# ORDINAL ANALYSIS WITHOUT PROOFS

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**Abstract.** An approach to ordinal analysis is presented which is finitary, but highlights the semantic content of the theories under consideration, rather than the syntactic structure of their proofs. In this paper the methods are applied to the analysis of theories extending Peano arithmetic with transfinite induction and transfinite arithmetic hierarchies.

**§1. Introduction.** As the name implies, in the field of proof theory one tends to focus on proofs. Nowhere is this emphasis more evident than in the field of ordinal analysis, where one typically designs procedures for "unwinding" derivations in appropriate deductive systems. One might wonder, however, if this emphasis is really necessary; after all, the results of an ordinal analysis describe a relationship between a system of ordinal notations and a theory, and it is natural to think of the latter as the set of semantic consequences of some axioms. From this point of view, it may seem disappointing that we have to choose a specific deductive system before we can begin the ordinal analysis.

In fact, Hilbert's epsilon substitution method, historically the first attempt at finding a finitary consistency proof for arithmetic, has a more semantic character. With this method one uses so-called epsilon terms to reduce arithmetic to a quantifier-free calculus, and then one looks for a procedure that assigns numerical values to any finite set of closed terms, in a manner consistent with the axioms. The first ordinal analysis of arithmetic using epsilon terms is due to Ackermann [1]; for further developments see, for example, [20].

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Dedicated to Solomon Feferman on the occasion of his 70th birthday.

It is an honor to be able to contribute a paper to this volume. Although I did my graduate work under Jack Silver at Berkeley, Sol served as an informal second advisor to me, and his thoughtful advice and guidance made frequent visits to Stanford both enjoyable and well worthwhile. Mathematical logic, as a discipline, is poised between philosophy and mathematics, and so has to answer to competing standards of philosophical relevance and mathematical elegance. Throughout his career, Sol has been able to strike a harmonious balance between the two, with work that is deeply satisfying on both counts. His style sets a high standard for future generations, and one that many of us will look to as a model.

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More recently, investigations of nonstandard models of arithmetic due to Paris and Kirby have given rise to another approach, which incorporates Ketonen and Solovay's finitary combinatorial notion of an $\alpha$-large set of natural numbers. Roughly speaking, to show that the proof-theoretic ordinal of a theory $T$ is bounded by $\alpha$, one uses an $\alpha$-large interval in a nonstandard model of arithmetic to construct a model of $T$. These methods are surveyed and extended in [3, 4]; some of the constructions found in the second paper are derived from model-theoretic methods due to Friedman [12, 13]. Sommer [28] has shown that one can avoid references to nonstandard models and instead view the methods as providing a way of building "finite approximations" to models of arithmetic, an idea which traces its way back to Herbrand [18]. (See also the introduction to [19].)

Finally, Quinsey has shown that a notion called *fulfillment*, due to Kripke, provides yet another way of using more semantic methods to obtain traditional proof-theoretic results. His Ph.D. thesis [23] is a tour-de-force, offering a wealth of applications in wide range of areas. Similar ideas have been developed independently by Carlson [10].

Each of the approaches just described has its own advantages and disadvantages. But with their emphasis on "building models" over "unwinding proofs," the similarities between them are more striking than the differences. And the persistence with which this point of view keeps resurfacing suggests that such methods may have something to offer to the development of proof theory.

In this paper, I have tried to fashion an approach to ordinal analysis which is in concord with these themes, incorporating and adapting ideas from all the sources mentioned above. Since these ideas have appeared in so many different contexts, often arising independently, trying to sort out the proper accreditations at each step along the way would be difficult; and so I hope this broad attribution is enough to acknowledge the general debt that this work owes to that which has come before. I should mention that I have also benefited a good deal from Buss' ordinal analysis of arithmetic, using the witnessing method, in [7]; from the emphasis on ordinal recursion and its properties, in Friedman and Sheard [11]; and, of course, from the traditional Gentzen-Schütte approach to ordinal analysis, surveyed in [21, 22, 24].

One aspect of the approach developed here is that Herbrand's theorem is used in a central way. One begins by embedding a classical theory in a universal one, with symbols describing functions that are nonconstructive in the intended interpretation. By Herbrand's theorem, to extract an appropriate witness from the proof of a $\Sigma_1$ sentence, one does not need to know the *correct* interpretation of the function symbols; one only needs an interpretation that is consistent with a finite set of axioms relevant to the proof.

Another aspect of the approach is that it is cumulative: once we have analyzed a theory $T^\alpha$, dependent on a parameter $\alpha$, we can work "in" that theory to analyze the next nonconstructive principle. That way, as we work our way up, we can leave behind the low-level combinatorial constructions, and carry on in a more familiar mathematical and logical framework.

Despite the semantic flavor of the approach, it is entirely finitary, in a sense that will be made precise in Section 4.

In this article, I will develop semantic analogues of the traditional tools and methods of predicative proof theory. In [2], I will extend the methods to analyze Kripke-Platek set theory, $KP\omega$. To my knowledge, the latter will provide the first ordinal analysis of a theory of that strength without the use of cut-elimination.

The outline of this paper is as follows. The first few sections provide the necessary background information: Section 2 discusses some weak fragments of arithmetic; Section 3 introduces a form of ordinal recursion, which we will use to define the proof-theoretic ordinal of a theory in Section 4; Section 5 describes the systems of ordinal notations that are needed to carry out the ordinal analysis; and Section 6 reviews Herbrand's theorem for first-order logic. The rest of the paper is concerned with bounding the proof-theoretic ordinals of various theories: primitive recursive arithmetic in Section 7, theories with $\Pi_1$ transfinite induction in Section 8, theories with arithmetic transfinite induction in Section 9, and, finally, theories of transfinite arithmetic hierarchies in 10.

**§2. Weak theories of arithmetic.** To get us off the ground, in this section I will introduce some weak theories of arithmetic. The theories, notations, and facts discussed are fairly standard. More information on the theories discussed below, including the formal representation of sequences and syntactic objects, can be found in [8, 17, 30]. More information on the elementary and primitive recursive functions and their properties can be found in [25].

I will take the language of arithmetic to be the first-order language with symbols $0$, $1$, $+$, $\times$, and $<$, and if $n$ is a natural number, I will use $\bar{n}$ to denote the corresponding numeral. Peano arithmetic, $PA$, consists of quantifier-free defining axioms for $+$, $\times$, and $<$, and the schema of induction,

$$\varphi(0) \wedge \forall x\ (\varphi(x) \rightarrow \varphi(x+1)) \rightarrow \forall x\ \varphi(x)$$

for arbitrary formulae $\varphi$. A formula is said to be $\Delta_0$ if every quantifier is bounded, that is, of the form $\forall x < t$ or $\exists x < t$, where these are interpreted in the usual way. A formula is $\Sigma_1$ (resp. $\Pi_1$) if it is obtained by prefixing existential (resp. universal) quantifiers to a $\Delta_0$ formula; more generally, with $\Sigma_n$ and $\Pi_n$ formulas one is allowed $n$ alternations of quantifiers. The theory of arithmetic in which induction is restricted to formulas in a set $\Gamma$ is denoted $I\Gamma$. Over a weak base theory, $\Sigma_n$ and $\Pi_n$ are induction are equivalent: e.g. given a $\Sigma_n$ formula $\varphi(x)$ satisfying the hypotheses of the induction axiom, if there is a $y$ satisfying $\neg\varphi(y)$, then $\Pi_n$ induction on $z$ implies $\forall z\ \neg\varphi(y \dot{-} z)$; but then $\neg\varphi(0)$, yielding a contradiction.

Theories like $I\Delta_0$ are sensitive to the choice of initial functions. The theory obtained by adding a function symbol $exp(x, y)$ for exponentiation, with the usual defining equations, is denoted $I\Delta_0^{exp}$. From a mathematical point of view, $I\Delta_0^{exp}$ is very weak, but from a finitary, computational point of view, it is fairly strong, as the following discussion will show.

Taking constants to be functions of arity 0, the set of *elementary functions* is defined to be the smallest set of functions on the natural numbers containing 0, $+$, $\times$, and *exp*, and projections, and closed under the operations of composition and *bounded recursion.* Using $\vec{z}$ to denote a finite sequence of variables $z_0, \ldots, z_k$, closure under bounded recursion means that whenever the functions $g(\vec{z})$, $h(x, y, \vec{z})$, and $b(x, \vec{z})$ are elementary, then so is the function $f(x, \vec{z})$, defined by the equations

$$f(0, \vec{z}) = g(\vec{z})$$
$$f(x+1, \vec{z}) = \begin{cases} h(x, f(x, \vec{z}), \vec{z}) & \text{if this is less than } b(x+1, \vec{z}) \\ 0 & \text{otherwise.} \end{cases}$$

Since bounded recursion does not allow us to introduce functions that grow faster than ones that have been previously defined, a moment's reflection shows that every elementary function is bounded by some finite iteration of the function $x \mapsto 2^x$. I will say that a relation $R(\vec{x})$ is elementary if its characteristic function, $\chi_R(\vec{x})$, is elementary, and for notational convenience I will write $R(\vec{x})$ instead of $\chi_R(\vec{x}) = 1$.

One can show that the set of elementary functions is closed under bounded sums, $f(x, \vec{z}) = \sum_{y<x} h(y, \vec{z})$, and bounded products, $g(x, \vec{z}) = \prod_{y<x} h(y, \vec{z})$. The set of elementary relations is closed under boolean operations and bounded quantification, and if $R(y, \vec{z})$ is an elementary relation, then the function $f(x, \vec{z}) = \mu y < x \; R(y, \vec{z})$ is also elementary, where the right hand side is defined to be the least $y$ less than $x$ satisfying $R(y, \vec{z})$ if there is one and zero otherwise. We can take $\mu y \leq x \; R(y, \vec{z})$ to abbreviate $\mu y < x + 1 \; R(y, \vec{z})$. One can define functions by cases: if $R(\vec{x})$, $f(\vec{x})$, and $g(\vec{x})$ are elementary, then so is the function

$$h(\vec{x}) = \begin{cases} f(\vec{x}) & \text{if } R(\vec{x}) \\ g(\vec{x}) & \text{otherwise.} \end{cases}$$

One can also code finite sequences of numbers as a single natural number in such a way that the usual operations on sequences are elementary. I will write $\langle x_0, \ldots, x_k \rangle$ to denote (the code for) the sequence with the elements shown; if $s$ is such a sequence, $length(s)$ to denote the length of $s$, $last(s)$ to denote the index of the last element of $s$, and $(s)_i$ to denote the $i$th element of $s$ (or 0 if $i > last(s)$). The concatenation of two sequences $s$ and $t$ will be denoted $s \hat{\ } t$. From the point of view of computational complexity, the set of elementary functions is quite large: it can be characterized alternatively as the set of functions Turing computable with time (and/or space) resources bounded by a finite iteration of an exponential function. For details, see [25].

Elementary recursive arithmetic, denoted *ERA*, is the "natural" first-order theory of the elementary functions. The language has a symbol for each such function, and its axioms include the corresponding defining equations, as well as axioms for $<$ and the usual axioms for equality. To these, one adds the schema of induction for quantifier-free formulas. *ERA* is essentially a definitional extension of $I\Delta_0^{exp}$, or Friedman's *elementary function arithmetic*, *EFA*.

PROPOSITION 2.1. *Every $\Delta_0^{exp}$ relation is equivalent to an elementary one, provably in ERA.*

PROOF (SKETCH). Use the facts that equality and less-than are elementary relations, and that the elementary relations are closed under boolean operations and bounded quantification, provably in *ERA*. ⊣

PROPOSITION 2.2. *ERA can be axiomatized by a set of universal sentences, and is a conservative extension of $I\Delta_0^{exp}$.*

PROOF (SKETCH). The defining equations for the function and relation symbols are universal, and one can replace the schema of induction with axioms of the form $R(x, \vec{z}) \rightarrow R(\mu y \leq x\ R(y, \vec{z}), \vec{z})$, where $R$ is any elementary relation. (If $R(\mu y \leq x\ R(y, \vec{z}), \vec{z})$, the defining equations guarantee that either $\mu y \leq x\ R(y, \vec{z})$ is 0 or $R(\cdot, \vec{z})$ does not hold of its predecessor.)

Proposition 2.1 guarantees that *ERA includes* $I\Delta_0^{exp}$. To show that *ERA* is a conservative extension, one shows that every elementary function is definable by a $\Delta_0^{exp}$ formula, provably in $I\Delta_0^{exp}$. See, for example, [8, 17]. ⊣

Moving on, the primitive recursive functions are obtained by dropping the bound requirement in the recursion schema; and primitive recursive arithmetic, or *PRA*, is the corresponding theory. Here we can omit the special treatment of $+$, $\times$, *exp*, and $<$, since these can defined using primitive recursion. Since bounded recursion can be seen as a special case of primitive recursion, we can view the language of *PRA* as including that of *ERA*. *PRA* is properly stronger: in *PRA* one can define iterated exponential functions, as well as a function which evaluates closed terms of *ERA*.

One can relativize the definitions of the elementary recursive and primitive recursive functions by adding additional functions to the initial set. On the axiomatic level, we will consider extensions of *ERA* and *PRA*, denoted by $ERA(f_0, \ldots, f_k)$ and $PRA(f_0, \ldots, f_k)$ respectively, obtained by adding new unary function symbols $f_0, \ldots, f_k$ to the underlying language. These are taken as initial functions in the inductively defined set of function symbols, and so may "appear" in the definitions of other functions; so, for example, there are symbols for functions defined using composition and bounded recursion (resp. primitive recursion) from these. Otherwise, however, there are no axioms governing their behavior. If $f$ is such an uninterpreted function symbol, I will write $t(\vec{x}, f)$ to indicate that the term $t$ depends on $f$, when $f$ occurs in $t$ or in the definition of one of the function symbols occurring in $t$. If $g(\vec{x}, f)$ is a $k$-ary function, and $h(y, \vec{z})$ is $l$-ary, I will use $g(\vec{x}, \lambda y\ h(y, \vec{z}))$ to denote the $k + l$-ary function obtained by replacing $f(y)$ by $h(y, \vec{z})$ everywhere in the definition of

$g$; and I will adopt a similar convention for terms and formulas. This notation is somewhat justified by the following:

LEMMA 2.3. *Suppose* $ERA(f)$ *proves* $\varphi(\vec{x}, f)$, *and* $h(y, \vec{z})$ *is a function symbol of* $ERA$. *Then* $ERA$ *proves* $\varphi(\vec{x}, \lambda y\; h(y, \vec{z}))$.

PROOF. A straightforward induction on derivations. ⊣

Primitive recursive arithmetic is often taken as a reasonable representation of Hilbert and Bernays' informal notion of "finitary" mathematical reasoning. For this purpose, it is better to view *PRA* as a quantifier-free theory, obtained by dropping the universal quantifiers from the axioms and replacing quantifier rules with a substitution rule, namely, from $\varphi(x)$ conclude $\varphi(t)$ for any term $t$ free for $x$ in $\varphi$. Herbrand's theorem, discussed in Section 6, implies that the first-order version of *PRA* is a conservative extension of the quantifier-free one. In fact, if the first-order version of *PRA* proves a sentence of the form $\forall x\; \exists y\; R(x, y)$, where $R(x, y)$ is a primitive recursive relation, then there is a function symbol $g$ such that the quantifier-free version proves $R(x, g(x))$; and similarly for *ERA*. In much the same way, the versions of these theories with extra function symbols may be better understood in terms of conservative second-order extensions, but developing the details for such a presentation would take us too far afield.

