vernacular is substantial:

- implicit arguments
- overloading
- type classes
- unification hints
- modules

Assertion languages

```
class semigroup (α : Type u) extends has_mul α :=
(mul_assoc : ∀ a b c, a * b * c = a * (b * c))

class monoid (α : Type u) extends semigroup α, has_one α :=
(one_mul : ∀ a, 1 * a = a) (mul_one : ∀ a, a * 1 = a)

def pow {α : Type u} [monoid α] (a : α) : ℕ → α
| 0     := 1
| (n+1) := a * pow n

theorem pow_add {α : Type u} [monoid α] (a : α) (m n : ℕ) :
  a^(m + n) = a^m * a^n :=
begin
  induction n with n ih,
  { simp [add_zero, pow_zero, mul_one] },
  rw [add_succ, pow_succ', ih, pow_succ', mul_assoc]
end

instance : linear_ordered_comm_ring int := ...
```

Proof languages

```
theorem sqrt_two_irrational {a b : ℕ} (co : coprime a b) :
  a^2 ≠ 2 * b^2 :=
assume h : a^2 = 2 * b^2,
have even (a^2),
  from even_of_exists (exists.intro _ h),
have even a,
  from even_of_even_pow this,
obtain (c : ℕ) (aeq : a = 2 * c),
  from exists_of_even this,
have 2 * (2 * c^2) = 2 * b^2,
  by rw [-h, aeq, *pow_two, mul.assoc, mul.left_comm c],
have 2 * c^2 = b^2,
  from eq_of_mul_eq_mul_left dec_trivial this,
have even (b^2),
  from even_of_exists (exists.intro _ (eq.symm this)),
have even b,
  from even_of_even_pow this,
have 2 | gcd a b,
  from dvd_gcd (dvd_of_even <even a>) (dvd_of_even <even b>),
have 2 | 1,
  by rw [gcd_eq_one_of_coprime co at this]; exact this,
show false,
  from absurd <2 | 1> dec_trivial
```

Lessons

Some of the things we have learned:

- Language is important.
- Notation is important.
- Definitions are important.
- Organization is important.
- Structure is important.
- Infrastructure is important.
- Matching and unification are important.
- Indexing and retrieval are important.
- Methods of reasoning are important.
- Heuristics are important.

The philosophy of mathematics should help us better understand how, and why.

Lessons

Designing a theorem prover involves designing a language (in a broad sense):

- axioms, rules
- syntax, notation
- semantics
- idioms
- concepts
- theories

A theorem prover and its libraries can be well designed, or poorly designed.

The same is true of a piece mathematics.

Strategies

Third strategy: look to the history of mathematics.

Find an important historical development (what Ken Manders calls a "big deal difference").

This suggests that we were in

- a certain epistemological state beforehand, and
- a certain epistemological state after,

and that they are different in some important way.

Explain the difference.

Dirichlet's theorem

In 1837, Dirichlet proved that there are infinitely many primes in any arithmetic progression in which the first two terms are relatively prime.

For example, there are no primes in the sequence

$$10, 25, 40, 55, 70, 85, \ldots$$

There are infinitely many primes in the sequence

$$4, 13, 22, 31, 40, 40, \ldots$$

Dirichlet's theorem

Contemporary presentations use *Dirichlet characters*, certain types of functions $\chi : \mathbb{N} \to \mathbb{C}$.

The proofs are higher order:

- One considers sets of characters.
- The characters modulo a positive integer m form a group.
- One considers functions $L(s, \chi)$ that take characters as arguments.
- One sums over sets of characters,

$$\sum_{\chi} \overline{\chi(m)} \log L(s, \chi) = \ldots$$

In short, functions are mathematical objects, like numbers.

Dirichlet's theorem

Rebecca Morris and I studied presentations of Dirichlet's theorem to understand how functions came to be treated in that way.

