Simplicity, extensionality, and functions as objects (or: Character and object)

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From methodology to metaphysics

The philosophy of mathematics has traditionally addressed issues regarding the nature of mathematical objects and the proper justification for mathematical knowledge.

As of late, some have begun to address a broader array of issues: What is mathematical understanding? Why are some proofs better than others? In what sense can a proof *explain* a result? What makes a concept fruitful, or a definition the "right" definition? Why are some historical advances especially important?

Sometimes "methodology" is distinguished from "metaphysics."

But it is a mistake to view these as disjoint.

From methodology to metaphysics

Some "metaphysical" questions: Are there infinite totalities? Are there complex numbers? Are there sets and functions, and what properties do they have? Are there infinitesimals?

Compare to: Is there a nontrivial root of Riemann zeta function with real part not equal to $1/2$? Are there any noncyclic simple groups of odd order?

This is reminiscent of the distinction between "internal" and "external" questions, but Quine also recognized important differences between the two.

In short, it seems uncontroversial to say that insofar as there are rational ways to address both sets of questions, the appropriate methods differ.

From methodology to metaphysics

Reductionism (set theory, possible points of space time):

- Justification of basic axioms is problematic.
- Focus on what is *allowed*, rather than what one *ought* to do.

Antirealism / realism:

- "Doublethink": there are prime numbers, but there aren't really numbers.
- There seems to be no fact of the matter in the debate.
- Focus on abstracta in general, but not in particular.

From methodology to metaphysics

Let's interpret the metaphysical questions as (reflective) scientific questions as to

- what objects we should admit into our ontology; and
- how we should reason about them.

"Metaphysics" should weigh the pros and cons.

From methodology to metaphysics

What has happened historically:

- Expansions are met with caution and concern.
- Sometimes expansions can be explained in terms of a more conservative theory (complex numbers as ordered pairs, algebraic proofs understood geometrically).
- Otherwise, sometimes the expansions can at least be explained away in particular instances (nonconstructive proofs can be constructivized, infinitesimals eliminated), and, moreover, come with clear rules of use.
- Over time, the expansions become more than useful shorthands, and the nonconservative aspects don't seem to cause problems.

(Cf.: the introduction of negative numbers, algebraic methods in geometry, infinitesimals in the calculus, points at infinity, abstract algebraic arguments, ideals, cosets, equivalence classes, nonconstructive definitions, infinitary objects, and so on.)

From methodology to metaphysics

This is rational!

Concerns:

- consistency / coherence / control
- loss of meaning / utility

Benefits:

- efficiency, economy of thought, ability to manage complex details by suppressing / ignoring irrelevant detail, making key features salient
- generality, ability to transfer to other domains.

“Metaphysics” should give us better wherewithal to evaluate the pros and the cons.

From methodology to metaphysics

Goal of this talk:

- To explore the evolution of the mathematical treatment of certain types of functions (characters) that arose in nineteenth century mathematics.
- To understand the concerns that accompanied these changes.
- To understand the benefits that accompanied these changes.

Overview

An outline of this talk:

- The concept of function in the nineteenth century
- Dirichlet's theorem on primes in an arithmetic progression
- Notable features of the modern understanding
- Frege on function and object
- An analysis of Frege's motivations
- The evolution of proofs of Dirichlet's theorem
  - Dirichlet 1837
  - Dedekind 1871
  - Hadamard 1896
  - de la Vallée-Poussin 1895–1897
- Analysis
- Conclusions

Functions in the nineteenth century

Nineteenth century instances of the function concept:

1. Functions defined on the continuum ($\mathbb{R}$ to $\mathbb{R}$, $\mathbb{C}$ to $\mathbb{C}$)
2. Sequences and series ($\mathbb{N}$ to $\mathbb{R}$ or $\mathbb{Q}$)
3. Number theoretic functions ($\mathbb{N}$ to $\mathbb{N}$)
4. Transformations of the plane
5. Permutations of a finite set A (bijections from A to A)
6. Characters ($\mathbb{Z}$ to $\mathbb{C}$, or $(\mathbb{Z}/m\mathbb{Z})^*$ to $\mathbb{C}$)
7. Arbitrary mappings, or correspondences, between domains

Functions in the nineteenth century

Some landmarks:

- In 1850, Eisenstein explicitly introduced the term "zahlentheoretische Funktion."
- Dedekind 1854: "Über die Einführung neuer Funktionen in der Mathematik; Habilitationsvortrag"
- In 1879, in the third edition of the *Vorlesungen*, Dedekind refers to characters on the class groups as functions.
- In 1879, in the *Begriffsschrift*, Frege introduces a very general notion of function.
- In 1888, Dedekind considers arbitrary mappings (*Abbildung*) between domains.