**§3. Computations with ordinal bounds.** To get a sense for the kinds of theorems we are after, consider the first ordinal analyses of arithmetic, due to Gentzen [14, 15]. Roughly speaking, Gentzen devised a means of "unwinding" proofs in arithmetic, using iterative procedures that "count down" through the ordinals below $\varepsilon_0$. In particular, given a proof of a $\Sigma_1$ sentence $\exists y\; \varphi(x, y, f)$ in Peano arithmetic (in a language augmented with a function symbol $f$), his analysis provides a procedure which, for any $x$ and $f$, finds a suitable value for $y$.

In the next section, I will use this informal characterization to provide a formal definition of the proof-theoretic ordinal of a theory. But first, we need to make the notion of an "iterative procedure which counts down through the ordinals" more precise.

Fix an elementary relation $\prec$ that happens to be a well ordering of an elementary subset of the natural numbers. Let variables $\alpha, \beta, \gamma, \ldots$ range over the field of $\prec$; think of these as *notations* for ordinals. A $\prec\alpha$*-iterative algorithm* is given by a notation $\beta$ less than $\alpha$ and elementary functions $start(\vec{x})$, $next(q)$, $norm(q)$, and $result(q)$. These data define a function $F(\vec{x})$, whose value at an input $x$ is computed by starting in a state given by $start(\vec{x})$; assuming the norm of this state is less than $\beta$, iterating the *next* function until the norm of the resulting state fails to decrease; and then returning the value of *result*, applied to the final state. This algorithm can be summarized as follows:

$$
\begin{array}{l}
\textit{clock} \leftarrow \beta \\
\textit{state} \leftarrow \textit{start}(\vec{x}) \\
\textsf{while } \textit{norm}(\textit{state}) \prec \textit{clock} \textsf{ do} \\
\qquad \textit{clock} \leftarrow \textit{norm}(\textit{state}) \\
\qquad \textit{state} \leftarrow \textit{next}(\textit{state}) \\
\textsf{return } \textit{result}(\textit{state})
\end{array}
$$

The fact that $\prec$ is a well ordering guarantees that the while loop always terminates.[1] To describe $F$ more formally, say that $s$ is a computation sequence for $F$ at $\vec{x}$, written $CompSeq_F(s, \vec{x})$, if $s$ is a sequence $\langle s_0, s_1, s_2, \ldots, s_k\rangle$ satisfying the following:

- $s_0 = \textit{start}(\vec{x})$
- for every $i < k$, $s_{i+1} = \textit{next}(s_i)$
- either
  $k = 0$ and $\textit{norm}(s_0) \succeq \beta$, or
  $k > 0$, $\textit{norm}(s_0) \prec \beta$, $\textit{norm}(s_{i+1}) \prec \textit{norm}(s_i)$ for every $i < k-1$, and $\textit{norm}(s_k) \succeq \textit{norm}(s_{k-1})$

Then $F(\vec{x}) = y$ means there is a sequence $s$ satisfying $CompSeq_F(s, \vec{x})$, and $\textit{result}(s_{last(s)}) = y$. I will say that a function $F(\vec{x})$ is $\prec\alpha$-recursive if it is defined by a $\prec\alpha$-iterative algorithm.

We can refer to specific $\prec\alpha$-recursive functions within *ERA*, by identifying such a function with the numeral $\bar{\beta}$ and function symbols *start*, *norm*, *next*, and *result* that define it. We have to be careful, since in *ERA* the relation $\prec$ may not be provably well ordered. In that case, we can use the notation $F(\vec{x})\downarrow$ to abbreviate $\exists s\ CompSeq_F(s, \vec{x})$, indicating that $F$ is "defined" at $\vec{x}$. One can show, in *ERA*, that there is at most one computation sequence for $F$ at $\vec{x}$, so $F(\vec{x}) = y$ is equivalent to saying that $F$ is defined at $\vec{x}$ and $y$ is the result of the corresponding computation sequence. More generally, we can use the notation $F(\vec{x}) \simeq G(\vec{x})$ to abbreviate the assertion that if either side is defined, then both sides are defined and equal. It is not hard to see that, again in *ERA*, $\simeq$ has the properties of an equivalence relation. As far as composition is concerned, we can take $F(G_0(\vec{x}), \ldots, G_k(\vec{x}))\downarrow$ to denote the assertion that there are computation sequences for $G_0, \ldots, G_k$ at $\vec{x}$, and a computation sequence for $F$ at the results of these computation sequences. Then $F(G_0(\vec{x}), \ldots, G_k(\vec{x})) = y$ means that $F(G_0(\vec{x}), \ldots, G_k(\vec{x}))$ is defined, and $y$ is the result of the corresponding computation sequence for $F$. Inscriptions like $F(G_0(\vec{x}), \ldots, G_k(\vec{x})) \simeq H(\vec{x})$ are to be interpreted similarly.

We also want a notion of iterative computation relative to a function $f$. The easiest thing to do would be to allow *start*, *norm*, *next*, and *result* to

[1] It is not hard to extend these notions to cases where $\prec$ is, more generally, a well-founded relation; assuming $\prec$ is a linear ordering simplifies the exposition.

be elementary functions relative to $f$, but we will need a slightly finer notion. This is obtained by requiring the basic functions to be purely elementary, as before, but allowing the computation to query a single value of $f$ at each step. That is, a *relativized* $\prec\alpha$-iterative algorithm is given by a notation $\beta$ less than $\alpha$ and elementary functions $\mathit{start}(\vec{x})$, $\mathit{query}(q)$, $\mathit{next}(q, z)$, $\mathit{norm}(q)$, and $\mathit{result}(q)$; values of the corresponding function $F(\vec{x}, f)$ are computed by replacing the second line of the while loop above with

$$\mathit{state} \leftarrow \mathit{next}(\mathit{state}, f(\mathit{query}(\mathit{state}))).$$

Let $\mathit{CompSeq}_F(s, \vec{x}, f)$ denote the resulting modification of $\mathit{CompSeq}(s, x)$, and then proceed as before. From now on I will say that a function $F(\vec{x}, f)$ is $\prec\alpha$-recursive if there is a $\prec\alpha$-iterative algorithm, relative to $f$, which computes it. More generally, $F(\vec{x}, f_0, \ldots, f_k)$ is $\prec\alpha$-recursive if it can be computed by an algorithm which, at each step, is allowed to pose a single query to each of the $f_i$.

Having set forth these definitions, I should warn the reader that there are, in fact, a number competing definitions of ordinal recursion in the literature: see, for example, the schematic presentation in [26], the $\prec\alpha$-descent recursive functions of [11], or the various characterizations in [7] and [25]. The good news is that the various definitions of the $\prec\alpha$-recursive functions usually coincide, with minimal assumptions on the system of notations and $\alpha$. (See, for example, [11, Proposition 1.9], [7, Section 3.2], [25], and Lemma 7.3 below.) I have chosen the presentation above because it is easy to work with, and convenient for our applications.

**§4. The proof-theoretic ordinal of a theory.** Following Gentzen's lead, we would like to say that the proof theoretic ordinal of $T$ is bounded by $\alpha$ when there is a finitary proof of the following:

> Whenever $T$ proves a $\Sigma_1(f)$ formula $\exists y\ \varphi(x, y, f)$, there is a $\prec\alpha$-recursive function $F$, such that for any $x$, $y$, and $f$, if $F(x, f) = y$ then $\varphi(x, y, f)$ is true.

Call this informal statement (*). Note that (*) only makes sense for theories in a language that includes the language of arithmetic and a function symbol $f$, or, more generally, theories for which we can interpret a notion of provability for $\Sigma_1(f)$ formulae. Note also that (*) does not imply that $F(x, f)$ is defined at every value of $x$ and $f$; only that when it *is* defined, it produces a suitable witness.

The rest of this paper is dedicated to proving (*) in a finitary way, for various theories $T$ and notations $\alpha$. This section is devoted to making the notion of "proving (*) in a finitary way" more precise, and explaining why this is a desirable goal. The reader that is already satisfied with the informal characterization should feel free to skip to Section 5.

To turn the informal statement into a mathematical one, we need a formal notion of "finitary proof," and an appropriate formalization of (*). For the former, let us take primitive recursive arithmetic relative to a function symbol $f$; below we will see that weaker theories will do. In $PRA(f)$, we can develop a theory of syntax, representing terms and formulae as numbers in an appropriate way; we can define the set of $\Sigma_1(f)$ formulas as well as names for the elementary functions; and we can identify $\prec\alpha$-recursive functions with the iterative algorithms that define them. In $PRA(f)$ we can also refer to the "value" of an elementary function at a given set of inputs, using an appropriate primitive recursive evaluation function for the set of elementary recursive functions, and we can refer to the truth value of a $\Delta_0(f)$ formula at a given set of parameters, using a truth predicate for $\Delta_0(f)$ formulas that is primitive recursive in $f$.

Expressed in greater detail, (*) asserts the following:

> For every proof $d$ of a $\Sigma_1(f)$ sentence $e$, there is a $\prec\alpha$-recursive function $F$, such that for every $x$, $y$, $f$, and computation sequence $s$ for $F$ at $x$ and $f$, if the result of the computation is $y$, then $y$ witnesses the truth of $e$ at $x$ and $f$.

We can get rid of the existential quantifier by requiring, more stringently, that we have a primitive recursive function $\mathcal{F}(d)$ which extracts $F$ from the proof $d$; and then we can leave the universal quantification over $d$, $e$, $x$, $y$, $f$, and $s$ implicit. Given a primitive recursive relation $Proof_T(d, e)$, an elementary well ordering $\prec$, and a notation $\alpha$, we can then take the statement "the proof-theoretic ordinal of $T$ is at most $\alpha$" to mean that there is a function symbol $\mathcal{F}$, and a $PRA(f)$-proof of an appropriate formalization of the following assertion:

> For every $d$, $\mathcal{F}(d)$ is a $\prec\alpha$-recursive function, and whenever
> - $e$ is a $\Sigma_1(f)$ formula with free variable $x$,
> - $Proof_T(d, e)$,
> - $CompSeq_{\mathcal{F}(d)}(s, x, f)$, and
> - $result(s_{last(s)}) = y$
>
> then $y$ witnesses the truth of $e$ at $x$ and $f$.

There is, no doubt, much to criticize in this choice of a definition, but let us consider some of the things one can say in its favor. To start with, it is *strong*, which is to say, it implies all the usual results of an ordinal analysis. Suppose the proof-theoretic ordinal of $T$ is at most $\alpha$, according to our definition. Then we have

1. A consistency proof for $T$
2. A characterization of $T$'s provably total computable functions
3. A characterization of $T$'s provably well-ordered computable relations

For the first, ignoring $x$ and $f$ and taking $e$ to be the sentence "$0 = 1$," we can conclude that the consistency of $T$ is provable in $PRA$ together with any principle that implies that every $\prec\alpha$-iterative procedure terminates. For the second, we can ignore $f$ and conclude that any $\Pi_2$ statement provable in $T$ has a $\prec\alpha$-recursive Skolem function. For the third, suppose $\prec'$ is a $\Sigma_1$-definable well ordering such that $T$ proves $\exists y\ (f(y+1) \not\prec' f(y))$. If the order type of $\prec'$ is greater than $\alpha$, there is an order-preserving embedding $g$ of notations less than $\alpha$ into the field of $\prec'$. We can use the conclusion of (*) to obtain a $\prec\alpha$-recursive function $H(g, h)$, which for any $h$ returns a value $y$ such that that $h(y+1) \not\prec h(y)$. With minor assumptions on $\alpha$, one can use this to define a $\prec\alpha$-recursive function $J(g, x)$ that diagonalizes the functions that are $\prec\alpha$-recursive in $g$, yielding a contradiction. (See [11] for additional information.) Incidentally, Rathjen [24] notes that for second-order theories that include arithmetic comprehension (or first-order theories that have such conservative extensions), 3 extends to arithmetically definable orderings; and for second-order theories that include the $\Sigma^1_1$ axiom of choice, 3 extends to the hyperarithmetically definable orderings as well.

The formalization of (*) is unpalatable, and it is tempting to take the assertion that the proof-theoretic ordinal of $T$ is at most $\alpha$ to mean that (*) is simply true. But this has the undesired consequence that adding arbitrary true $\Pi_1$ statements to $T$ (like consistency statements) does not increase the proof-theoretic ordinal. Similarly, defining the proof-theoretic ordinal in terms of provable well orderings means that the ordinal does not change when one adds arbitrary true $\Sigma^1_1$ sentences to the theory; see [24] for a discussion. This points to a second advantage of the definition above: it is *immune* to these objections.

Our formalized version of (*) expresses a relationship between a primitive recursive *representation* of $T$, and a system of *notations* for ordinals. It is well known that one can always cook up representations of theories and ordinals which render the ordinal analysis trivial, or meaningless; in this respect the definition above is *honest*, since it is really the "natural" representations that we care about. Some logicians are disturbed by the absence of a formal definition of naturality, and so prefer to characterize the proof-theoretic ordinal as the least upper bound to the theory's provable well orderings; this characterization is independent of the representations, but has the drawbacks mentioned above. But the absence of such a formal definition should not concern us much. The natural representations of theories and ordinals are just those for which the provability of (*) is *interesting*; and very few mathematicians have formal criteria which tell them which theorems of their subject have this property.[2]

Bounding the proof-theoretic ordinal of a theory has two aspects: proving (*), and doing so in a finitary way. In the sections that follow, I will focus on the first; but I will proceed with the implicit understanding that once we have specified an appropriate ordinal notation system (with properties described in the next section), every definition, theorem, and proof can be formalized in $PRA(f)$. The exception is this: when I state as a theorem that "the proof-theoretic ordinal of $T$ is at most $\alpha$," I mean simply that (*) holds, again with

[2]Another approach to dealing with the various definitions of "proof theoretic ordinal" in the literature is to embrace the multiplicity, and explore the relationships between them. See [6] for a development along these lines, as well as the discussion in [24].

the implicit understanding that the formalized version of (*) can be proved in $PRA(f)$.

Let me close this section with two notes. First, the choice of $PRA(f)$ is not crucial. We need a metatheory that is strong enough to formalize syntax and quantify over elementary functions, and is strong enough to prove Herbrand's theorem. For these purposes, $I\Delta_0(f)$ together with the assertion that an iterated exponential function is total will suffice. If one uses a weaker class of functions in defining the ordinal-iterative algorithms, one can get by with even weaker theories, by "pushing" the work involved in satisfying Herbrand's theorem into the computation of the $\prec\alpha$-recursive function. The possibility of using a weaker metatheory is interesting from a foundational point of view, but it does not seem to help with the analysis of weaker theories. (For evidence of this, see [29]. For a more fruitful approach to the analysis of weak theories, see the use of "dynamic ordinals" in [5].)

The second note has to do with lower bounds. One can take the statement "the proof-theoretic ordinal of $T$ is *exactly* $\alpha$" to mean that the proof-theoretic ordinal of $T$ is at most $\alpha$, but it is not at most $\beta$ for any $\beta$ less than $\alpha$. This takes us outside our finitary metatheory, since it requires us to show that for any such $\beta$ there is *no* proof of the formalized version of (*) in $PRA(f)$. But, on the assumption that $PRA(f)$ is consistent, one obtains the desired conclusion by giving a finitary proof that for every $\beta$ less than $\alpha$, there is a provable $\Sigma_1(f)$ formula that is not witnessed by any $\prec\beta$-recursive function; and one typically achieves this goal by developing a theory of transfinite recursion below $\alpha$ in $T$. In this paper I will focus on the upper bounds, but, in fact, all the upper bounds I provide will be sharp in this sense. (See [21, 22] for more information on establishing the lower bounds.)