- Dirichlet 1837: Dirichlet's original proof
- Dirichlet 1840, 1841: extensions to Gaussian integers, quadratic forms
- Dedekind 1863: presentation of Dirichlet's theorem
- Dedekind 1879, Weber 1882: characters on arbitrary abelian groups
- Hadamard 1896: Dirichlet's theorem and extensions
- de la Vallée Poussin 1897: Dirichlet's theorem and extensions
- Kronecker (1901, really 1870's and 1880's): constructive, quantitative proof of Dirichlet's theorem
- Landau 1909, 1927: Dirichlet's theorem and extensions

Lessons

Some things we learned:

- Through the middle of the nineteenth century, the word "function" was used exclusively for functions on the reals or complexes.
- Even the phrase "number-theoretic function" was novel in 1850.
- There was no general concept of function until around 1879.

Lessons

Regarding the characters:

- There is no (explicit) notion of character in Dirichlet's proof.
- Early authors were reluctant to treat them as arguments to functions.
- Early authors were reluctant to sum over them.
- They were often treated as *intensional* objects, i.e. as expressions.

Modern features appeared in fits and starts.

Lessons

Treating functions as objects brings benefits.

- Expressions are simplified.
- Proofs become modular.
- The reader has to keep track of less information when parsing expressions.
- The reader has to keep track of less information when reading a proof.
- The relevant data and relations are made more salient.
- Lemmas and definitions can be reused elsewhere.
- Lemmas and definitions can be modified and adapted.
- Abstraction leads to greater generality.

Lessons

In other words:

- Dependencies between components are minimized.
- The mathematics become easier to understand.
- It becomes easier to ensure correctness.
- Components are adaptable, reusable, and generalizable.
- Proofs can be modified and varied more easily.

These are exactly the benefits associated with modularity in software engineering.

REFACTORING

IMPROVING THE DESIGN OF EXISTING CODE

MARTIN FOWLER

With Contributions by Kent Beck, John Brant, William Opdyke, and Don Roberts

Foreword by Erich Gamma
Object Technology International Inc.

OBJECT TECHNOLOGY SERIES
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Lessons

Why did it take so long to arrive at the contemporary treatment of functions?

Reading mathematics involves a good deal of tacit knowledge.

When I publish a proof, my intention is that you will read it, understand it, and accept it as correct.

Lessons

Concerns raised by any methodological expansion:

- Are the new methods, concepts, and notations meaningful?
- Do they come with clear rules of use?
- Are they appropriate to the mathematics?
- Do they answer the questions we have asked?
- Do they provide the information we want?
- Are they reliable?
- Are they likely to cause error or confusion?

Changes to the practice have to be accepted by the *community*.

The philosophy of mathematics can help us weigh the pros and cons.

Concluding remarks

Recall the outline of this talk:

- General epistemological questions
- Intuitions
- Towards a theory of mathematical understanding
  - Stay grounded in syntax
  - Think of mathematics in terms of design
  - Find more focused questions and projects
- Strategies
  - Look to mathematical practice
  - Look to interactive theorem proving
  - Look to the history of mathematics

Concluding remarks

We care about mathematics.

- We subject our children to countless hours of mathematical training.
- We put a lot of faith in mathematical results.
- We applaud mathematical achievements.

The subject deserves philosophical study that helps us understand what it means to do mathematics, and helps us do it better.

Concluding remarks

Strategy:

- Keep the general questions in mind.
- Find more precise, focused questions.
- Look to domains of application, such as
  - formal verification and automated reasoning
  - mathematical pedagogy and cognitive science
  - history (and historiography) of mathematics
  - mathematics itself

Over time, small but concrete advances will hopefully come together to give us a coherent theory of mathematical understanding.

Concluding remarks

And what if they don't?

Then we will have merely contributed to the conceptual foundations of automated reasoning, cognitive science, pedagogy, history of science, and so on — and learned some interesting things about mathematics as well.