Functions in the nineteenth century

For the concept of function, one moved away from the necessity of an analytic connection, and began to view its essence (of that concept) in the tabular "composition" of a row of values associated with the values of one or several variables. Thus, it became possible, to categorize those functions under the concept that—due to "conditions" of an arithmetic nature—receive a determinate sense only when the variables occurring in them have integral values or only for certain value-combinations arising from the natural number series. For intermediate values, such functions remain either indeterminate and arbitrary or without any meaning. (Eisenstein, 1950; translated with help from Wilfried Sieg)

Functions in the nineteenth century

. . . the function $\chi(\mathfrak{a})$ *also posseses the property that it takes the same value on all ideals* $\mathfrak{a}$ *belonging to the same class* A*; this value is therefore appropriately denoted by* $\chi(A)$ *and is clearly always an hth root of unity. Such functions* χ*, which in an extended sense can be termed characters, always exist; and indeed it follows easily from the theorems mentioned at the conclusion of* §*149 that the class number h is also the number of all distinct characters* $\chi_1, \chi_2, \ldots, \chi_h$ *and that every class* A *is completely characterized, i.e. is distinguished from all other classes, by the h values* $\chi_1(A), \chi_2(A), \ldots, \chi_h(A)$*. (Dedekind 1879, translation by Hawkins)*

Dirichlet's theorem

Dirichlet's famous theorem of 1837:

Theorem

Let a and d be relatively prime. Then the arithmetic progression $a, a+d, a+2d, \ldots$ contains infinitely many primes.

If G is a finite abelian group, a *character χ on G* is a homomorphism from G to the nonzero complex numbers: $\chi(g_1 g_2) = \chi(g_1)\chi(g_2)$ for every g_1 and g_2 in G.

The set of characters on G forms a group, denoted $\hat{G}$.

Dirichlet's theorem

If $G = \{1, g, g^2, \ldots, g^{n-1}\}$ is cyclic, $\chi(g)$ must be an nth root of 1, and each such root determines a unique character.

So $\hat{G}$ is also cyclic, and isomorphic to G.

In the more general case, write $G \simeq G_1 \times \ldots \times G_k$, with each G_i cyclic.

Can show $\hat{G} \simeq \hat{G}_1 \times \ldots \times \hat{G}_k$.

Dirichlet's theorem

The following two "orthogonality" relations hold:

$$\sum_{g \in G} \chi(g) = \begin{cases} |G| & \text{if } \chi = \chi_0 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\sum_{\chi \in \hat{G}} \chi(g) = \begin{cases} |G| & \text{if } g = 1 \\ 0 & \text{otherwise} \end{cases}$$

For the first, pick h such that $\chi(h) \neq 1$ and note

$$\chi(h) \sum_{g \in G} \chi(g) = \sum_{g \in G} \chi(hg) = \sum_{g \in G} \chi(g).$$

This makes it possible to do "finite Fourier analysis": if $\hat{f}(\chi) = \sum_g f(g)\chi(g)$, then $f = \frac{1}{|G|} \sum_\chi \hat{f}(\chi)\chi$.

Dirichlet's theorem

Fix m, and "lift" the characters on $(\mathbb{Z}/m\mathbb{Z})^*$ to functions on $\mathbb{N}$.

Define

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.$$

Euler product expansion:

$$L(s,\chi) = \prod_{p} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1} = \prod_{p \nmid m} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}$$

This converges when $re(s) > 1$.