**§5. Systems of ordinal notations.** As noted in the last section, ordinal analysis, as understood here, involves calibrating the strength of various theories relative to an elementary recursive system of ordinal notations. In this section I will discuss the properties that our system of notations, $\prec$, needs to satisfy, provably in $ERA(f)$. For more information on ordinal notations, see, for example, [3, 21, 22, 24].

For most of the results below, we only need to assume that $\prec$ is a linear ordering, with elementary functions $+$, $\cdot$, and $\alpha \mapsto \omega^\alpha$, for which the "usual properties" hold. In other words, I will assume that $\prec$ is an elementary recursive ordinal notation system (ERONS) in the terminology of [11], such that the given functions are everywhere defined; a list of the "usual properties" they are to satisfy appears in [11, Section 1]. In particular, we will need to use the fact that any $\alpha$ can be written in Cantor normal form, $\alpha = \omega^{\beta_1} + \ldots + \omega^{\beta_k}$ with $\beta_1 \succeq \ldots \succeq \beta_k$. If $\alpha' = \omega^{\beta'_1} + \ldots + \omega^{\beta'_{k'}}$ is also in Cantor normal form, then the symmetric sum of $\alpha$ and $\alpha'$, written $\alpha \# \alpha'$, is equal to $\omega^{\gamma_1} + \ldots + \omega^{\gamma_{k+k'}}$, where the $\gamma_i$'s list the $\beta_i$'s and the $\beta'_i$'s in decreasing order. Unlike ordinary ordinal addition, the symmetric sum is strictly monotone in both arguments.

Classically, $\varepsilon_0$ is defined to be the least fixed-point of the function $\alpha \mapsto \omega^\alpha$; equivalently, it is the limit of the sequence $\langle \omega_n \rangle_{n\in\omega}$, where $\omega_0 = 1$ and $\omega_{n+1} = \omega^{\omega_n}$. The statement of Theorem 9.7 assumes that there is a notation with these properties.

More generally, the sequence of Veblen functions on any regular cardinal is defined by letting $\varphi_0(\beta) = \omega^\beta$, and otherwise letting $\varphi_\alpha$ enumerate the simultaneous fixed points of $\{\varphi_\gamma \mid \gamma < \alpha\}$. In Section 10, we need to assume that there is a binary elementary function $\varphi(\alpha, \beta)$, defined on the system of notations, representing the Veblen functions. Writing $\varphi_\alpha(\beta)$ instead of $\varphi(\alpha, \beta)$, we will assume that for every $\alpha$, $\beta$, $\gamma$, and $\delta$, $\varphi_\alpha(\beta)$ is less than $\varphi_\gamma(\delta)$ if and only if either

- $\alpha \prec \gamma$ and $\beta \prec \varphi_\gamma(\delta)$,
- $\alpha = \gamma$ and $\beta \prec \delta$, or
- $\alpha \succ \gamma$ and $\varphi_\alpha(\beta) \prec \delta$.

I will say that an ordinal notation $\alpha$ is infinite if it is greater than or equal to $\omega$. Many of the lemmata and theorems below are stated most cleanly by assuming closure properties on a notation $\alpha$. For reference, here are some equivalent characterizations.

Proposition 5.1. *Let $\alpha$ be infinite. Then*

1. *$\alpha$ is closed under addition if and only if it is equal to $\omega^\gamma$, for some $\gamma$.*
2. *$\alpha$ is closed under the function $\beta \mapsto \omega \cdot \beta$ if and only if it is equal to $\omega^\omega \cdot \gamma$, for some $\gamma$.*
3. *$\alpha$ is closed under multiplication if and only if it is equal to $\omega^{(\omega^\gamma)}$, for some $\gamma$.*
4. *$\alpha$ is closed under the function $\beta \mapsto \omega^\beta$ if and only if it is equal to $\varepsilon_\gamma$ (that is, $\varphi_1(\gamma)$), for some $\gamma$.*

The proof is an exercise in ordinal arithmetic (see [21]).

## §6. Herbrand's theorem.

Herbrand's theorem can be stated as follows:

Theorem 6.1. *Let $L$ be a language with at least one constant symbol, let $\varphi(\vec{x})$ be a quantifier-free formula in $L$, and suppose $\exists \vec{x}\ \varphi(\vec{x})$ is provable in classical first-order logic with equality. Then there are sequences of terms $\vec{t}_0, \ldots, \vec{t}_k$, whose free variables are among those of $\exists \vec{x}\ \varphi(\vec{x})$, such that $\varphi(\vec{t}_1) \lor \varphi(\vec{t}_2) \lor \ldots \lor \varphi(\vec{t}_k)$ is provable in propositional logic from substitution instances of the equality axioms.*

This theorem, which effectively enables us to extract additional information from proofs of existential sentences, will form a cornerstone to our investigations. There are model-theoretic proofs of Herbrand's theorem: if the conclusion fails, the set $\{\neg\varphi(t) \mid t \text{ is a closed term}\}$ is propositionally consistent with the set of all substitution instances of the equality axioms; and from a satisfying truth assignment, one can build a model of $\forall x\ \neg\varphi(x)$. But Herbrand's theorem is also an easy consequence of the cut-elimination theorem (see [8, 26]), and hence provable in our finitary metatheory. See also [27, 23] for alternative syntactic proofs, and [19] for Herbrand's original proof.

I will say that a theory $T$ is *universal* if it can be axiomatized by a universal set of sentences (or, equivalently, a quantifier-free set of formulae, since $\forall \vec{y}\, \psi(\vec{y})$ follows from an axiom $\psi(\vec{y})$).

COROLLARY 6.2. *Let $L$ and $\varphi(\vec{x})$ be as above, and let $T$ be a universal theory in $L$. If $T$ proves $\exists \vec{x}\, \varphi(\vec{x})$, then there are sequences of terms $\vec{t}_0, \ldots, \vec{t}_k$, whose free variables are among those of $\exists \vec{x}\, \varphi(\vec{x})$, such that $T$ proves $\varphi(\vec{t}_1) \lor \varphi(\vec{t}_2) \lor \ldots \lor \varphi(\vec{t}_k)$. Moreover, we can assume this formula is provable in propositional logic from substitution instances of axioms of $T$ and equality axioms.*

PROOF. Suppose $T$ proves $\exists \vec{x}\, \varphi(\vec{x})$. Then there are (universal closures of) axioms of $T$, $\psi_1, \ldots, \psi_k$, such that $\exists \vec{x}\, \varphi(\vec{x})$ is provable from $\psi_1, \ldots, \psi_k$. By the deduction theorem, $\psi_1 \land \ldots \land \psi_k \to \exists \vec{x}\, \varphi(\vec{x})$ is provable in first-order logic. Bring all the quantifiers to the front, and apply Herbrand's theorem. ⊣

In many cases (but not all the ones we will consider), the theory $T$ will be rich enough so that for every sequence of terms $t_1(\vec{x}), \ldots, t_k(\vec{x})$ and quantifier-free formulae $\varphi_1(\vec{x}), \ldots, \varphi_{k-1}(\vec{x})$, there is a function symbol $f(\vec{x})$ such that $T$ proves

$$f(\vec{x}) = \begin{cases} t_1(\vec{x}) & \text{if } \varphi_1(\vec{x}) \\ t_2(\vec{x}) & \text{if } \neg\varphi_1(\vec{x}) \land \varphi_2(\vec{x}) \\ \vdots & \\ t_k(\vec{x}) & \text{otherwise.} \end{cases}$$

In cases like this, we can replace the sequence of terms $t_1, \ldots, t_k$ in the Corollary 6.2 with a single function symbol $f$.

Recall that if $G(x, f)$ is a $\prec\alpha$-recursive function, we will interpret references to $G$ in the context of $ERA(f)$ as references to the elements $\beta$, *start*, *norm*, *next*, *query*, and *result* that define the iterative algorithms that computes it. As an application of Herbrand's theorem, we have the following:

THEOREM 6.3. *Suppose $\theta(x, y, f)$ is a $\Delta_0(f)$ formula with the free variables shown, and suppose there is an $\alpha$-recursive function $G(x, f)$ such that $ERA(f)$ proves $G(x, f) = y \to \theta(x, y, f)$. For any $x$ and $y$, if $G(x, f)$ is defined and equal to $y$, then $\theta(x, y, f)$ is true.*

PROOF. The conclusion follows from the soundness of $ERA(f)$, but we have to take care to make sure that our proof is finitary. The following proof uses an evaluation function for the set of functions that are elementary recursive in $f$, and a truth predicate for $\Delta_0(f)$ sentences.

Suppose $ERA(f)$ proves $G(x, f) = y \to \theta(x, y, f)$. From the definition of $G(x, f) = y$, we see that it also proves $CompSeq_G(s, x, f) \land result((s)_{last(s)}) = y \to \theta(x, y, f)$. Now, suppose $G(x, f)$ is defined and equal to $y$, so that in addition, there is an $s$ satisfying $CompSeq_G(s, x, f)$ and $result((s)_{last(s)}) = y$. Then for this particular $s$, $x$, and $y$, $ERA(f)$ proves $CompSeq_G(\bar{s}, \bar{x}, f) \land$

$result((\bar{s})_{last(\bar{s})}) = \bar{y} \to \theta(\bar{x}, \bar{y}, f)$. By Herbrand's theorem, there is a propositional proof of this fact from closed instances of equality axioms and axioms of $ERA(f)$. The axioms of $ERA(f)$ are true; so, by induction on the length of the proof, the conclusion is also true. As a result, we have that $CompSeq_G(s, x, f)$ and $result(s)_{last(s)}) = y$ imply $\theta(x, y, f)$. So $\theta(x, y, f)$ is true. ⊣

This theorem seems minor, but it will play a central role. It enables us to show that the ordinal of a theory $T$ is less than or equal to $\alpha$, by showing that whenever $T$ proves a statement of the form $\exists y\, \theta(x, y, f)$, then there is a $\prec\alpha$-recursive function $G(x, f)$ which, provably in $ERA(f)$, finds a witness. We will do this repeatedly, providing explicit translations; this is what makes the account finitary. But we will proceed in steps, successively reducing more "abstract" theories to more "concrete" ones, and working "in" the target theory as much as possible.

**§7. Primitive recursion.** In this section we will bound the proof-theoretic ordinal of primitive recursive arithmetic. To do so, we will first show that with sufficient conditions on $\alpha$, the $\prec\alpha$-recursive functions have nice closure properties, provably in $ERA(f)$. In particular, if $\alpha$ is $\omega^\omega$, we will see that one can use a single $\prec\alpha$-recursive function to assign "correct" values to a finite set of terms in $PRA(f)$, again provably in $ERA(f)$. Applying Theorem 6.3 will then yield the desired upper bound.

The first lemma states that for $\alpha$ greater than 1, the $\prec\alpha$-recursive functions in $f$ include both the purely elementary functions and $f$ itself.

LEMMA 7.1. *Suppose $\alpha$ is greater than 0. Then for every elementary function $g(\vec{x})$ (not involving the function $f$), there is a $\prec\alpha$-recursive function $G(\vec{x}, f)$ such that $ERA(f)$ proves $G(\vec{x}, f) \simeq g(\vec{x})$. Also, if $\alpha$ is greater than 1, there is a $\prec\alpha$-recursive function $H(x, f)$ such that $ERA(f)$ proves $H(x, f) \simeq f(x)$.*

PROOF. For the first claim, let the algorithm for $H$ store $\vec{x}$ in the state, and then return $g(\vec{x})$ immediately. In other words, assuming $g$ is arity $k$, take $\beta = 0$, $start(\vec{x}) = \langle \vec{x} \rangle$, $norm(q) = 0$, $result(q) = g((q)_0, \ldots, (q)_k)$.

For the second claim, let the algorithm for $H$ store $x$ in the state, query $f$, and then return the result. That is, take $\beta = 1$, $start(x) = x$, $query(q) = q$, $next(q, z) = z$, $norm(q) = 0$, $result(q) = q$. ⊣

The next lemma gives conditions under which the $\prec\alpha$-recursive functions are closed under composition, again provably in $ERA(f)$.

LEMMA 7.2. *Suppose $\alpha$ is infinite and closed under addition, and suppose that $F_0(\vec{x}, f), \ldots, F_k(\vec{x}, f)$ and $G(z_0, \ldots, z_k)$ are $\prec\alpha$-recursive functions. Then there is a $\prec\alpha$-recursive function $H(\vec{x}, f)$ such that $ERA(f)$ proves $H(\vec{x}, f) \simeq G(F_0(\vec{x}, f), \ldots, F_k(\vec{x}, f))$.*

PROOF. Let the algorithm for $G$ carry out the algorithms for $F_0$ through $F_k$ on input $\vec{x}$, and then send the result to the algorithm for $G$.

In more detail, suppose the algorithm for each $F_i$ is given by the data $\beta_i$, $\mathit{start}_i$, $\mathit{query}_i$, $\mathit{next}_i$, and $\mathit{result}_i$, and suppose the algorithm for $G$ is given by $\beta_{k+1}$, $\mathit{start}_{k+1}$, $\mathit{query}_{k+1}$, $\mathit{next}_{k+1}$, and $\mathit{result}_{k+1}$. Take the states of $H$ to code tuples of the form $\langle i, c, s, u, v\rangle$, where $i$ indicates the current subalgorithm, $c$ is the setting of an ordinal "clock," $s$ is the state in the subalgorithm, $u$ stores the original input, and $v$ stores the results which have been computed so far. The algorithm for $H$ then corresponds to the data $\beta$, *start*, *query*, *next*, and *result*, given as follows. First, set $\beta = \beta_{k+1} + \beta_k + \ldots + \beta_0 + 1$, and

$$\mathit{start}(\vec{x}) = \langle 0, \beta_0, \mathit{start}_0(\vec{x}), \emptyset, \langle \vec{x}\rangle\rangle.$$

Assuming $q$ is of the form $\langle i, c, s, u, v\rangle$, set

$$\mathit{norm}(q) = \beta_{k+1} + \ldots + \beta_i + c$$

and

$$\mathit{query}(q) = \mathit{query}_i(s).$$

Then define $\mathit{next}(q, z)$ by cases, again assuming that $q$ is of the form $\langle i, c, s, u, v\rangle$:

1. If $\mathit{norm}_i(s) \prec c$, we are in the middle of the computation of the $i$th algorithm. In that case, set
$$\mathit{next}(q, z) = \langle i, \mathit{norm}_i(s), \mathit{next}_i(s, z), u, v\rangle.$$
2. If $\mathit{norm}_i(s) \succeq c$, we have completed the computation of the $i$th algorithm.
   (a) If $i < k$, store the result and begin algorithm $i + 1$: set
   $$\mathit{next}(q, z) = \langle i + 1, \beta_{i+1}, \mathit{start}_{i+1}((u)_0, \ldots , (u)_l), u, v\hat{\ }\langle \mathit{result}_i(s)\rangle\rangle,$$
   where $l$ is the arity of the $F_i$'s.
   (b) If $i = k$, begin the computation of $G$: set
   $$\mathit{next}(q, z) = \langle k + 1, \beta_{k+1}, \mathit{start}_{k+1}((v)_0, \ldots , (v)_{k-1}, \mathit{result}_k(s)), \emptyset, \emptyset\rangle$$
   (c) If $i = k + 1$, we are done. Set $\mathit{next}(q, z) = q$ to flag this fact.

Finally, set $\mathit{result}(q) = \mathit{result}_{k+1}(s)$.

It is not hard to show, in $\mathit{ERA}(f)$, that from a computation sequence for $H$ at $\vec{x}$ and $f$ one can extract computation sequences for $F_1, \ldots , F_k$ at $\vec{x}$ and $f$, and a computation sequence for $G$ at the result of those computations. $\dashv$

From now on, I will rely on less formal descriptions of the algorithms, and leave the details of the implementation to the reader. The next lemma shows that assuming that $\alpha$ is closed under multiplication, the set of $\prec\alpha$-recursive functions is closed under a schema of $\prec\alpha$-recursion, in which the functions defining the algorithm are themselves $\prec\alpha$-recursive. Notice that the condition on $s$ in the statement of the lemma is identical to $\mathit{CompSeq}_F(s, \vec{x})$, except that the functions defining the algorithm is no longer assumed to be elementary.

