A Simple Characterization of Supply Correspondences*

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Abstract

We prove that supply correspondences are characterized by two properties: the law of supply and being homogeneous of degree zero.

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1 Introduction

We prove that supply correspondences are characterized by two simple properties: *the law of supply* and being *homogeneous of degree zero*. These two properties are both basic properties of profit maximization behavior: The first one captures the intuition that the supply decisions should change in the same direction as prices change. The second property states that if all prices of inputs and outputs are proportionally changed, then the firm supply decision should not change (see Section 5.C in Mas-Colell, Whinston, and Green, 1995).

To address the question of when a given supply correspondence is consistent with profit maximization behavior, we consider two concepts of rationalizability. A supply correspondence is *rationalizable* if there exists a convex closed production set such that for every price vector each supply decision is profit maximizing. A supply correspondence is *strongly rationalizable* if it consists of all possible maximizers. Our first result shows that if a supply correspondence satisfies the law of supply and is homogeneous of degree zero, then it is rationalizable. To obtain the strong form of rationalizability, we strengthen the law of supply property. The law of supply is equivalent

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to a supply correspondence being monotone. Consider the set of all monotone correspondences together with their graphs. A *maximal monotone* correspondence is a monotone correspondence with the graph that cannot be a strict subset of the graph of any other monotone correspondence; that is, its graph is maximal by set inclusion. This is an important and well-known concept in mathematical literature, where it has been extensively studied (see, e.g., Phelps, 1997; Rockafellar, 1970a; Simons, 2006). We show that for a supply correspondence to be strongly rationalizable, it is necessary and sufficient for it to be maximal monotone and homogeneous of degree zero.

The previous literature has provided several characterizations of rationalizable supply functions.[1] Most of these characterizations make strong assumptions about the differentiability of supply function. For example, Proposition 8 in Levin, Migrom, and Segal (2016) states that a continuously differentiable supply function y is rationalizable if and only if it satisfies $Dy(p)p = 0$ and its Jacobian $Dy(p)$ is symmetric, positive semidefinite.[2] The condition $Dy(p)p = 0$ is nothing but Euler's law applied to a homogeneous of degree zero supply function. The condition on Jacobian is an analog of the cyclical monotonicity condition. Our characterization applies to supply correspondences. When supply correspondence satisfies the law of supply, we show that the cyclical monotonicity condition could be dropped or reduced to the maximal monotonicity condition, depending on what rationalizability concept one is interested in.[3] Both conditions are easy to verify. Also, our characterizations does not require any assumptions of differentiablity on the supply function.

Our characterization of supply correspondences is also closely related to the characterization of combinatorial demand correspondences in the recent paper by Chambers and Echenique (2018). However, we note several important differences. First, our characterization does not require the image of supply correspondence to be finite (combinatorial). Second, we have different properties: we characterize supply correspondences in terms of the law of supply and homogeneity of degree zero, whereas Chambers and Echenique (2018) characterize combinatorial demand correspondences in terms of the law of demand and upper hemicontinuity. While both the law of demand and the law of supply are two versions of the monotonicity condition with opposite signs, the homogeneity of degree zero condition is absent in consumer choice theory, as the maximization objective is not linear. We also illustrate in Section 4 why the maximal monotonicity condition cannot be replaced

[1] See Samuelson (1948), Hanoch and Rothschild (1972), Varian (1984) for earlier important contributions to production analysis.

[2] For a similar statement, see Proposition 7.9 in Kreps (2013). Jehle and Reny (2011) in Chapter 3 reference this result as integrability theorem for supply functions, which is parallel to a similar result in consumer choice theory. See also Mas-Colell, Whinston, and Green (1995) and Varian (1992).

[3] An important result in convex analysis is that if a correspondence is maximal cyclically monotone, then it is the subdifferential of some proper convex lower semicontinuous function (i.e., the profit function) (Rockafellar, 1970a). In relation to this result, we show that maximal cyclical monotonicity condition could be reduced to maximal monotonicity condition if the function is homogeneous of degree zero. Homogeneous of degree zero is a strong condition, but it is natural in the context of production theory. See Section 4 for a more detailed discussion.

with the upper hemicontinuity condition when the image of supply correspondence is not finite.

2 Notation

Let us consider an economy with $N \geq 2$ commodities. A typical production plan is a vector $z = (z_1, ..., z_N) \in \mathbb{R}^N$, where an output has $z_n > 0$ and an input has $z_n < 0$. The vector of prices is denoted as $p \in \mathbb{R}^N$. We say that for $p, p' \in \mathbb{R}^N$, $p \leq p'$ if each coordinate of p is smaller than the corresponding coordinate of p'.