Dirichlet's theorem

Taking logarithms of both sides yields

$$\log L(s,\chi) = \sum_{p \nmid m} \frac{\chi(p)}{p^s} + O(1).$$

Multiply both sides by $\overline{\chi(a)}$ and sum over χ.

$$\sum_{\chi} \overline{\chi(a)} \log L(s,\chi) = \sum_{\chi} \sum_{p \nmid m} \overline{\chi(a)} \frac{\chi(p)}{p^s} + O(1).$$

Using the orthogonality relations,

$$\sum_{\chi} \overline{\chi(a)} \log L(s,\chi) = \phi(m) \sum_{p \equiv a \bmod m} \frac{1}{p^s} + O(1).$$

Dirichlet's theorem

We have, when $re(s) > 1$:

$$\sum_{\chi} \overline{\chi(a)} \log L(s, \chi) = \phi(m) \sum_{p \equiv a \bmod m} \frac{1}{p^s} + O(1).$$

Let $s \to 1$ from above.

Divide the characters into three types:

1. The trivial character, χ_0.
2. The nontrivial real-valued characters.
3. The (properly) complex characters.

Show:

- $L(s, \chi_0)$ has a simple pole at $s = 1$.
- For $\chi \neq \chi_0$, $L(s, \chi)$ has a nonzero limit at $s \to 1$.

This yields the result.

The most difficult case involves the nontrivial real-valued characters.

Functions as objects

Nineteenth century: methodological changes

1. Unification of the function concept
2. Generalization of the function concept
3. Liberalization of the function concept
4. Extensionalization of functions
5. Reification of the function concept

Functions as objects

"Reification of the function concept" is vague.

Some aspects:

1. Extensionalization and independence of representation
2. Sending functions as arguments to other functions, $F(f)$.
3. Forming collections of functions.
4. Quantifying over functions (in definitions, in theorems).

Functions as objects

In the modern presentation of Dirichlet's theorem:

- The notion of a character is defined.
- One determines some of their properties.
- Characters appear as arguments to other functions.
- One sums over sets of characters, without having representations for any particular one.
- One carries out proofs (in fact, one has to!) without making reference to any particular representation.
- One characterizes sets of characters extensionally.

These are the main points of contrast with the historical sources.

Function and object in Frege

In modern simple type theory:

- Predicates on the natural numbers have type $\mathbb{N} \to \mathbb{B}$ (type 1)
- Binary relation have type $\mathbb{N} \times \mathbb{N} \to \mathbb{B}$ (type 1)
- Sequences of natural numbers have type $\mathbb{N} \to \mathbb{N}$ (type 1)
- The real numbers, $\mathbb{R}$, are type 1
- Functions from $\mathbb{R}$ to $\mathbb{R}$ are type 2
- Sets of functions from $\mathbb{R}$ to $\mathbb{R}$ are type 3
- Sets of sets of reals are type 3
- For example, a set of measures on the Borel sets of $\mathbb{R}$ is a type 4 object.
- And so on...

Function and object in Frege

. . . it will not do to call a general concept word the name of a thing. That leads straight to the illusion that the number is a property of a thing. (Grundlagen)

We may say in brief, taking "subject" and "predicate" in the linguistic sense: a concept is the Bedeutung of a predicate; and object is something that can never be the whole Bedeutung of a predicate, but can be the Bedeutung of a subject. (Concept and object)

Function and object in Frege

It must indeed be recognized that we are confronted by an awkwardness of language... if we say that the concept horse is not a concept... (Concept and object)

The business of a general concept word is precisely to signify a concept. Only when conjoined with the definite article or a demonstrative pronoun can it be counted as the proper name of a thing, but in that case it ceases to count as a concept word. The name of a thing is a proper name. (Grundlagen)

If we keep it in mind that in my way of speaking expressions like 'the concept F' designate not concepts but objects, most of Kerry's objections already collapse. (Concept and object)

Function and object in Frege

To the question, what the number 1 is, or what the sign 1 refers to, one mostly receives the answer, "Well, now, a thing." (Grundlagen)

...our concern here is to arrive at a concept of number usable for the purposes of science; we should not, therefore, be deterred by the fact that in the language of everyday life number appears also in attributive constructions. That can always be got round. (Grundlagen)

Function and object in Frege

After discussing the historical introduction of derivatives:

> *Now at this point people had particular second-level functions, but lacked the conception of what we have called second-level functions. By forming that, we make the next step forwards. One might think that this would go on. But probably this last step is not so rich in consequences as the earlier ones; for instead of second-level functions one can deal, in further advances, with first-level functions – as shall be shown elsewhere. (Function and Concept)*

Function and object in Frege

Other aspects of Frege's development:

- There is only one basic type.
- There is no identity between elements of the higher types.
- His system does *not* include the axiom $\forall x \ \varphi(x) \rightarrow \varphi(t)$ for any x beyond type 1.
- For elements of type 1, he only uses it once!