LEMMA 7.3. *Suppose $\alpha$ is infinite and closed under multiplication. Given $\beta$ less than $\alpha$ and $\prec\alpha$-recursive functions $\mathit{Start}(\vec{x}, f)$, $\mathit{Norm}(q, f)$, $\mathit{Next}(q, f)$,* *and* $Result(q, f)$*, there is a* $\prec\alpha$*-recursive function* $F(x, f)$*, such that* $ERA(f)$ *proves*

$$
\begin{aligned}
&F(\vec{x}, f) = y \leftrightarrow \exists s\ ( \\
&\quad (s)_0 = Start(\vec{x}, f) \wedge \\
&\quad \forall i < length(s)\ ((s)_{i+1} = Next((s)_i, f)) \wedge \\
&\quad ((length(s) = 1 \wedge Norm((s)_0, f) \succeq \beta) \vee \\
&\qquad (length(s) > 1 \wedge Norm((s)_0, f) \prec \beta \wedge \\
&\qquad \forall i < (last(s) - 1)\ (Norm((s)_{i+1}, f) \prec Norm((s)_i, f)) \wedge \\
&\qquad Norm((s)_{last(s)}, f) \succeq Norm((s)_{last(s)-1}, f))) \wedge \\
&\quad Result((s)_{last(s)}) = y).
\end{aligned}
$$

Proof. The proof is similar to the preceding one. The algorithm for $F$ first computes $Start(\vec{x}, f)$; then iteratively computes $Norm$ and $Next$, until the norm of the state fails to decrease; and then computes $Result$. Assuming the algorithms for $Start$, $Norm$, $Next$, and $Result$ are, respectively, $\gamma$-, $\delta$-, $\epsilon$-, $\zeta$-recursive, the algorithm for $H$ can be made $\eta$-recursive, where $\eta = \zeta + (\delta + \epsilon) \cdot \beta + \gamma + 1$. ⊣

Using $\omega$-recursion, we can simulate ordinary primitive recursion.

Lemma 7.4. *Suppose* $\alpha$ *is infinite and closed under the function* $\gamma \mapsto \gamma \cdot \omega$*. Let* $F_0(\vec{z}, f)$ *and* $F_1(x, w, \vec{z}, f)$ *be* $\prec\alpha$*-recursive. Then there is is a* $\prec\alpha$*-recursive function* $G(x, \vec{z}, f)$ *such that* $ERA(f)$ *proves* $G(0, \vec{z}, f) \simeq F_0(\vec{z}, f)$ *and*

$$G(x + 1, \vec{z}, f) \simeq F_1(x, G(x, \vec{z}, f), \vec{z}, f).$$

*Furthermore, we can define* $G$ *in such a way that* $ERA(f)$ *proves that whenever* $G(x, \vec{z}, f)$ *is defined, there is a sequence of computation sequences* $\langle s_0, \ldots, s_x\rangle$*, such that*

- $s_0$ *is a computation sequence for* $F_0$ *at* $\vec{z}, f$*.*
- *If* $x$ *is greater than or equal to* $1$*,* $s_1$ *is a computation sequence for* $F_1$ *at* $(1, result_0((s_0)_{last(s_0)}), \vec{z}), f$*.*
- *For each* $i$ *such that* $0 < i < x$*,* $s_{i+1}$ *is a computation sequence for* $F_1$ *at* $(i, result_1((s_i)_{last(s_i)}), \vec{z}), f$*.*
- $G(x, \vec{z}, f) = result_1((s_x)_{last(s_x)})$*.*

Proof. As in the previous proof, with $\beta = \omega$, we can design an algorithm that successively computes $G(0, \vec{z}, f)$, $G(1, \vec{z}, f)$, … $G(x, \vec{z}, f)$. ⊣

Lemmata 7.1–7.4 imply that we can assign to each function symbol $g(\vec{x}, f)$ of $PRA(f)$ a $\prec\omega^\omega$-recursive function $F_g(\vec{x}, f)$, in such a way that $ERA(f)$ proves that the axioms of $PRA(f)$ are satisfied by these functions, at least at arguments where they are defined. Recall that we can take the language of $PRA(f)$ to include that of $ERA(f)$; below we will need to know that the translation $g \mapsto F_g$ preserves elementary functions, in the following sense.

LEMMA 7.5. *Let $g(\vec{x}, f)$ be an elementary function in $f$. Then $ERA(f)$ proves*

$$F_g(\vec{x}, f)\downarrow \rightarrow F_g(\vec{x}, f) = g(\vec{x}, f).$$

PROOF. By induction on the definition of $g$, using the additional information in Lemma 7.4. $\dashv$

In fact, in $ERA(f)$ one can prove the existence of suitable computation sequences, and therefore show $\forall \vec{x}\; F_g(\vec{x}, f)\downarrow$. We will not, however, need this fact below.

We can now show that the proof-theoretic ordinal of $PRA(f)$ is at most $\omega^\omega$. The observations following Lemma 7.4 show that one can interpret $PRA(f)$ in $ERA(f)$ together the assumptions that each $\prec\omega^\omega$-recursive function is everywhere defined. In order to use Theorem 6.3, however, we need to show that a *single* $\prec\omega^\omega$-recursive function suffices. The idea is this: we will show that given any proof in $PRA(f)$, one can use a single $\prec\omega^\omega$-recursive function to assign correct values to all the terms appearing in the proof; and furthermore, that we can do this "within" $ERA(f)$.

Say a sequence of terms $t_0, \ldots, t_k$ in $PRA(f)$ is a *formation sequence* if each $t_i$ is either a constant or variable, or the result of applying a function symbol of $PRA(f)$ to previous terms in the sequence. To each formation sequence $S$ in which no variable other than $x$ occurs, the following definition assigns a formula $Eval_S(e, x, f)$ in the language of $ERA(f)$, which asserts that the sequence $e$ assigns the correct values to the members of $S$, when the symbols $x$ and $f$ are interpreted as $x$ and $f$, respectively.

DEFINITION 7.6. *For each formation sequence $S$ in which no variable other than $x$ occurs, let $Eval_S(e, x, f)$ be the formula in the language of $ERA(f)$, defined inductively as follows:*

- *$Eval_\emptyset(e, x, f)$ is the sentence $0 = 0$*
- *If $t_k$ is the variable $x$, then $Eval_{\langle t_0,\ldots,t_k\rangle}(e, x, f)$ is defined to be*

$$(e)_{\bar{k}} = x \wedge Eval_{\langle t_0,\ldots,t_{k-1}\rangle}(e, x, f).$$

- *If $t_k$ is of the form $g(t_{i_0}, \ldots, t_{i_l}, f)$, where $g$ is a function symbol of $PRA(f)$, then $Eval_{\langle t_0,\ldots,t_k\rangle}(e, x, f)$ is*

$$(e)_{\bar{k}} = F_g((e)_{\bar{i}_0}, \ldots, (e)_{\bar{i}_l}, f) \wedge Eval_{\langle t_0,\ldots,t_{k-1}\rangle}(e, x, f).$$

LEMMA 7.7. *Let $S$ be a formation sequence of terms in the language of $PRA(f)$ in which at most the variable $x$ is free. Then there is a $\prec\omega^\omega$-recursive function $G(x, f)$ such that $ERA(f)$ proves $G(x, f) = e \rightarrow Eval_S(e, x, f)$.*

PROOF. By induction on the length of $S$, using Lemmata 7.1–7.4. $\dashv$

Given a proof in the quantifier-free version of $PRA(f)$, we can use Lemma 7.7 to find a correct evaluation of the terms appearing in the proof.

Lemma 7.8. *Suppose $PRA(f)$ proves $\exists y\ \theta(x, y, f)$, where $\theta$ is $\Delta_0(f)$. Then there is a $\prec\omega^\omega$-recursive function $H(x, f)$ such that $ERA(f)$ proves $H(x, f) = y \to \theta(x, y, f)$.*

Proof. Suppose $PRA(f)$ proves $\exists y\ \theta(x, y, f)$. Then it also proves the formula $\exists y\ (\chi_\theta(x, y, f) = 1)$, where $\chi_\theta$ is a an elementary recursive characteristic function representing $\theta$. By Herbrand's theorem, there is a function symbol $g(x, f)$ of primitive recursive arithmetic, and a proof $d$ of $\chi_\theta(x, g(x, f), f) = 1$ in propositional logic, from substitution instances of the equality axioms and axioms of $PRA(f)$. For example, we may take $d$ to be a sequence of quantifier-free formulae in the language of $PRA(f)$ such that each line either is an instance of an axiom of $PRA(f)$, is an instance of an axiom of equality, is an instance of a propositional tautology, or follows from previous lines by modus ponens (or other valid propositional inferences).

Let $S$ be a formation sequence that includes all the terms occurring in $d$. Each line of $d$ is a boolean combination of atomic formulae of the form $t = s$, where $t$ and $s$ are terms occurring in $S$. If $\varphi$ is such a formula, let $\varphi^e$ denote the formula obtained by replacing each term $t_i$ by $(e)_{\bar{i}}$. Then by induction, one can show that for each line $\varphi$ of $d$, then $ERA(f)$ proves

$$Eval_S(e, x, f) \to \varphi^e.$$

When $\varphi$ is an axiom of equality or $PRA(f)$, this follows from the definition of $Eval_S(e, x, f)$; otherwise, the propositional axioms and inferences of $d$ can be mirrored in $ERA(f)$.

In particular, suppose $g(x, f)$ is the $k$th term in $S$ and $\chi_\theta(x, g(x, f), f)$ is the $l$th. Since the conclusion of $d$ is $\chi_\theta(x, g(x, f), f) = 1$, in $ERA(f)$ one can prove

$$Eval_S(e, x, f) \to (e)_{\bar{l}} = 1.$$

But if $e$ evaluates terms correctly, $(e)_{\bar{l}}$ is equal to $F_{\chi_\theta}(x, (e)_{\bar{k}}, f)$; so $ERA(f)$ proves

$$Eval_S(e, x, f) \to (e)_{\bar{l}} = F_{\chi_\theta}(x, (e)_{\bar{k}}, f),$$

and hence $Eval_S(e, x, f) \to F_{\chi_\theta}(x, (e)_{\bar{k}}, f) = 1$. But by Lemma 7.5, $ERA(f)$ proves that $F_{\chi_\theta}(x, (e)_{\bar{k}}, f) = 1$ is equivalent to $\theta(x, (e)_{\bar{k}}, f)$.

In short, in $ERA(f)$ we can prove

$$Eval_S(e, x, f) \to \theta(x, (e)_{\bar{k}}, f).$$

Using Lemma 7.7, let $G(x, f)$ be a $\prec\alpha$-recursive function which returns an $e$ satisfying $Eval_S(e, x, f)$. Using Lemmata 7.1 and 7.2 let $H(x, f)$ be a $\prec\alpha$-recursive function such that $ERA(f)$ proves $H(x, f) \simeq (G(x, f))_{\bar{k}}$. Putting it all together, we have that $ERA(f)$ proves $H(x, f) = y \to \theta(x, y, f)$, as desired. ⊣

By Theorem 6.3 this yields

Theorem 7.9. *The proof-theoretic ordinal of $PRA(f)$ is at most $\omega^\omega$.*

At this point, we could extend the analysis to various forms of primitive recursion on the ordinals, and use similar methods to obtain ordinal analyses of various extensions of $PRA(f)$. But instead of pursuing that, let us turn instead to theories of transfinite induction.

**§8.** $\Pi_1$ **Transfinite induction.** In the last section, we saw that ordinal recursion can be used to simulate ordinary primitive recursion; but this should not have been very surprising. In this section we will be somewhat bolder: we will augment our basic theory with function symbols that are intended to denote *noncomputable* functions, allowing us to prove a form of transfinite induction. A judicious application of Herbrand's theorem will then enable us to extract constructive information from proofs in the augmented theory.

By Proposition 2.2, we can represent our system of notations in the language of $I\Delta_0^{exp}$. If $\varphi(x)$ is any formula in this language and $\beta$ is any ordinal notation, then the *principle of transfinite induction* on $\beta$ for $\varphi$ is

$$\forall\gamma \prec \bar{\beta}\ (\forall\delta \prec \gamma\ \varphi(\delta) \to \varphi(\gamma)) \to \forall\gamma \prec \bar{\beta}\ \varphi(\gamma).$$

In words, this reads "if $\varphi(x)$ is progressive on $\beta$, then it holds for every ordinal less than $\beta$." Its contrapositive,

$$\exists\gamma \prec \bar{\beta}\ \neg\varphi(\gamma) \to \exists\gamma \prec \bar{\beta}\ (\neg\varphi(\gamma) \wedge \forall\delta \prec \gamma\ \varphi(\delta))$$

is the *least-element principle* on $\beta$ for $\neg\varphi$. If $\Gamma$ is any set of formulae, then $TI(\beta, \Gamma)$ and $LEP(\beta, \Gamma)$ denote, respectively, the principle of transfinite induction and the least-element principle on $\beta$, restricted to formulae in $\Gamma$. Similarly, $TI(\prec\alpha, \Gamma)$ and $LEP(\prec\alpha, \Gamma)$ denote these principles for arbitrary $\beta$ less than $\alpha$.

Our goal here is to provide ordinal analysis of the theories for the form

$$I\Delta_0^{exp}(f) + TI(\prec\alpha, \Pi_1(f)).$$

The following lemma gives some equivalent characterizations.

LEMMA 8.1. *Assume $\alpha$ is closed under the function $\beta \mapsto \omega \cdot \beta$. Then over $I\Delta_0^{exp}(f)$, the following schemata are equivalent:*

1. $TI(\prec\alpha, \Pi_1(f))$
2. $LEP(\prec\alpha, \Sigma_1(f))$
3. $TI(\prec\alpha, \Delta_0^{exp}(f))$
4. $LEP(\prec\alpha, \Delta_0^{exp}(f))$

PROOF. The contrapositive of any instance of 1 is equivalent to an instance of 2, and vice-versa; similarly for 3 and 4. Clearly 2 implies 4, so it suffices to show that 4 implies 2.

Let $\theta(\gamma, x)$ be $\Delta_0^{exp}$ and let $\beta$ be a notation less than $\alpha$. Arguing in $I\Delta_0^{exp}(f) + LEP(\prec\alpha, \Delta_0^{exp}(f))$, let us prove the least-element principle on $\beta$ for $\exists x\ \theta(\gamma, x)$. Let $\theta'(\delta)$ be a formula which asserts that, if $\delta$ is written in the form $\omega \cdot \delta' + y$, then $\theta(\delta', y)$. Now suppose $\theta(\gamma, x)$. Then $\theta'(\omega \cdot \gamma + x)$. By the least-element principle on $\omega \cdot \beta$ for $\theta'$, there is a least $\delta$ satisfying $\theta'(\delta)$. But if $\delta = \omega \cdot \delta' + y$, then $\delta'$ is the least element satisfying $\exists x\, \theta(\delta', x)$. $\dashv$

Now let us add function symbols to $ERA(f)$ that enable us to interpret the new axioms. Using the last characterization in Lemma 8.1, it is sufficient to have, for each notation $\beta$ less than $\alpha$ and elementary relation $R(\vec{x}, y, f)$, a function $g(\vec{x}, f)$ which returns the least $\gamma$ less than $\beta$ satisfying $R(\vec{x}, \gamma, f)$, whenever such a $\gamma$ exists. The approach we will take is slightly more general, but not more difficult.

