Inverting the Furstenberg correspondence

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Szemerédi’s theorem

Theorem
For every $k$ and $\delta > 0$, there is an $n$ large enough, such that if $A$ is any subset of $\{0, \ldots, n-1\}$ with density at least $\delta$, then $A$ has an arithmetic progression of length $k$.

In 1977, Furstenberg showed that this is a consequence of the following:

Theorem
For every $k$, measure-preserving system $(X, \mathcal{X}, \nu, U)$, and measurable set $E$ with $\nu(E) > 0$, there is an $i > 0$ such that $\nu(E \cap U^{-i}E \cap U^{-2i}E \cap \ldots \cap U^{-(k-1)i}E) > 0$.

The idea: by compactness, a sequence of counterexamples to Szemerédi’s theorem yields a counterexample $(X, \mathcal{X}, \nu, U)$ and $E$. 
Let $n = \{0, \ldots, n - 1\}$. Identify subsets of $n$ with binary sequences. For example, 0110101100 denotes $\{1, 2, 4, 6, 7\} \subseteq 10$.

For $A \subseteq n$, let $D_A(\sigma)$ be the number of times $\sigma$ occurs in $A$ (allowing wraparound), divided by $n$.

Let $2^\omega$ denote Cantor space, with Borel sets $B$.

Let $[\sigma] = \{f \mid \sigma \subset f\}$. So $B$ is generated by $\{[\sigma] \mid \sigma \in 2^{<\omega}\}$.

Let $T$ denote the shift-left map on $2^\omega$, $(Tf)(n) = f(n + 1)$.

For every $A$, $\mu_A([\sigma]) = D_A(\sigma)$ defines a $T$-invariant measure.

But the Furstenberg recurrence theorem is trivially true for $(2^\omega, B, \mu_A, T)$. 
Theorem

Given \((A_n)\) with \(A_n \subseteq n\), there are a \(T\)-invariant measure \(\mu\) on \(2^\omega\) and a subsequence \((A_{n_i})\) such that

\[
\mu([\sigma]) = \lim_{i \to \infty} D_{A_{n_i}}(\sigma),
\]

for every \(\sigma\).

Proof: let \(\mu\) be a limit point of the set \(\{\mu_{A_i}\}\) in the vague topology.

In more elementary terms: iteratively thin out the original sequence to ensure that each \(D_{A_{n_i}}(\sigma)\) converges.

(So apply the measure-theoretic statement with \(E = [1]\).)
In fact, any shift-invariant $\mu$ on $2^\omega$ arises in this way.

**Theorem**

Let $\mu$ be any $T$-invariant measure on $2^\omega$. Then for each $j$ and $\varepsilon$, there are an $n \leq 2^{O(j/\varepsilon)}$ and an $A \subseteq n$, such that for every $\sigma$ of length at most $j$,

$$|\mu([\sigma]) - D_A(\sigma)| < \varepsilon.$$ 

Moreover, if $n$ is sufficiently large (independent of $\mu$), there is always a set $A$ with this property.
Inverting the correspondence

To prove this, fix $j$ and $\varepsilon > 0$, choose $k$ large, and consider $\{[\tau] \mid \text{len}(\tau) = k\}$.

$$
\mu([\sigma]) = \frac{1}{k} \sum_{i < k} \mu(T^{-i}[\sigma])
$$

$$
= \frac{1}{k} \sum_{i < k} \sum_{\tau} \mu(T^{-i}[\sigma] \cap \tau)
$$

$$
= \sum_{\tau} \frac{1}{k} \sum_{i < k} \mu(T^{-i}[\sigma] \cap \tau)
$$

$$
= \sum_{\tau} \frac{1}{k} (N_\sigma(\tau) + O(j))\mu(\tau)
$$

$$
= \sum_{\tau} D_\sigma(\tau)\mu(\tau) + O(j/k).
$$

Build $A$ by concatenating $\tau$’s in the right proportion.
A uniform version of the recurrence theorem

**Theorem**
For every $k$, $(X, \mathcal{X}, \nu, U)$, and $E$ with $\nu(E) > 0$, there is an $i > 0$ such that $\nu(E \cap U^{-i}E \cap U^{-2i}E \cap \ldots \cap U^{-(k-1)i}E) > 0$.

**Theorem**
For every $k$ and $\delta > 0$, there are $n$ and $\eta > 0$ such that for every $(X, \mathcal{X}, \nu, U)$ and $E$ with $\nu(E) \geq \delta$, there is an $i$ such that $0 < i \leq n$ and $\nu(E \cap U^{-i}E \cap U^{-2i}E \cap \ldots \cap U^{(k-1)i}E) \geq \eta$.

In the second, $\eta$ and $n$ depend only on $k$ and $\delta > 0$, and not $(X, \mathcal{X}, \nu, U)$ or $E$. 
Thus we have three versions of the theorem:

1. the finitary one
2. the measure-theoretic one
3. the uniform measure-theoretic one

1 and 3 can be proved equivalent *without compactness*.

1 $\Rightarrow$ 3: use a finite approximation to the measure.

3 $\Rightarrow$ 1: specialize the measure to $\mu_A$, for a finite set $A$.

On the other hand, one can use compactness to pass from 2 to *either* 1 or 3.
The moral

study of patterns in $\mathbb{Z}/n\mathbb{Z} = \text{uniform, complexity bounded ergodic theory}$

Compactness is not needed to mediate the passage from finite to infinite, but, rather, to obtain uniform bounds on complexity.

The bad news: one cannot always bound the complexity of an ergodic-theoretic construction.
Unpredictably noisy limits

Let $B_k$ be the factor of $2^\omega$ generated by $\{[\sigma] \mid \text{len}(\sigma) = k\}$.

Say $B_k$-measurable sets and functions are \textit{simple}, with \textit{complexity at most} $k$.

**Proposition**

Let $\mu$ be the uniform measure on $2^\omega$. There is an $f$ such that

- $f$ is the limit of a computable sequence of simple functions $f_n$.
- There is no computable bound on the rate of convergence of $E(f|B_k)$ to $f$. 
Theorem (V’yugin)

There is a computable shift-invariant measure $\mu$ on $2^\omega$ such that there is no computable bound on the rate of convergence of $A_n 1_{[1]}$.

V’yugin’s construction doesn’t have the property on the previous slide. But:

Theorem

There is a computable shift-invariant measure $\mu$ on $2^\omega$ such that if $f = \lim_n A_n 1_{[1]}$, there is no computable bound on the rate of convergence of $E(f | B_k)$ to $f$.

Corollary

There is no bound on the complexity of $f$ that is uniform in $\mu$. 
Tao has presented an quantitative ergodic-theoretic proof of 
Szemerédi’s theorem in the language of $\mathbb{Z}/n\mathbb{Z}$.

The proof can be viewed as a uniform, complexity bounded version 
of a (by now) straightforward ergodic-theoretic argument, 
specialized to finite measures $\mu_A$.

The arguments used to obtain the necessary uniformities are, 
however, quite delicate and subtle.

Given that these uniformities form the nexus between infinitary and 
discrete methods, it seems important to understand how and when 
they can be obtained in the ergodic-theoretic setting.