The type of objects is then a lot like the universe of sets. Higher type functions are only linguistic glue.

Function and object in Frege

Marco Panza writes:

> *. . . according to Frege, appealing to functions is indispensable in order to fix the way his formal language is to run, but functions are not as such actual components of the language. More generally, functions manifest themselves in our referring to objects—either concrete or abstract—and making statements about them, but they are not as such actual inhabitants of some world of concreta and abstracta. Briefly: Frege's formal language, as well as ordinary one[s], display functions, but there are no functions as such.*

Analysis

Striking features:

1. functions (in the logical and linguistic sense) are not objects
2. functions (in mathematical usage) are objects, and extensional.

Two possibilities:

1. Frege determined they should be separate on broad metaphysical grounds, and then designed the logic accordingly.
2. Frege designed the logic, determined it worked out best with a separation of individuals and functions, and read off the metaphysics.

But this way of framing the issue misses the point.

Analysis

Better questions:

1. What considerations push Frege to maintain the sharp distinction between function and object?
2. What considerations push Frege to identify mathematical entities as objects?

Analysis

If "one man" could be understood similarly as "wise man," then one could think that "one" could also be used as a predicate, so that as one says "Solon was wise," one could also say "Solon was one" or "Solon was a one." Although the last expression can occur, it is nevertheless not understandable in itself alone. It can, for example, mean "Solon was a wise man," if "wise man" can be supplied from the context. But alone is appears that "one" could not be a predicate. This shows itself even more clearly in the plural. Whereas one can combine "Solon was wise" and "Thales was wise" into "Solon and Thales were wise," one cannot say that "Solon and Thales were one." From this the impossibility is not understandable, if "one" in the same as "wise" were as much a property of Solon as it is also of Thales. (Grundlagen, §29)

Analysis

. . . a number does not vary; for we have nothing of which we could predicate the variation. A cube never turns into a prime number; an irrational number never becomes rational. (What is a Function, 1904)

In other connections, indeed, we say that an object assumes a property, here the number must play both parts; as an object it is called a variable or a variable magnitude, and as a property it is called a value. That is why people prefer the word 'magnitude' to the word 'number'; they have to deceive themselves about the fact that the variable magnitude and the value it is said to assume are essentially the same thing, that in this case we have not got an object assuming different properties in succession, and that therefore there can be no question of a variation. (ibid.)

Analysis

The endeavor to be brief has introduced many inexact expressions into mathematical language, and these have reacted by obscuring thought and producing faulty definitions. Mathematics ought properly to be a model of logical clarity. In actual fact there are perhaps no scientific works where you will find more wrong expressions, and consequently wrong thoughts, than in mathematical ones. Logical correctness should never be sacrificed to brevity of expression. It is therefore highly important to devise a mathematical language that combines the most rigorous accuracy with the greatest possible brevity. To this end a symbolic language would be best adapted, by means of which we could directly express thoughts in written or printed symbols without the intervention of spoken language. (ibid.)

Analysis

Why insist that (mathematical) functions are objects?

Consider the following statements:

- "there are two truth values"
- "there are two natural numbers strictly between 5 and 8"
- "there are two constant functions taking values among the truth values"
- "there are two characters on $(\mathbb{Z}/4\mathbb{Z})^*$"
- "there are two subsets of a singleton set."

Compare to Dirichlet's theorem:

- We want to sum over finite sets of numbers.
- We want to sum over finite sets of characters.

Analysis

In contemporary mathematics, we can speak of:

- the group of units modulo m
- the group of characters of a finite abelian group
- the group of automorphisms of another group
- homormorphisms between any two groups

It is important that all groups are on the same level!

Otherwise, we would have to speak of the type $i+1$ group of automorphisms of a type i group, and homomorphisms from a type i group to a type j group.

This also speaks to treating functions extensionally: "there are $\varphi(m)$ characters on $(\mathbb{Z}/m\mathbb{Z})^*$" would be false otherwise.