Given an elementary function $norm(\vec{x}, z, f)$, let "$z$ minimizes $norm(\vec{x}, \cdot, f)$ below $\beta$" denote the following formula:

$$\forall w\ (norm(\vec{x}, w, f) \prec \beta \rightarrow norm(\vec{x}, z, f) \preceq norm(\vec{z}, w, f)).$$

In words, if anything has a norm less than $\beta$, then $z$ has the smallest possible norm. Let

$$ERA(f) + min(\prec\alpha, \mathcal{E}(f))$$

be the theory obtained by adding, for each elementary function $norm(\vec{x}, z, f)$ and $\beta$ less than $\alpha$, a new function symbol, $min_{norm,\beta}(\vec{x}, f)$, to the language, and an axiom

$$\text{“}min_{norm,\beta}(\vec{x}, f) \text{ minimizes } norm(\vec{x}, \cdot, f) \text{ below } \bar{\beta}.\text{”}$$

In the name of the theory, the "$\mathcal{E}(f)$" indicates that the norm functions are required to be elementary in $f$; note that that the theory does not have symbols, say, for elementary functions or minimization functions defined from the ones we have just added.

Even for $\beta = 1$, a function $min_{norm,\beta}$ may be nonconstructive. For example, let $T(x, y, f)$ be an elementary relation such that $\exists y\, T(x, y, f)$ is a complete $\Sigma_1(f)$ formula; more precisely, assume $T$ has the property that for any $\Sigma_1(f)$ formula $\varphi(\vec{w}, f)$ there is a natural number $n$, such that $\varphi(\vec{w}, f)$ is equivalent, in $ERA(f)$, to the formula $\exists y\ T(\langle \bar{n}, \vec{w}\rangle, y, f)$. (Think of $T(\langle \bar{n}, \vec{w}\rangle, y, f)$ as asserting that $y$ witnesses the truth of the $\Sigma_1(f)$ formula coded by $n$, at the parameters $\vec{w}$; or $T$ may be a version of Kleene's $T$ predicate, asserting that $y$ is a halting computation of the $n$th Turing machine with oracle $f$, on input $\vec{w}$. Below, we will also assume that $n$ can be computed in an elementary way from a Gödel number of $\varphi$.) Let $\beta = 1$, and let

$$norm(x, z, f) = \begin{cases} 0 & \text{if } T(x, z, f) \\ 1 & \text{otherwise.} \end{cases}$$

Then $min_{norm,\beta}(\vec{x})$ is guaranteed to return a witness $y$ to $T(x, y, f)$, if there is one; this enables us, for example, to solve the halting problem.

Lemma 8.2. *For each $\alpha$, $ERA(f) + min(\prec\alpha, \mathcal{E}(f))$ is a universal theory containing $I\Delta_0^{exp}(f) + TI(\prec\alpha, \Pi_1(f))$.*

Proof. The formula "$min_{norm,\beta}(\vec{x}, f)$ minimizes $norm(\vec{x}, \cdot, f)$ below $\bar{\beta}$" is universal, so it suffices to show that instances of the $\Sigma_1$ least-element principle are derivable from these axioms. Given a $\Delta_0^{exp}$ formula $\theta(y,\gamma,\vec{x},f)$ with the free variables shown and a notation $\beta$, let

$$norm(\vec{x},z,f) = \begin{cases} (z)_1 & \text{if } (z)_1 \prec \beta \text{ and } \theta((z)_0,(z)_1,\vec{x},f) \\ \beta & \text{otherwise} \end{cases}$$

Arguing in $ERA(f)+min(\prec\alpha,\mathcal{E}(f))$, if there is any $\gamma$ satisfying $\exists y\, \theta(y,\gamma,\vec{x},f)$, then $(min_{norm,\beta}(\vec{x}))_1$ is a least such one. ⊣

We will carry out the ordinal analysis of $ERA(f) + min(\prec\alpha,\mathcal{E}(f))$ in two steps. First, we will show that one can reduce the problem of assigning the correct values to a set of terms in this theory to the problem of finding a value minimizing an appropriate norm, provably in $ERA(f)$. Then we will use Herbrand's theorem to replace the latter problem with a $\prec\alpha$-recursive calculation.

If $\varphi(\vec{x},z)$ is any formula in the language of $ERA(f)$ and $\beta$ is any notation, let us say that $ERA(f)$ proves that $\varphi(\vec{x},y)$ is *solvable* (for $y$) *by $\beta$-minimization* if there are elementary functions $norm(\vec{x},z,f)$ and $result(\vec{x},z,f)$ such that $ERA(f)$ proves

$$\text{“}z \text{ minimizes } norm(\vec{x},\cdot) \text{ below } \bar{\beta}\text{”} \rightarrow \varphi(\vec{x}, result(\vec{x},z,f)).$$

Say that $ERA(f)$ proves that $\varphi(\vec{x},y)$ is solvable by *$\prec\alpha$-minimization* if it proves that $\varphi(\vec{x},y)$ is solvable by $\beta$-minimization for some $\beta$ less than $\alpha$. Note that the "solution" to $\varphi(\vec{x},z)$ may not be unique. If $ERA(f)$ also proves $\varphi(\vec{x},z) \wedge \varphi(\vec{x},z') \rightarrow z = z'$ it makes sense to say that $ERA(f)$ proves that $\varphi(\vec{x},z)$ defines a function that is computable by $\prec\alpha$-minimization; but for our purposes the more general notion is more useful.

The next few lemmata provide closure properties on the kinds of problems that are solvable by $\prec\alpha$-minimization.

LEMMA 8.3. *For any $\alpha$, if $g(\vec{x},f)$ is an elementary function, then $ERA(f)$ proves that the relation $g(\vec{x},f) = y$ is solvable by $\prec\alpha$-minimization.*

PROOF. Let $norm(\vec{x},z,f)$ be arbitrary, and let $result(\vec{x},z,f) = g(\vec{x},f)$. ⊣

LEMMA 8.4. *Suppose $\beta$ is less than $\alpha$, and $norm(\vec{x},y,f)$ is an elementary function in $f$. Then $ERA(f)$ proves that the relation "$z$ minimizes $norm(\vec{x},\cdot,f)$ below $\beta$" is solvable by $\prec\alpha$-minimization.*

PROOF. Leave $\beta$ and $norm$ alone, and let $result(\vec{x},z,f) = z$. ⊣

LEMMA 8.5. *Let $\alpha$ be closed under addition. If $\varphi_0(\vec{x},y,f),\ldots,\varphi_k(\vec{x},y,f)$ are all solvable by $\prec\alpha$-minimization, provably in $ERA(f)$, then so is the formula $\varphi_0(\vec{x},(y)_0,f) \wedge \ldots \wedge \varphi_k(\vec{x},(y)_k,f)$.*

PROOF. Suppose each $\varphi_i$ is solvable by $\beta_i$, $norm_i(\vec{x},z,f)$, and $result_i(\vec{x},z,f)$. We can assume without loss of generality that $ERA(f)$ proves that for every $\vec{x}$ and $z$, $norm_i(\vec{x},z,f)$ is less than or equal to $\beta_i$, by replacing $norm_i(\vec{x},z,f)$ with $min(\beta_i, norm_i(\vec{x},z,f))$ if necessary. Let $\beta = \beta_0 \# \ldots \# \beta_k$, let

$$norm(\vec{x},z,f) = norm_0(\vec{x},(z)_0,f) \# \ldots \# norm_k(\vec{x},(z)_k,f),$$

and let

$$result(\vec{x}, z, f) = \langle result_0(\vec{x}, (z)_0, f), \dots, result_k(\vec{x}, (z)_k, f)\rangle.$$

It is not hard to see that if $z$ minimizes $norm(\vec{x}, \cdot, f)$ below $\beta$, then each $(z)_i$ minimizes $norm_i(\vec{x}, \cdot, f)$ below $\beta_i$. ⊣

Lemma 8.6. *Let $\alpha$ be infinite and closed under multiplication. Suppose $\varphi_0(\vec{x}, w, f)$ and $\varphi_1(\vec{x}, w, y, f)$ are solvable by $\prec\alpha$-minimization, for $w$ and $y$ respectively, provably in ERA($f$). Then the formula*

$$\varphi_0(\vec{x}, (y)_0) \land \varphi_1(\vec{x}, (y)_0, (y)_1),$$

*and the formula*

$$\exists w\ (\varphi_0(\vec{x}, w) \land \varphi_1(\vec{x}, w, y)),$$

*are solvable for $y$ by $\prec\alpha$ minimization, provably in ERA($f$).*

Proof. Suppose $\varphi_0$ is solved by $\beta_0$, $norm_0(\vec{x}, z, f)$, and $result_0(\vec{x}, z, f)$, and $\varphi_1$ is solved by $\beta_1$, $norm_1(\vec{x}, w, z)$, and $result_1(\vec{x}, w, f)$. As in the previous lemma, we can assume that $norm_0$ and $norm_1$ are bounded by $\beta_0$ and $\beta_1$ respectively. Let $\beta = (\beta_1 + 1)(\beta_0 + 1)$ and let

$$norm(\vec{x}, z, f) = (\beta_1 + 1) \cdot norm_0(\vec{x}, (z)_0, f) + norm_1(\vec{x}, result_0(\vec{x}, (z)_0, f), (z)_1).$$

Suppose $z$ minimizes $norm(\vec{x}, \cdot, f)$ below $\beta$. Then $(z)_0$ minimizes $norm_0(\vec{x}, \cdot, f)$ below $\beta_0$; otherwise we could change $(z)_0$ and decrease the value of sum above, independent of the behavior of the second term. Fixing $(z)_0$, we also see that $(z)_1$ minimizes $norm_1(\vec{x}, result_0(\vec{x}, (z)_0, f), \cdot, f)$ below $\beta_1$, because otherwise we could change $(z)_1$ and decrease the value of the sum above. To solve the first formula, take

$$result(\vec{x}, z, f) = \langle result_0(\vec{x}, (z)_0, f), result_1(\vec{x}, result_0(\vec{x}, (z)_0, f), (z)_1, f)\rangle.$$

To solve the second formula, take

$$result(\vec{x}, z, f) = result_1(\vec{x}, result_0(\vec{x}, (z)_0, f), (z)_1, f).$$

This completes the proof. ⊣

Taken together, the lemmata above imply that for infinite $\alpha$ closed under multiplication, the functions that are computable by $\prec\alpha$-minimization are closed under composition. As an exercise, the reader can try to prove that if $F(\vec{x}, f)$ is a $\prec\alpha$-recursive function, then it is computable by $\prec\alpha$-minimization. (See also the first proof of Lemma 9.2 below.)

The next step is to show that given any formation sequence $S$ for terms in $ERA(f) + min(\prec\alpha, \mathcal{E}(f))$, the problem of finding an appropriate evaluation of the terms in $S$ is solvable by $\prec\alpha$-minimization. We have to be careful, since, in general, there may be more than one value that we can assign to a term of the form $min_{norm,\beta}(t_1, \dots, t_k, f)$; so when more than one term of this form appears in the formation sequence, we have to make sure that the evaluation assigns values to these terms consistently.

DEFINITION 8.7. *For each formulation sequence $S$ for terms in the theory $ERA(f) + min(\prec\alpha, \mathcal{E}(f))$ with no variable other than $x$, let $Eval_S(e, x, f)$ be the formula in the language of $ERA(f)$, defined inductively as follows:*

- *$Eval_\emptyset(e, x, f)$ is the sentence $0 = 0$*
- *If $t_k$ is the variable $x$, then $Eval_{\langle t_0,\dots,t_k\rangle}(e, x, f)$ is*

$$(e)_{\bar{k}} = x \wedge Eval_{\langle t_0,\dots,t_{k-1}\rangle}(e, x, f).$$

- *If $t_k$ is of the form $g(t_{i_0}, \dots, t_{i_l}, f)$, where $g$ is a function symbol of $ERA(f)$, then $Eval_{\langle t_0,\dots,t_k\rangle}(e, x, f)$ is*

$$(e)_{\bar{k}} = g((e)_{\bar{i}_0}, \dots, (e)_{\bar{i}_l}, f) \wedge Eval_{\langle t_0,\dots,t_{k-1}\rangle}(e, x, f).$$

- *If $t_k$ is of the form $min_{norm,\beta}(t_{i_0}, \dots, t_{i_l}, f)$, let $t_{j_0}, \dots, t_{j_m}$ enumerate all the terms before $t_k$ in $S$ that are also of this form; and for $u = 0, \dots, m$, suppose $t_{j_u}$ is the term $min_{norm,\beta}(t_{i_{u,0}}, \dots, t_{i_{u,l}}, f)$. Then $Eval_{\langle t_0,\dots,t_k\rangle}(e, x, f)$ is the conjunction of the following:*
  - *"$(e)_{\bar{k}}$ minimizes $norm((e)_{\bar{i}_0}, \dots, (e)_{\bar{i}_l}, \cdot, f)$ below $\bar{\beta}$"*
  - *$Eval_{\langle t_0,\dots,t_{k-1}\rangle}(e, x, f)$*
  - *$\bigwedge_{u=0}^{m}((e)_{\bar{i}_0} = (e)_{\bar{i}_{u,0}} \wedge \dots \wedge (e)_{\bar{i}_l} = (e)_{\bar{i}_{u,l}}) \rightarrow (e)_{\bar{k}} = (e)_{\bar{j}_u})$.*

LEMMA 8.8. *Let $\alpha$ be infinite and closed under multiplication, and let $S$ be a formation sequence of terms in the language of $ERA(f) + min(\prec\alpha, \mathcal{E}(f))$. Then $ERA(f)$ proves that the relation $Eval_S(e, x, f)$ is solvable for $e$ by $\prec\alpha$-minimization.*

PROOF. By induction on the length of $S$, using Lemmata 8.3–8.6. Consider the case where $S$ is the sequence $\langle t_0, \dots, t_k\rangle$ and $t_k$ is of the form $min_{norm,\beta}(t_{i_0}, \dots, t_{i_l}, f)$. By the induction hypothesis, $ERA(f)$ proves that the relation $Eval_{\langle t_0,\dots,t_{k-1}\rangle}(e', x, f)$ is solvable for $e'$ by $\prec\alpha$-minimization. By Lemma 8.4, $ERA(f)$ also proves that the problem of finding a value $v$ minimizing $norm((e)_{\bar{i}_0}, \dots, (e)_{\bar{i}_l}, \cdot, f)$ below $\beta$ is solvable by $\prec\alpha$-minimization. There is an elementary function which, given $e'$ and $v$, checks the values that $e'$ assigns to previous terms in $S$, decides whether to assign one of these values or $v$ to $t_k$, and returns the resulting assignment. By Lemmata 8.3 and 8.6, this provides a solution to $Eval_S(e, x, f)$. ⊣

LEMMA 8.9. *Let $\alpha$ be infinite and closed under multiplication. If $\theta(x, y, f)$ is a $\Delta_0(f)$ formula such that $ERA(f) + min(\prec\alpha, \mathcal{E}(f))$ proves $\exists y\ \theta(x, y, f)$, then $ERA(f)$ proves that $\theta(x, y, f)$ is solvable for $y$ by $\prec\alpha$-minimization.*

PROOF. Just as in the proof of Lemma 7.8. If $ERA(f) + min(\prec\alpha, \mathcal{E}(f))$ proves $\exists y\ \theta(x, y, f)$, then there is a sequence of terms $r_0, \dots, r_m$ and a propositional proof of $\chi_\theta(x, r_0, f) = 1 \vee \dots \vee \chi_\theta(x, r_m, f) = 1$ from substitution instances of equality axioms and axioms of $ERA(f) + min(\prec\alpha, \mathcal{E}(f))$. In $ERA(f)$, given $x$ we can use $\prec\alpha$-minimization to evaluate all the terms occurring in the proof and choose one satisfying $\theta$. $\dashv$

The key use of Herbrand's theorem is in the proof of the following lemma.

Lemma 8.10. *Suppose $\alpha$ is infinite and closed under multiplication, and let $\theta(x, y, f)$ be a $\Delta_0$ formula such that $ERA(f)$ proves $\theta(x, y, f)$ is solvable by $\prec\alpha$-minimization. Then there is a $\prec\alpha$-recursive function $F(x, f)$ such that $ERA(f)$ proves $F(x, f) = y \to \theta(x, y, f)$.*

Proof. The hypothesis of the lemma means that there is a $\beta$ less than $\alpha$, and functions *norm* and *result*, such that $ERA(f)$ proves

$$\forall z\ (\forall w\ (norm(x, w, f) \prec \bar{\beta} \to norm(x, z, f) \preceq norm(x, w, f)) \to \theta(x, result(x, z, f), f)).$$

Leaving the universal quantification over $z$ implicit, bringing the universal quantification over $w$ to the front (where it becomes existential), and rewriting the formula slightly, we have that $ERA(f)$ proves

$$\exists w\ (norm(x, w, f) \not\prec \beta \lor norm(x, w, f) \not\prec norm(x, z, f) \to \theta(x, result(x, z, f), f)).$$

By Herbrand's theorem, there is a an elementary function $g(x, z, f)$ such that $ERA(f)$ proves

$$norm(x, g(x, z, f), f) \not\prec \beta \lor norm(x, g(x, z, f), f) \not\prec norm(x, z, f) \to \theta(x, result(x, z, f), f).$$

In other words, if the norm of $g(x, z, f)$ is not less than both $\beta$ and the norm of $z$, then $result(x, z, f)$ is the witness we are after. Our job, then, is to find a $z$ satisfying the antecedent of this implication. An obvious iterative algorithm suggests itself: First set $z_0 = 0$. If $norm(x, g(x, z_0, f), f) \not\prec \beta$ or $norm(x, g(x, z_0, f), f) \not\prec norm(x, z_0, f)$, we are done. Otherwise, iteratively set $z_{i+1}$ equal to $g(x, z_i, f)$, until $norm(x, z_{i+1}, f) \not\prec norm(x, z_i, f)$. Then $result(x, z_i, f)$ is the value we are after.

By Lemma 7.5, $norm(x, z, f)$ and $result(x, z, f)$ are $\prec\alpha$-recursive. The algorithm just described is a $\prec\alpha$-iterative algorithm using $\prec\alpha$-recursive functions; by Lemma 7.3, it can be carried out by a single $\prec\alpha$-recursive function, provably in $ERA(f)$. $\dashv$

Putting the last two lemmata together yield the following

Lemma 8.11. *Let $\alpha$ be infinite and closed under multiplication. If $\theta(x, y, f)$ is a $\Delta_0(f)$ formula such that $ERA(f) + min(\prec\alpha, \mathcal{E}(f))$ proves $\exists y\ \theta(x, y, f)$, then there is a $\prec\alpha$-recursive function $F(x, f)$ such that $ERA(f)$ proves $F(x, f) = y \to \theta(x, y, f)$.*

Below we will need to know that Lemma 8.11 still holds with additional function symbols, $f_0, \ldots, f_k$. To see this, note that every proof in this section can easily be generalized in this respect; alternatively, one can take $f$ to code a finite sequence of function symbols $f_0, \dots, f_k$, and use variant of Lemma 2.3 to reduce the more general statement to that of Lemma 8.11.

Together with Theorem 6.3, Lemma 8.11 yields

THEOREM 8.12. *Let $\alpha$ be infinite and closed under multiplication. Then the proof-theoretic ordinal of $I\Delta_0(f) + TI(\prec\alpha, \Pi_1(f))$ is at most $\alpha$.*

Since induction on the natural numbers corresponds to transfinite induction on $\omega$, and $\Pi_1$ and $\Sigma_1$ induction on the natural numbers are equivalent, $TI(\prec\omega^\omega, \Pi_1(f))$ includes $I\Sigma_1$, and we have

THEOREM 8.13. *The proof-theoretic ordinal of $I\Sigma_1(f)$ is at most $\omega^\omega$.*

These bounds are sharp, and, more generally, the proof-theoretic ordinal of $I\Delta_0(f) + TI(\prec\alpha, \Pi_1(f))$ is the least $\alpha'$ greater than or equal to $\alpha$ that is closed under multiplication.

**§9. Arithmetic transfinite induction.** Having dealt with $\Pi_1$ transfinite induction, let us now extend the analysis to theories of transfinite induction for arbitrary arithmetic formulae. Once again, the first step is to express these principles in a suitable universal theory. We will do this in a straightforward way: we will use Skolem functions to reduce any arithmetic formula to an elementary relation, and then use minimization, as in the last section.

To start with, consider $\Pi_2$ transfinite induction. Let $\exists y\ T(x, y, f)$ be the complete $\Sigma_1(f)$ formula introduced in the last section, and let $wit_1$ be a new function symbol with defining equation

$$(wit_1(f)) \qquad \forall y\ (T(x, y, f) \rightarrow T(x, wit_1(x), f)).$$

In words, if there is any $y$ satisfying $T(x, y, f)$, $wit_1$ returns such a one.[3] Let

$$ERA(f, wit_1) + (wit_1(f)) + min(\prec\alpha, \mathcal{E}(f, wit_1))$$

denote the theory extending $ERA(f, wit_1)$ with the defining axiom for $wit_1$, and minimization for function symbols in the language of $ERA(f, wit_1)$.

LEMMA 9.1. *The theory $ERA(f, wit_1) + (wit_1(f)) + min(\prec\alpha, \mathcal{E}(f, wit_1))$ is a universal theory containing $I\Delta_0^{exp}(f) + TI(\prec\alpha, \Pi_2(f))$.*

PROOF. If $\varphi(\vec{z}, f)$ is a $\Sigma_1(f)$ formula of the form $\exists y\ \theta(y, \vec{z}, f)$, there is a numeral $n$ such that $\varphi(\vec{z}, f)$ is equivalent in $ERA(f, wit_1) + (wit_1)$ to the $\mathcal{E}(f, wit_1)$ relation $T(\langle \bar{n}, \vec{z}\rangle, wit_1(\langle \bar{n}, \vec{z}\rangle), f)$. $\Pi_2(f)$ formulae are then equivalent to formulae that are $\Pi_1(f, wit_1)$. As in the proof of Lemma 8.2, from $(min(\prec\alpha, \mathcal{E}(f, wit_1)))$ one can derive instances of transfinite induction for formulae of this kind. ⊣

[3] We could go further to fix the interpretation of $wit_1$ by requiring it to return the *least* such $y$, if there is one, and 0 otherwise; but the additional generality will be useful in [2].

The following lemma enables us to bound the ordinal of the theory of Lemma 9.1.

Lemma 9.2. *Let $\alpha$ be infinite and closed under multiplication. If $\theta(x,y,f)$ is a $\Delta_0(f)$ formula such that $ERA(f, wit_1) + (wit_1(f)) + min(\prec\alpha, \mathcal{E}(f, wit_1))$ proves $\exists y\, \theta(x,y,f)$, then there is a $\prec\omega^\alpha$-recursive function $H(x,f)$ such that $ERA(f)$ proves $H(x,f) = y \to \theta(x,y,f)$.*

Proof. Suppose the hypothesis of the lemma holds. By the deduction theorem, the theory $ERA(f, wit_1) + min(\prec\alpha, \mathcal{E}(f, wit_1))$ proves

$$\forall u, v\, (T(u,v,f) \to T(u, wit_1(u), f)) \to \exists y\, \theta(x,y,f).$$

Letting $y$ code the pair $\langle u, v\rangle$ and rewriting yields

$$\exists y\, (T((y)_0, (y)_1, f) \wedge \neg T((y)_0, wit_1((y)_0), f)) \vee \exists y\, \theta(x,y,f).$$

Bringing the existential quantifiers to the front and combining them yields

$$\exists y\, ((T((y)_0, (y)_1, f) \to T((y)_0, wit_1((y)_0), f)) \to \theta(x,y,f)).$$

By Lemma 8.11, there is a $\prec\alpha$-recursive function $F$ such that $ERA(f, wit_1)$ proves

$$F(x, f, wit_1) = y \wedge (T((y)_0, (y)_1, f) \to T((y)_0, wit_1((y)_0), f)) \to \theta(x,y,f).$$

In words, $ERA(f, wit_1)$ proves that if $F(x, f, wit_1)$ is defined, it returns either a $y$ showing that $wit_1$ fails to satisfy its defining axiom at $(y)_0$, or a $y$ satisfying $\theta(x,y,f)$.

The rest of the proof hinges on finding a finite interpretation of $wit_1$ that is robust enough to carry out the computation of $F$ and pass the final test at the end. Towards that goal, note that one can code a finite *partial* function from the natural numbers to the natural numbers with a natural number. For example, one can take the number $m$ to code the partial function

$$\tilde{m}(x) = \begin{cases} (m)_x - 1 & \text{if } x \le last(m) \text{ and } (m)_x > 0 \\ \text{undefined} & \text{otherwise} \end{cases}$$

Let $\hat{m}(x)$ denote the extension of $\tilde{m}$ to the natural numbers, such that $\hat{m}(x) = 0$ where $\tilde{m}$ is undefined. Finally, let $eval(m,x)$ be the elementary function which returns $\hat{m}(x)$. If we now take $m$ to be a variable in the language of $ERA(f)$, we can interpret references to $\hat{m}(x)$ as $eval(m,x)$. Returning to the conclusion of the last paragraph, using Lemma 2.3 to replace $wit_1$ by $\lambda x\, eval(m,x)$, we see that in $ERA(f)$ there is a proof of

$$F(x, f, \hat{m}) = y \wedge (T((y)_0, (y)_1, f) \to T((y)_0, \hat{m}((y)_0), f)) \to \theta(x,y,f).$$

Expanding the definition of $F(x, f, \hat{m}) = y$, this yields a proof of

$$\begin{aligned}(&CompSeq_F(s, x, f, \hat{m}) \wedge result_F((s)_{last(s)}) = y \wedge \\ &\qquad (T((y)_0, (y)_1, f) \to T((y)_0, \hat{m}((y)_0), f))) \to \theta(x,y,f).\end{aligned}$$

We have therefore reduced the problem to finding a $\prec\omega^\alpha$-recursive function $G(x, f)$ which returns a pair $\langle s, m\rangle$ satisfying the antecedent of this last formula, provably in $ERA(f)$. In other words, we are looking for a $\prec\omega^\alpha$-recursive function that returns a finite interpretation $m$ for $wit_1$ and a computation sequence $s$ for $F$ at $x$, $f$, and $\hat{m}$, such that if $y = result_F((s)_{last(s)})$, then $\hat{m}$ satisfies the defining equation for $wit_1$ at the pair coded by $y$. If we then let $H(x, f) \simeq result_F((G(x, f)_0)_{last(G(x,f)_0)})$, $H(x, f)$ satisfies the conclusion of the lemma.

I will now describe two ways of finding such a function $G$. The first is used in the proof of [7, Lemma 12]; the second is more explicit.

For the first method, note that by Lemma 8.11 it is sufficient to show that the existence of the pair $s, m$ is provable in $ERA(f) + min(\prec\omega^\alpha, \mathcal{E}(f))$. Arguing in this theory, then, let us show how to find $s$ and $m$ using ordinal minimization. Without loss of generality, we can assume that if $F(x, f)$ returns $y$ then $F$ queries $\hat{m}$ at $(y)_0$ in the last step of its computation, since we can always replace the $\beta$-iterative algorithm for $F$ with a $1 + \beta$-iterative algorithm which does so. Say that $m$ is a *sound* interpretation of $wit_1$ if $T(x, \hat{m}(x), f)$ holds for every $x$ in the domain of $m$. Let

$$norm'(m, x) = \begin{cases} 1 & \text{if } m \text{ is undefined at } x \\ 0 & \text{otherwise.} \end{cases}$$

Finally, say that $s$ is a *partial computation sequence* for $F$ at $x$, $f$, and $\hat{m}$ if $s$ is a proper initial segment of a computation sequence for $F$ at $x$, $f$, and $\hat{m}$. More explicitly, this amounts to saying that $(s)_0 = start_F(x)$, $norm_F((s)_0) \prec \beta$, and

$$\forall i < k - 1\ ((s)_{i+1} = next_F((s)_i, f(query_{F,0}((s)_i)), \hat{m}(query_{F,1}((s)_i))) \wedge \\ norm_F((s)_{i+1}) \prec norm_F((s)_i)).$$

Now define $norm(x, z, f)$ as follows:

- If $z$ codes a pair $\langle s, m\rangle$, $m$ is a sound interpretation of $wit_1$, and $s$ is a partial computation sequence for $F$ at $x$, $f$, and $\hat{m}$, set $norm(x, z, f)$ to

$$\omega^{norm_F((s)_0)} \cdot norm'(m, query_{F,1}((s)_0)) + \dots \\ + \omega^{norm_F((s)_{last(s)-1})} \cdot norm'(m, query_{F,1}((s)_{last(s)-1})) \\ + \omega^{norm_F((s)_{last(s)})} \cdot 2$$

  (Interpret this as 0 if $last(s) = 0$.)
- If $z$ codes a pair $\langle s, m\rangle$, $m$ is a sound interpretation of $wit_1$, and $s$ is a computation sequence for $F$ at $x$, $f$, and $\hat{m}$, set $norm(z)$ to

$$\omega^{norm_F((s)_0)} \cdot norm'(m, query_{F,1}((s)_0)) + \dots \\ + \omega^{norm_F((s)_{last(s)-1})} \cdot norm'(m, query_{F,1}((s)_{last(s)-1}))$$

- Otherwise, set $norm(x, z, f) = \omega^\beta$.

Let $z$ be a value minimizing $norm(x, \cdot, f)$ below $\omega^\beta$. We know that for this $z$ either the first or second case must hold, since taking $s$ to be the sequence $\langle start_F(\vec{x})\rangle$ and $m$ to be the partial function that is nowhere defined satisfies one of these two cases, and hence yields a norm less than $\omega^\beta$. In fact, $z$ must fall under the second case: if $s$ is a partial computation sequence, extending it by appending $next_F((s)_{last(s)}, f(query_0((s)_{last(s)})), \hat{m}(query_1((s)_{last(s)})))$ yields either a computation sequence or partial computation sequence with a smaller norm. Finally, let us show that if $result_F((s)_{last(s)})$ is equal to $y$, we have $T((y)_0, (y)_1, f) \to T((y)_0, \hat{m}((y)_0), f)$. Suppose otherwise; then $T((y)_0, (y)_1, f)$ holds, but not $T((y)_0, \hat{m}((y)_0), f)$. Since $m$ is a sound interpretation of $wit_1$, this means that $(y)_0$ is not in the domain of $\tilde{m}$. Let $i$ be the least value less than $last(s)$ such that $query_1((s)_i) = (y)_0$; there is at least one such $i$, since we are assuming that $F$ queries $\hat{m}$ at this value before the end of its computation. Note that we then have $norm'(m, query_{F,1}((s)_i)) = 1$. Let $m'$ represent the partial function extending $\tilde{m}$ with $\tilde{m}'((y)_0) = (y)_1$, and let $s'$ be the partial computation sequence which is an initial segment of $s$ of length $i$. Then the norm of the pair $\langle s', m'\rangle$ is less than the norm of the pair $\langle s, m\rangle$, contradicting the fact that $\langle s, m\rangle$ is supposed to minimize $norm(x, \cdot, f)$.