Analysis

In sum, Frege is concerned with:

- control: having clear, consistent rules of use
- efficacy: having feasible and efficient ways of carrying out the mathematics
- meaning: having some sense of how basic terms are interpreted

Two main goals:

1. We need a way of dealing with mathematical objects uniformly, since mathematical constructions and operations have to be applied to many sorts of objects, many of which cannot be foreseen in advance.
2. We need a flexible but rigorous way of talking about higher-type entities, like functions and predicates, without falling into inconsistency.

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Dirichlet 1837

Dirichlet does not introduce a notation for characters; rather he uses explicit expressions.

In the case where p is prime,

- Let c be a primitive element modulo p.
- For every n coprime to p, let γ_n be such that $c^{\gamma_n} \equiv n \bmod p$.
- Characters χ correspond to $p-1$st roots of unity ω, where $\chi(c^{\gamma_n} \bmod p) = \omega^n$.
- Dirichlet writes ω^{γ_n} where we would write $\chi(n)$.

Pick a generator Ω of the $p-1$st roots of unity, $\{\Omega^0, \ldots, \Omega^{p-2}\}$.

L_m is the L-series corresponding to the root Ω^m. Dirichlet sums over m, rather than χ.

Dirichlet 1837

After demonstrating that the Euler product formula,

$$\prod \frac{1}{1-\omega^{\gamma}\frac{1}{q^{s}}} = \sum \omega^{\gamma}\frac{1}{n^{s}} = L,$$

Dirichlet writes:

> *The equation just found represents $p-1$ different equations that result if we put for ω its $p-1$ values. It is known that these $p-1$ different values can be written using powers of the same Ω when it is chosen correctly, to wit:*
>
> $$\Omega^0,\ \Omega^1,\ \Omega^2,\ \ldots, \Omega^{p-2}$$
>
> *According to this notation, we will write the different values L of the series or product as:*
>
> $$L_0,\ L_1,\ L_2, \ldots,\ L_{p-2}\ \ldots$$
>
> *(Dirichlet 1837, 3)*

Dirichlet 1837

In the case where the modulus k is not prime,

- Decompose $(\mathbb{Z}/k\mathbb{Z})^*$ into a product of cycles.
- Choose generators for each cyclic group.
- A number n modulo k has indices α_n, β_n, γ_n, $\gamma'_n, \ldots$
- Each character corresponds to a choice of roots of unity, $\theta, \varphi, \omega, \omega', \ldots$
- Dirichlet writes $\theta^\alpha \varphi^\beta \omega^\gamma \omega'^{\gamma'} \ldots$ where we would write $\chi(n)$.

Notice that the dependence on n is left implicit.

Moreover, as before, if we choose appropriate primitive roots of unity, each character is given by a list of indices $a, b, c, c', \ldots$.

Thus Dirichlet writes $L_{a,b,c,c',\ldots}$ in "a comfortable way" where we would write $L(s, \chi)$.

Dirichlet 1837

Summing over characters: Dirichlet writes

$$\log L_0 + \Omega^{-\gamma_m} \log L_1 + \Omega^{-2\gamma_m} \log L_2 + \ldots + \Omega^{-(p-2)\gamma_m} \log L_{p-2}$$

where we would write $\sum_\chi \overline{\chi(m)} \log L(s, \chi)$.

For composite k, he writes

$$\sum \Theta^{-\alpha_m a}\ \Phi^{-\beta_m b} \Omega^{-\gamma_m c} \Omega^{-\gamma_{m'} c'} \ldots \log L_{a,b,c,c' \ldots}$$

where the sum is over all combinations of $a, b, c, c' \ldots$.

Dirichlet 1837

Dirichlet divides the L functions into three classes:

- the one in which all the roots are 1
- the ones in which all the roots are real (± 1)
- those in which at least one of the roots is not real

This is an *intensional* characterization.

Summary:

- Dirichlet does not name or identify "characters."
- The L functions depend on a sequence of natural numbers ($L_{a,b,c,c' \ldots}$ rather than $L(s,\chi)$)
- Instead of summing over L functions, he sums over these sequences.
- The L functions are classified intensionally.

Dedekind 1871

Dedekind begins by sketching the proof, and introduces "a class of infinite series of the form $L = \sum \psi(n) \ldots$" where ψ is a real or complex function such that $\psi(n)\psi(n') = \psi(nn')$ and $\psi(1) \neq 0$.

He shows that the Euler product formula holds for such series.