For the second proof, note that we can extract an explicit description of $G$ from the preceding argument. The iterative algorithm for $G$ starts with the pair $\langle s, m\rangle$, where $s$ is the sequence $\langle start_F(x)\rangle$ and $m$ is nowhere defined, and $norm_G$ is the function $norm$ above. As long as the current state of $G$ is a pair $\langle s, m\rangle$, where $s$ is either a partial computation sequence or a computation sequence for $F$ at $x$, $\hat{m}$, and $f$, and $m$ is a sound interpretation of $wit_1$, the argument above provides a recipe $next_G$ for finding a pair $\langle s, m\rangle$ with a smaller norm, unless $s$ is, in fact, a computation sequence and $\hat{m}$ satisfies the axiom for $wit_1$ at $result_F((s)_{last(s)})$; and in that case, $result_G$ can just return the pair $\langle s, m\rangle$. By Lemma 7.5, $T$ is $\prec\alpha$-recursive; so $start_G$, $norm_G$, $next_G$, and $result_G$ are all $\prec\alpha$-recursive functions. By Lemma 7.3, $G$ is be $\prec\omega^\alpha$-recursive, and the relevant properties can be proved in $ERA(f)$. ⊣

Putting this all together, we have

Corollary 9.3. *Let $\alpha$ be infinite and closed under multiplication. Then the proof-theoretic ordinal of $I\Delta_0^{exp}(f) + TI(\prec\alpha, \Pi_2(f))$ is at most $\omega^\alpha$.*

More generally, let

$$ERA(wit_1, \ldots, wit_n, f) + (wit^n(f)) + min(\prec\alpha, \mathcal{E}(wit_1, \ldots, wit_n, f))$$

denote the theory with $n$ witnessing functions, axioms asserting that each $wit_{i+1}$ returns witnesses to a complete $\Sigma_1(wit_1, \ldots, wit_i, f)$ formula, and minimization for functions that are elementary in $wit_1, \ldots, wit_n, f$. Adapting the proof of Lemma 9.2 yields

Lemma 9.4. *Let $\alpha$ be infinite and closed under multiplication. Suppose $\theta(x, y, wit_1, \ldots, wit_n, f)$ is a $\Delta_0(wit_1, \ldots, wit_n, f)$ formula such that*

$$ERA(wit_1, \ldots, wit_{n+1}, f) + (wit^{n+1}(f)) + min(\prec\alpha, \mathcal{E}(wit_1, \ldots, wit_{n+1}, f))$$

*proves* $\exists y\, \theta(x, y, wit_1, \ldots, wit_n, f)$. *Then*

$$ERA(wit_1, \ldots, wit_n, f) + (wit^n(f)) + min(\prec\omega^\alpha, \mathcal{E}(wit_1, \ldots, wit_n, f))$$

*proves it as well.*

Let $\omega_0^\alpha = \alpha$, and $\omega_{n+1}^\alpha = \omega^{\omega_n^\alpha}$. By induction, we have

THEOREM 9.5. *Let* $\alpha$ *be infinite and closed under multiplication, and let* $n$ *be greater than or equal to* 0. *Then the proof-theoretic ordinal of the theory* $I\Delta_0^{exp}(f) + TI(\prec\alpha, \Pi_{n+1}(f))$ *is at most* $\omega_n^\alpha$.

In particular, for $\alpha = \omega^\omega$, we have

THEOREM 9.6. *For* $n$ *greater than or equal to 1, the proof-theoretic ordinal of* $I\Sigma_n(f)$ *is at most* $\omega_{n+1}$.

Also, since any proof in $PA(f)$ is a proof in $I\Sigma_n(f)$, for some $n$, we have

THEOREM 9.7. *The proof-theoretic ordinal of* $PA(f)$ *is at most* $\varepsilon_0$.

More generally, let $PA(f) + TI(\prec\alpha)$ denote Peano arithmetic together with transfinite induction principles for arbitrary notations $\beta$ less than $\alpha$ and arbitrary formulae in the language.

THEOREM 9.8. *Let* $\alpha$ *be infinite and closed under* $\beta \mapsto \omega^\beta$. *Then the proof-theoretic ordinal of* $PA(f) + TI(\prec\alpha)$ *is at most* $\alpha$.

In all these theorems, the bounds are sharp.

**§10. Transfinite arithmetic hierarchies.** In this final section, we will consider theories with transfinite arithmetic hierarchies. We will continue to use functions as our basic second-order objects, and interpret references to a set $X$ of natural numbers as references to its characteristic function, $\chi_X$. If $X$ is any set, the Turing jump of $X$, written $X'$, is defined to be the set $\{x \mid \exists y\, T(x, y, \chi_X)\}$ where $\exists y\, T(x, y, f)$ is the complete $\Sigma_1(f)$ formula from Section 8. We can use a set $X$ to code a sequence of such sets, by interpreting $z \in X_y$ as $\langle y, z\rangle \in X$. If $\beta$ is an ordinal notation and $\langle H_\gamma\rangle_{\gamma\prec\beta}$ is a sequence of sets, then $H_{\prec\beta}$ is a set which codes this sequence (with $H_y = \emptyset$ if $y$ is the notation for the ordinal 0, or $y$ is not a notation less than $\beta$). Given an ordinal notation $\beta$, a jump hierarchy on $\beta$ is defined to be a set $H$, such that for every $\gamma$ less than $\beta$, $H_\gamma = (H_{\prec\gamma})'$.

Given a notation for a limit ordinal $\alpha$, we can extend the language of $PA$ with new symbols $H^\beta$, for $\beta$ less than $\alpha$. Our goal is then to bound the proof-theoretic ordinal of theories of the form

$$PA(\mathcal{H}) + H(\prec\alpha) + TI(\prec\alpha),$$

given by the following set of axioms: the axioms of $PA$, extended to the language with the new symbols; for each $\beta$ less than $\alpha$, an axiom asserting that $H^\beta$ is a jump hierarchy on $\beta$; and transfinite induction up to $\alpha$, for arbitrary formulae in new language. For an ordinal analysis, we really need to consider versions of these theories in which the induction axioms and transfinite hierarchies are relativized to a function symbol $f$; but to simplify the exposition I will drop the references to $f$, with the implicit understanding that all the lemmata and theorems in this section are easily generalized in this way.

We can simplify the theories above in a number of ways. For one thing, in $I\Delta_0^{exp}(\mathcal{H})$ one can show that if $H^\beta$ is a jump hierarchy on $\beta$, then for each $\gamma$ less than $\beta$, $H^\beta_{\prec\gamma}$ is a transfinite hierarchy on $\gamma$. This means that for any particular proof of an arithmetic statement, it suffices to use only a single new symbol $H^\beta$, for a sufficiently large $\beta$ less than $\alpha$. Also, if $\varphi(H^\beta)$ is an arithmetic formula relative to one of these hierarchies, there is a natural number $n$ such that $I\Delta_0^{exp}(\mathcal{H})$ proves that $\varphi(H^\beta)$ is equivalent to a formula that is $\Delta_0(H^{\beta+n})$. This means that nothing is lost of we restrict the transfinite induction principles to formulae that are $\Delta_0$ in the new symbols. Finally, since the transfinite induction axioms include ordinary induction, one can replace $PA(\mathcal{H})$ by $I\Delta_0^{exp}(\mathcal{H})$.

In carrying out the ordinal analysis we will need to make use of Kleene's recursion theorem. Let $R_0, R_1, R_2, \ldots$ denote a standard enumeration of the partial computable functions (of various arities). The recursion theorem says that for any partial computable function $f(x,e)$, there is an index $e$ for a unary partial computable function $R_e$, satisfying $R_e(x) \simeq f(e,x)$ for every $e$ and $x$. Since there is a universal partial computable function, this allows us to define a partial computable function $R_e$ in terms of itself, and its own index. A theory of computable functions strong enough to prove the recursion theorem can be developed in $ERA$ or $I\Delta_0^{exp}$, interpreting references to such functions by references to their indices. If I say that a function $f$ is "computable" instead of "partial computable," I mean, implicitly, that $f$ is total.

Once again, the first step in our ordinal analysis is to embed the theories we are interested in appropriate universal theories. An obvious strategy is to extend the $\Sigma_n$ witness functions of the previous section to transfinite ones. We can take a single function $f$ to represent a sequence of functions $f_x$, where $f_x(y) = f(\langle x, y\rangle)$; and given a function $f$ and a notation $\gamma$, we can take $f\restriction_\gamma$ to denote the function

$$f\restriction_\gamma(z) = \begin{cases} f(z) & \text{if } (z)_0 \prec \gamma \\ 0 & \text{otherwise} \end{cases}$$

Then a transfinite witness function $wit$ on $\beta$ satisfies the following:

$$\forall\gamma \prec \beta\, \forall x, y\, (T(x, y, wit\restriction_\gamma) \to T(x, wit_\gamma(x), wit\restriction_\gamma)).$$

Now, for each $\beta$ less than $\alpha$, we can add a function symbol $wit^\beta$, to denote a witness function on $\beta$. Then

$$ERA(\mathcal{W}) + wit(\prec\alpha) + min(\prec\alpha, \mathcal{E}(\mathcal{W}))$$

denotes the theory that extends elementary computable arithmetic (relative to the new function symbols) with axioms that assert that the new symbols denote transfinite witness functions for each $\beta$ less than $\alpha$, and minimization for functions in the language of $ERA(\mathcal{W})$. In $ERA(\mathcal{W})$ one can show that if $wit$ is a transfinite witness function up to $\beta$ and $\gamma$ is less than $\beta$, then $wit \upharpoonright_\gamma$ is a transfinite witness function up to $\gamma$; so for any proof, it suffices to have a witness function for a single $\beta$ less than $\alpha$ that is large enough.

In the end, it makes little difference whether one has a transfinite jump hierarchy or a transfinite hierarchy of witnesses, since each one can be obtained, effectively, from the other. The following lemma makes use of this fact.

LEMMA 10.1. *Let $\alpha$ be a limit. Then $ERA(\mathcal{W}) + wit(\prec\alpha) + min(\prec\alpha, \mathcal{E}(\mathcal{W}))$ is a universal theory, and proves any arithmetic formula that is provable in $PA(\mathcal{H}) + H(\prec\alpha) + TI(\prec\alpha)$.*

PROOF. It is clear that the axioms for the witness functions are universal. By the observations above, to prove the second statement, we only need to interpret references to $H^\beta$ in the language of $ERA(\mathcal{W})$, for each $\beta$ less than $\alpha$. Fix $\beta$, write $H$ for $H^\beta$, and write $wit$ for $wit^\beta$. We will use the recursion theorem to define a total computable function $R_e(\gamma, x)$, whereby we can interpret $x \in H_\gamma$ as $T(R_e(\gamma, x), wit_\gamma(R_e(\gamma, x)), wit \upharpoonright_\gamma)$. In other words, we aim to find an index $e$ that allows us to interpret $\chi_H$ as

$$\chi_H(w) = \begin{cases} 1 & \text{if } (w)_0 \prec \beta \text{ and} \\ & T(R_e((w)_0, (w)_1), wit_{(w)_0}(R_e((w)_0, (w)_1)), wit \upharpoonright_{(w)_0}) \\ 0 & \text{otherwise.} \end{cases}$$

For the moment, consider the right hand side as a function of $e$, as well as $w$; with this interpretation of $H$, for $\gamma$ less than $\beta$, $\chi_{H\upharpoonright_\gamma}(w)$ is a function of $e$, $w$, $\gamma$, and $wit \upharpoonright_\gamma$, given by

$$h(w, e, \gamma, wit \upharpoonright_\gamma) = \begin{cases} 1 & \text{if } (w)_0 \prec \gamma \text{ and } T(R_e((w)_0, (w)_1), \\ & \quad (wit \upharpoonright_\gamma)_{(w)_0}(R_e((w)_0, (w)_1)), (wit \upharpoonright_\gamma) \upharpoonright_{(w)_0}) \\ 0 & \text{otherwise.} \end{cases}$$

To verify the defining axiom for $H$, we need to satisfy

$$T(R_e(\gamma, x), wit_\gamma(R_e(\gamma, x)), wit \upharpoonright_\gamma) \leftrightarrow \exists y\, T(R_e(\gamma, x), y, \lambda w\, h(w, e, \gamma, wit \upharpoonright_\gamma))$$

whenever $\gamma$ is less than $\beta$. But for each $e$, the right hand side is $\Sigma_1$ in $wit \upharpoonright_\gamma$; so there is a code $n(e, \gamma)$, effectively obtained from $e$ and $\gamma$, such that this formula is equivalent to

$$\exists y\, T(\langle n(e, \gamma), x\rangle, y, wit \upharpoonright_\gamma).$$

By the defining axiom for $wit$, this is equivalent to

$$T(\langle n(e, \gamma), x\rangle, wit_\gamma(\langle n(e, \gamma), x\rangle), wit \upharpoonright_\gamma).$$

Using the recursion theorem to find an $e$ such that

$$R_e(\gamma, x) = \langle n(e, \gamma), x\rangle$$

we have the equivalence we are after. $\dashv$

Most of the rest of this section is devoted to proving the following:

Lemma 10.2. *Let $\alpha$ be infinite and closed under multiplication. If $\theta(x, y)$ is a $\Delta_0$ formula and $ERA(\mathcal{W}) + wit(\prec\alpha) + min(\prec\alpha, \mathcal{E}(\mathcal{W}))$ proves $\exists y\, \theta(x, y)$, then $PA + TI(\prec\varphi_\alpha(0))$ proves it as well.*

From now on, fix $\alpha$, and let us work in $PA + TI(\prec\varphi_\alpha(0))$. In this theory we can refer to computable functions by identifying them with their indices, and we can refer to $\Delta_0$ formulae by identifying them with their Gödel numbers. The definitions which follow should be seen as taking place in $PA + TI(\prec\varphi_\alpha(0))$, with these conventions.

In the last section, we saw, roughly speaking, that given a $\prec\eta$-computable function that queries $wit_1$, and, with "adequate" responses, finds a witness to a $\Sigma_1$ formula, we can effectively obtain a $\prec\omega^\eta$-computable function that finds a witness outright. In the current setting, this will let us transform a $\prec\eta$-iterative computation that queries a witness hierarchy on $\mu + 1$, to a $\prec\omega^\eta$-iterative computation which queries a witness hierarchy on $\mu$. To extend this reduction to the transfinite, we need to generalize the notion of an iterative computation.

A *$\prec\eta$-computable* (rather than recursive) *functional* $F(f)$ is given by a notation $\mu_F \prec \eta$, a value $start_F$, and computable (rather than elementary) functions $norm_F(q)$, $next_F(q, z)$, $query_F(q)$, and $result_F(q)$. Given $f$, the notions $CompSeq_F(s, f)$, $F(f)\downarrow$, $F(f) = y$, and so on, are defined as in Section 3.

Using sequences, we can express the assertion that $f$ is a witness hierarchy up to $\mu$ with a formula $\forall y\, W(y, \mu, f)$, where $W(y, \mu, f)$ is given by

$$(y)_0 \prec \mu \wedge T((y)_1, (y)_2, f\restriction_{(y)_0}) \to T((y)_1, f((y)_1), f\restriction_{(y)_0}).$$

Think of $W(y, \mu, f)$ as asserting that $f$ looks like a transfinite witness hierarchy on $\mu$, "locally," at $y$. Let $\theta(y, f)$ be (the code of) a formula that is $\Delta_0(f)$, let $\eta$, $\mu$, and $\gamma$ be notations with $\gamma$ less than $\mu$, and let $F(f)$ be a $\prec\eta$-computable function. By way of notation, take

$$F \vdash^\eta_\mu \exists y\, \theta(y, wit^\gamma)$$

to mean

$$\forall m, y\, (F(\hat{m}) = y \wedge W(y, \mu, \hat{m}) \to Tr_{\Delta_0}(\ulcorner\theta(y, \hat{m}\restriction_\gamma)\urcorner)),$$

where $Tr_{\Delta_0}$ is a truth predicate for $\Delta_0$ formulae. In words, $F \vdash^\eta_\mu \exists y\, \theta(y, wit_\gamma)$ means that if $F(\hat{m})$ is equal to $y$ and, at $y$, $\hat{m}$ looks like a witness hierarchy on $\mu$ at $y$, then $\theta(y, \hat{m}\restriction_\gamma)$ is true. Take $\vdash^\eta_\mu \exists y\, \theta(y, wit^\gamma)$ to assert the existence of such an $F$.

(Stepping outside $PA + TI(\prec\varphi_\alpha(0))$ for a moment, we can say that $\vdash^\eta_\mu \exists y\, \theta(y, wit^\gamma)$ implies that $\exists y\, \theta(y, wit^\gamma)$ holds for any transfinite witness function $wit^\gamma$. Reason as follows: given $F$ satisfying $\vdash^\eta_\mu \exists y\, \theta(y, wit^\gamma)$ and a transfinite witness function $wit^\gamma$, let $wit^\mu$ be a transfinite witness function with $wit^\mu\restriction_\gamma = wit^\gamma$, and let $m$ be a finite partial function agreeing with $wit^\mu$, defined at enough values to carry out the computation of $F$, satisfy $W_\mu$ at the result of the computation (call it $y$), and evaluate the truth of $\theta(y, wit^{\mu} \restriction_{\gamma})$. Since we have assumed $F \vdash^{\eta}_{\mu} \exists y\, \theta(y, wit^{\gamma})$, we have $\theta(y, \hat{m})$, and hence $\theta(y, wit^{\gamma})$.)

Still working in $PA + TI(\prec\varphi_{\alpha}(0))$, note that if $\varepsilon$ is less than $\alpha$, then transfinite induction up to $\varphi_{\varepsilon}(0)$ is available to us in this theory; and $\varphi_{\varepsilon}(0)$ is closed under the functions $\varphi_{\gamma}$, for $\gamma$ less than $\varepsilon$. With the notation we have just introduced, the two lemmata below bear a strong resemblance to the elimination lemmata of [21, Section 18] and [22, Section 2.1.2], as well as to [4, Lemma 3.6].

LEMMA 10.3. *There is a computable function $R_j(\eta, \mu, \theta, F)$, such that the following is provable in $PA + TI(\prec\varphi_{\alpha}(0))$: for any $\eta$, $\mu$, $F$, and $\Delta_0(f)$ formula $\theta(x, f)$, if*

$$F \vdash^{\eta}_{\mu+1} \theta(y, wit^{\mu})$$

*then*

$$R_j(\eta, \mu, \theta, F) \vdash^{\omega^{\eta}}_{\mu} \theta(y, wit^{\mu}).$$

PROOF. This is a straightforward adaptation of the proof of Lemma 9.2. ⊣

LEMMA 10.4. *Let $\varepsilon$ be less than $\alpha$. Then there is a computable function $R_e(\rho, \eta, \mu, \theta, F)$, such that the following is provable in $PA + TI(\prec\varphi_{\alpha}(0))$: for any $\eta$ and $\mu$ less than $\varphi_{\varepsilon}(0)$, any $\rho$ less than $\varepsilon$, any $F$, and any $\Delta_0(f)$ formula $\theta(y, f)$, if*

$$F \vdash^{\eta}_{\mu+\omega^{\rho}} \exists y\, \theta(y, wit^{\mu})$$

*then*

$$R_e(\rho, \eta, \mu, \theta, F) \vdash^{\varphi_{\rho}(\eta)}_{\mu} \exists y\, \theta(y, wit^{\mu}).$$

PROOF. We will use effective transfinite recursion to define $R_e$, with a primary recursion on $\rho$ and a secondary recursion on $\eta$; and then we will use nested instances of transfinite induction in $PA + TI(\prec\varphi_{\alpha}(0))$ to prove that $R_e$ is total and satisfies the conclusion of the lemma. In other words, we will use the recursion theorem to define $R_e(\rho, \eta, \mu, \theta, F)$ in terms of $\rho$, $\eta$, $\mu$, $\theta$, $F$, and the index $e$; and then for each $\rho$ and $\eta$, we will verify the correctness of $R_e$ at $\rho$ and $\eta$, assuming it behaves correctly for arguments $\rho'$, $\eta'$, $\mu'$, $\theta'$, and $F'$ whenever either either $\rho' \prec \rho$ or $\rho' = \rho$ and $\eta' \prec \eta$. For expository purposes, I will combine these two steps, defining $R_e$ by cases and showing that in each case, assuming the induction hypotheses are met, we have the desired conclusion. The proof is adapted from [21, 22].

First, note that if $\rho$ is equal to 0, then we can set $R_e(\rho, \eta, \mu, \theta, F)$ equal to $R_j(\eta, \mu, \theta, F)$, where $R_j$ is as in the previous lemma.

Next, suppose $\rho$ is greater than 0, and

$$F \vdash^{\eta}_{\mu+\omega^{\rho}} \exists y\, \theta(y, wit^{\mu})$$

where $F$ is an $\eta'$-computable function and $\eta'$ is less than $\eta$. Consider the computation of $F$ at $\hat{m}$. If $norm_F(start_F) \not\prec \eta'$, then the computation halts immediately. In that case, set $y = \mathit{result}_F(\mathit{start}_F)$; then we have

$$W(y, \mu + \omega^\rho, \hat{m}) \to \theta(y, \hat{m} \restriction_\mu).$$

But for this particular $y$, $W(y, \mu + \omega^\rho, \hat{m})$ only depends on values of $\hat{m}$ whose first component is at most $(y)_0$, assuming $(y)_0 \prec \mu + \omega^\rho$; so there is a $\rho' \prec \rho$ and $n$ such that $W(\mu + \omega^\rho, y, \hat{m})$ holds if and only if $W(\mu + \omega^{\rho'} \cdot n, y, \hat{m})$ does. In other words, for this $F$, we have

$$F \vdash^\eta_{\mu + \omega^{\rho'} \cdot n} \exists y\, \theta(y, \mathit{wit}^\mu).$$

Applying, successively,

$$R_e(\rho', \eta, \mu, \theta, \cdot), R_e(\rho', \varphi_{\rho'}(\eta), \mu, \theta, \cdot), \ldots, R_e(\rho', \varphi^{n-1}_{\rho'}(\eta), \mu, \theta, \cdot)$$

yields a $G$ satisfying

$$G \vdash^{\varphi^n_{\rho'}(\eta)}_\mu \exists y\, \theta(y, \mathit{wit}^\mu).$$

Since $\eta \preceq \varphi_\rho(\eta)$ and $\rho' \prec \rho$ we have $\varphi^n_{\rho'}(\eta) \preceq \varphi_\rho(\eta)$, so

$$G \vdash^{\varphi_\rho(\eta)}_\mu \exists y\, \theta(y, \mathit{wit}^\mu).$$

as required.

Finally, consider the case where $\mathit{norm}_F(\mathit{start}_F) \prec \eta'$. Consider the next step in the computation of $F$: if the first state, $q_0$, is $\mathit{start}_F$, and $\mathit{query}_F(\mathit{start}_F)$ is equal to $k$, then the second state, $q_1$, is $\mathit{next}_F(\mathit{start}_F, \hat{m}(k))$. If $(k)_0$ is less than $\mu + \omega^\rho$, let $n$ and $\rho'$ be such that $(k)_0$ is less than $\mu + \omega^{\rho' \cdot n}$. (If $(k)_0$ is not less than $\mu + \omega^\rho$, then the value of $\hat{m}(k)$ is irrelevant, and we can take $n$ and $\rho'$ to be 0 in the argument which follows.) For each $l$, let $G_l$ be the $\eta'$-computable function which continues the computation of $F$, assuming that $\hat{m}(k)$ returns $l$; in other words, $\mathit{start}_{G_l}$ is equal to $\mathit{next}_F(\mathit{start}_F, l)$, while $\mathit{norm}_{G_l}$, $\mathit{next}_{G_l}$, $\mathit{query}_{G_l}$, and $\mathit{result}_{G_l}$ are just $\mathit{norm}_F$, $\mathit{next}_F$, $\mathit{query}_F$, and $\mathit{result}_F$, respectively. Then it is not hard to see that for each $l$,

$$G_l \vdash^{\eta'}_{\mu + \omega^\rho} \exists y\, (\mathit{wit}^{\mu + \omega^{\rho'} \cdot n}(\bar{k}) = \bar{l} \to \theta(y, \mathit{wit}^{\mu + \omega^{\rho'} \cdot n} \restriction_\mu)).$$

After all, for any number $m$, if $s$ is a computation sequence for $G_l$ at $\hat{m}$, $\mathit{result}_{G_l}((s)_{\mathit{last}(s)})) = y$, $W(y, \mu + \omega^\rho, \hat{m})$, and $\hat{m} \restriction_{\mu + \omega^{\rho'} \cdot n} (k) = l$, then prepending $\mathit{start}_F$ to $s$ yields a computation sequence for $F$ at $\hat{m}$ returning the same result; and since $F \vdash^\eta_{\mu + \omega^\rho} \theta(y, \mathit{wit}^\mu)$, we have $\theta(y, \hat{m} \restriction_\mu)$. Now, note that $\mu + \omega^\rho$ is equal to $(\mu + \omega^{\rho'} \cdot n) + \omega^\rho$, since, in general, $\alpha + \omega^\beta = \omega^\beta$ whenever $\alpha \prec \omega^\beta$. For each $l$, let $\theta_l$ be the formula

$$f(\bar{k}) = \bar{l} \to \theta(y, f \restriction_\mu)$$

and let

$$\hat{G}_l = R_e(\rho, \eta', \mu + \omega^{\rho'} \cdot n, \theta_l, G_l).$$

By the inductive hypothesis we have that for each $l$,

$$\hat{G}_l \vdash^{\varphi_\rho(\eta')}_{\mu + \omega^{\rho'} \cdot n} \exists y\, (\mathit{wit}^{\mu + \omega^{\rho'} \cdot n}(\bar{k}) = \bar{l} \to \theta(y, \mathit{wit}^{\mu + \omega^{\rho'} \cdot n} \restriction_\mu)).$$

Let $G'(f)$ be the $\varphi_\rho(\eta')$-computable function which, in the first step, evaluates $f$ at $k$; and if the result is $l$, continues with $\hat{G}_l$. Then we have

$$G' \vdash^{\varphi_\rho(\eta')+1}_{\mu+\omega^{\rho'}\cdot n} \exists y\ \theta(y, wit^\mu),$$

since if $s$ is a computation sequence for $G'$ at $\hat{m}$ and $l = \hat{m}(k)$, then dropping the first element of $s$ yields a computation sequence for $\hat{G}_l$ at $\hat{m}$ with the same result. Finally, applying $R_e$ $n$ times with first argument $\rho'$, we obtain a $G$ satisfying

$$G \vdash^{\varphi^n_{\rho'}(\varphi_\rho(\eta')+1)}_{\mu} \exists y\ \theta(y, wit^\mu).$$

Since $\varphi^n_{\rho'}(\varphi_\rho(\eta')+1)$ is less than $\varphi_\rho(\eta)$, we have $G \vdash^{\varphi_\rho(\eta)}_{\mu} \exists y\ \theta(y, wit^\mu)$, as desired.

To justify the use of the recursion theorem, we only need to verify that in each of the three paragraphs above, $G$ was obtained effectively from $\eta$, $\mu$, $\rho$, $\theta$, $F$, and the index $e$. This is straightforward. Then, using transfinite induction up to $\varepsilon$ on $\rho$, with a secondary transfinite induction up to $\varphi_\varepsilon(0)$ on $\eta$, we need to verify in $PA + TI(\prec\varphi_\alpha(0))$ that for every $\rho$, $\eta$, $\mu$, $\theta$, and $F$, $R_e$ is defined, and if $F$ satisfies the hypothesis of the lemma, then $R_e$ returns a $G$ witnessing the conclusion. But this is the argument we have just sketched. $\dashv$

Proof of Lemma 10.2. Suppose the theory $PA(\mathcal{H}) + H(\prec\alpha) + TI(\prec\alpha)$ proves $\exists y\ \theta(x, y)$. By Lemma 10.1, we have that $\exists y\ \theta(x, y)$ is also provable in $ERA(\mathcal{W}) + wit(\prec\alpha) + min(\prec\alpha)$. By the observations above, $\exists y\ \theta(x, y)$ is therefore provable in the theory $ERA(wit^{\alpha'}) + \forall y\ W(y, \bar{\alpha}', wit^{\alpha'}) + min(\prec\alpha)$, for an $\alpha'$ less than $\alpha$ that is large enough. By the deduction theorem and Lemma 8.11, there is a $\prec\alpha$-recursive function $F$ such that $ERA(wit^{\alpha'})$ proves

$$F(x, wit^{\alpha'}) = y \wedge W(y, \bar{\alpha}', wit^{\alpha'}) \rightarrow \theta(x, y).$$

Suppose $F$ is $\alpha''$-recursive, where $\alpha''$ is less than $\alpha$. Pick $\varepsilon$ large enough so that we have $\alpha', \alpha'' \prec \varepsilon \prec \alpha$.

Now argue in $PA + TI(\prec\varphi_\alpha(0))$. For each $x$, let $F_x(f)$ be $\alpha''$-computable functional which, for each $f$, returns $F(x, f)$. Since $ERA(f)$ is sound, $F_x$ satisfies

$$F_x \vdash^{\alpha''+1}_{\alpha'+1} \exists y\ \theta(\bar{x}, y).$$

Using Lemma 10.4 , we have that for each $x$ there is a function $G_x(f)$ satisfying

$$G_x \vdash^{\varphi_{\alpha'+1}(\alpha''+1)}_{0} \exists y\ \theta(\bar{x}, y).$$

In other words, for any $m$, if $G_x(\hat{m})$ is defined then it returns a value $y$ satisfying $\theta(\bar{x}, y)$ outright. Let $m$ be the partial function that is nowhere defined; using transfinite induction up to $\varphi_{\alpha'+1}(\alpha''+1)$ we can show that $G_x(\hat{m})$ is defined, and so $\exists y\ \theta(\bar{x}, y)$ is true. $\dashv$

Relativizing this theorem to an additional function symbol $f$, we have

THEOREM 10.5. *Let $\alpha$ be infinite and closed under multiplication. Then the proof-theoretic ordinal of $PA(\mathcal{H}, f) + H(\prec\alpha) + TI(\prec\alpha)$ is at most $\varphi(\prec\alpha, 0)$.*

If $\Gamma_0$ is the least fixed point of the function $\alpha \mapsto \varphi(\alpha, 0)$, we have the following:

THEOREM 10.6. *$PA(\mathcal{H}, f) + H(\prec\Gamma_0) + TI(\prec\Gamma_0)$ has proof-theoretic ordinal at most $\Gamma_0$.*

Also, relativizing the proof of Theorem 10.5 to finite witness functions, and then interpreting references to these witness functions in $PA(f)$, yields

THEOREM 10.7. *Let $\alpha$ be infinite and closed under multiplication. Then any formula that is arithmetic in $f$ and provable in $PA(\mathcal{H}, f) + H(\prec\alpha) + TI(\prec\alpha)$ is also provable in $PA(f) + TI(\prec\varphi_\alpha(0))$.*

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