After the introduction, Dedekind focuses on the L series relevant here, namely those for which

$$\psi(n) = \frac{\theta^{\alpha}\nu^{\beta}\omega^{\gamma}\omega'^{\gamma'} \ldots}{n^s}.$$

In a footnote, he denoted the numerator of $\phi(n)$ by $\chi(n)$.

Dedekind 1871

We divide these series L into three classes:
In the first class there is only one series L_1, namely the one for which all the roots of unity θ, ν, ω, ω',... have the value 1.
In the second class we include all the remaining series L_2 for which the roots of unity are real, and hence equal to ± 1.
In the third class we include all remaining series L_3, that is, those for which even at least one of the roots of unity is imaginary. The number of these series is even, since they can be grouped in conjugate pairs-if one such series L_3 corresponds to the roots θ, ν, ω, ω',..., then there is a second series corresponding to the roots θ^{-1}, ν^{-1}, ω^{-1}, ω'^{-1},..., and these two systems of roots are not identical.

This division is still intensional.

Dedekind 1871

Recall Dirichlet's notation $L_{a,b,c,c'\ldots}$ for the L series.

Dedekind does not includes this information. Instead, he wrote L_1, L_2, L_3 for L-functions that fall under the first, second and third categories respectively and wrote:

$$\log L_1 + \sum \log(L_2) + \sum \log(L_3 L_3').$$

But when it comes to calculations, Dedekind construes the sums as sums over the explicit data (systems of roots of unity) representing them.

Dedekind 1871

Summary:

1. Dedekind's outline abstracts and generalizes.
2. He attributed conditions and properties to the characters and real or complex functions more generally.
3. He used functional notation $(\chi(n), \phi(n))$ to represent the characters and functions involving the characters.
4. Dedekind's notation for the L-functions included less data concerning their construction.
5. Dedekind was willing to sum over classes of L-functions.

Yet:

1. His classification of the L-functions was intensional.
2. Although he wrote e.g. $\sum \chi \log L$, the sum was not taken over the characters, but, rather, over complex numbers used to construct them.

Hadamard 1896

Citing Dirichlet, Hadamard introduced the characters as follows:

$$\psi_v(n) = \begin{cases} 0 & \text{if } n \text{ is not coprime to } k \\ \theta^{\alpha}\eta^{\beta}\omega^{\gamma}\omega'^{\gamma'}\cdots & \text{if } n \text{ is prime to } k \end{cases}$$

Hadamard 1896

He then defined the L-series as

$$L_v(s) = \sum_{n=1}^{\infty} \frac{\psi_v(n)}{n^s}$$

He showed

$$\sum_v \frac{\log L_v(s)}{\psi_v(m)} = \phi(k)(\sum \frac{1}{q^s} + \frac{1}{2}\sum{}' \frac{1}{q^{2s}} + \frac{1}{3}\sum{}'' \frac{1}{q^{3s}} + \ldots)$$

$$\log \prod_v L_v(s) = \phi(k)(\sum \frac{1}{q^s} + \frac{1}{2}\sum{}' \frac{1}{q^{2s}} + \frac{1}{3}\sum{}'' \frac{1}{q^{3s}} + \ldots)$$

Compare the first to

$$\sum_\chi \overline{\chi(m)} \log L(s,\chi) = \phi(k) \sum_{p \equiv m (mod \ \ q)} \frac{1}{q^s} + O(1).$$

Hadamard 1896

Still classified L series intensionally:

1. Class 1: Consists of only one series, L_1, which corresponds to when $\theta = \eta = \omega = \omega^{'} = 1$
2. Class 2: Consists of those series such that θ, η, etc are all equal to $+1$ or -1, excluding L_1
3. Class 3: Consists of those series where at least one of the numbers are imaginary.

Hadamard 1896

Summary:

1. Hadamard introduced the characters as functions $\psi_\nu(n)$.
2. The L-series L_ν is parameterized by ν
3. He sums over characters by summing over ν (a natural number).
4. Gave an intensional classification of the L-series.

de la Vallée Poussin 1895–1897

de la Vallée Poussin defined the characters

$$\chi(n) = \omega_1^{\nu_1}\omega_2^{\nu_2}\dots$$

and indexed them, $\chi_1, \chi_2, \dots$

> *The characters (mod M) enjoy the following five properties:*
>
> 1. *For two numbers n and n' prime to M, one has*
>
> $$\chi(n)\chi(n') = \chi(nn');$$
>
> 2. *If $n \equiv n' \pmod{M}$ one has the relation*
>
> $$\chi(n) = \chi(n').$$

de la Vallée Poussin 1895–1897

3. *The sum* $\sum'$ *extending over all the numbers prime to and less than* M*, one has, in the case of the principal character,*

$$\sum_n{}' \chi_0 = \phi(M)$$

and for any other character,

$$\sum_n{}' \chi(n) = 0$$

de la Vallée Poussin 1895–1897

4. *The sum S extending over the totality of characters, one has, for any number n prime to M,*

$$S_\chi \chi(n) = 0,$$

except if $n \equiv 1 \pmod{M}$, in which case

$$S_\chi \chi(n) = \phi(m)$$

5. *If M and M' are two numbers prime to each other and prime to n, one has*

$$\chi(n, \bmod\ MM') = \chi(n, \bmod\ M)\chi(n, \bmod\ M')$$

(de la Vallée Poussin 1897; footnote omitted)

de la Vallée Poussin 1895–1897

de la Vallée Poussin still characterized the classes of L series intensionally, though in the simpler case where the modulus is prime also gives the extensional characterization.

In 1897 he denoted the L-functions by $Z(s, \chi)$.

Later in the same work he used more general series:

$$\phi(M) \lim_{s=1} (s-1) \sum_{q} [k(c_q) + k(c_q^{-1})] \frac{lq_1}{q_1^s} = -\lim (s-1) S_\chi \frac{L'(s,k,\chi)}{L(s,k,\chi)}$$

de la Vallée Poussin 1895–1897

Summary:

1. de la Vallée Poussin used functional notation $\chi(n)$ for characters.
2. He determined general properties that the characters satisfy.
3. He used a particular notation, S_χ to signify summation over characters.
4. His also gave an extensional classification of the characters.
5. He used the notation $Z(s, \chi)$.
6. Moreover, he took summations over characters when they occurred in the argument position.

de la Vallée Poussin 1895–1897

And yet, at times, he seems strikingly old fashioned.

After deriving a key identity involving the series $L(s, \chi)$, he writes

> *. . . and this equation* (E) *represents, in reality,* $\varphi(M)$ *distinct equations, obtained by exchanging the different characters among them.*

Analysis

Changes:

- Characters are named, occur in expressions, and as arguments to other functions.
- Their properties are abstracted, characterized.
- One sums over characters rather than representing data.
- One begins to make distinctions based on values rather than representations.

Effects:

- Expressions become simpler.
- Proofs become more modular.
- One needs to keep track of less information at any point in the proof.
- It becomes easy to generalize (e.g. to arbitrary group characters).

Conclusions

We argued that “doing metaphysics” should amount to weighing benefits and concerns.

We have seen some benefits of the modern understanding of characters.

Concerns:

- Loss of concrete, computational meaning.
- Neglect of additional, potentially useful information.
- The language has to be adapted to support the abstractions, for example, to allow summation of characters.

Conclusions

We saw similar concerns in Frege:

- Want to carry out mathematical constructions and operations uniformly.
- Traditional methods have to be extended (carefully) to support this.

"Metaphysics" involves balancing

- consistency, coherence, control, appropriateness, applicability, preservation of meaning, with
- efficiency, ease of use, generalizability, uniformity, convenience, cognitive reach.

Conclusions

We haven't provided a simple answer to the questions "do characters exist?" and "what properties do they have?".

But we have explored some of the advantages and disadvantages of the modern understanding.

This, we hope, sheds some light on modern mathematics and its objects.

Conclusions

The origins of set theory are often located in Cantor's work on the infinite.

We have located additional motivations in algebra.

Algebraic / axiomatic methods:

- Reduce detail manifested in particular expressions and calculations.
- Support modularity, thereby simplifying the "proof context."
- Support generalization and reuse.

Conclusions

Reaping all the benefits requires a language and framework that allows one to:

- "see" algebraic structure in sets of numbers, functions, equivalence classes, transformations, . . . ;
- construct increasingly elaborate instances of such structures;
- treat these structures uniformly;
- do all this in a clear, coherent, consistent way.

This is (in large part) what set theory was designed to do.

In sum, Mic (echoing Ernst Mach) is right: mathematics is all about introducing ideal objects to support economy of thought and expression.