A *supply correspondence* is a mapping $y : P \rightrightarrows \mathbb{R}^N$, giving for each price vector p a set of possible production plans. Here, $P \subseteq \mathbb{R}^N$ is any nonempty convex cone containing the origin, i.e., a nonempty convex subset of $\mathbb{R}^N$ such that $x \in P \Rightarrow \lambda x \in P$ for all $\lambda > 0$. A typical economically relevant example for P is the set of non-negative prices $\mathbb{R}^N_+$. If y is single-valued, we refer to it as a supply function. Let $\text{Im}(y) = \{z \in \mathbb{R}^n \mid \exists p \in P \text{ s.t. } z \in y(p)\}$. We consider two concepts of rationalizability.

Definition 1. Supply correspondence $y : P \rightrightarrows \mathbb{R}^N$ is *rationalizable* if there exists a convex and closed production set Y such that for all $p \in P$, $y(p) \subseteq \{z \in Y \mid p \cdot z = \sup_{z' \in Y} p \cdot z'\}$.

We note that the requirement for production set Y to be convex and closed is innocuous, because if we find some production set Y to satisfy the maximization condition, the closure of its convex hull will rationalize supply correspondence y as well.

Definition 2. Supply correspondence $y : P \rightrightarrows \mathbb{R}^N$ is *strongly rationalizable* if there exists a convex and closed production set Y such that for all $p \in P$, $y(p) = \{z \in Y \mid p \cdot z = \sup_{z' \in Y} p \cdot z'\}$.

The first concept requires a supply correspondence to be a subset of all optimal production plans. The second concept demands that the supply correspondence cover all optimal production plans. The requirement for production set to be convex and closed is no longer innocuous in this case (as we discuss in Section 4). We also consider three relevant properties for supply correspondences.

Definition 3 (Law of Supply; Monotone)**.** A correspondence $y : P \rightrightarrows \mathbb{R}^N$ satisfies *the law of supply (is monotone)* if for all $p, p' \in P$, and all $z \in y(p), z' \in y(p')$, we have $(p - p') \cdot (z - z') \geq 0$.

The law of supply is a basic property of the profit maximization behavior corresponding to the intuition that the quantities should change in the same direction as prices change (see Mas-Colell, Whinston, and Green, 1995). At the same time, correspondences that satisfy the law of supply are called simply *monotone* correspondences in the mathematical literature (see Phelps, 1997). One could clearly see a justification for this in an application to functions in one-dimensional settings.

For one-dimensional settings, the law of supply applied to a single-valued function requires it to be non-decreasing. One could consider the set of all possible monotone correspondences defined on P. It is possible to provide a partial order on these correspondences using their graphs and define their maximal elements.

Definition 4 (Maximal Monotone)**.** A subset G of $\mathbb{R}^N \times P$ is said to be *monotone* provided $(z - z') \cdot (p - p') \geq 0$ whenever $(z, p), (z', p') \in G$. A correspondence $y : P \rightrightarrows \mathbb{R}^N$ is *monotone* if and only if its graph

$$G(y) = \{(z, p) : z \in y(p))\}$$

is *a monotone set*. A monotone set is said to be *maximal monotone* if it is maximal in the family of monotone subsets of $\mathbb{R}^N \times P$, ordered by inclusion. We say that a correspondence y is *maximal monotone* provided its graph is a maximal monotone set.

Finally, we state one more standard property.

Definition 5 (Homogeneity of degree 0)**.** Supply correspondence y is *homogenous of degree 0* if and only if for all $p \in P$, $\lambda > 0$, we have $y(\lambda p) = y(p)$. Note that $p \in P$ implies $\lambda p \in P$ since P is a cone.

We use the above properties in the next section to characterize supply correspondences.

3 Results

In this section, we present our main results. First, we show that the law of demand and homogeneity of degree zero are sufficient for rationalizability of supply correspondences. Second, we show that the extension of the law of supply to the maximal monotone condition, and to homogeneity of degree zero are necessary and sufficient for strong rationalizability supply correspondences.

Theorem 1. A correspondence $y : P \rightrightarrows \mathbb{R}^N$ is *rationalizable* if it satisfies *the law of supply* and is *homogeneous of degree zero.*[4]

Proof. We establish that if y satisfies the law of supply and is homogeneous of degree zero, it must satisfy the weak axiom of profit maximization; that is, for any $p, p' \in P$, $z \in y(p)$ and $z' \in y(p')$, we must have $p \cdot z \geq p \cdot z'$ (see Varian, 1984). The law of supply and homogeneity of degree zero imply that $(p - \lambda p') \cdot (z - z') \geq 0$ for any $\lambda > 0$ or

$$p \cdot (z - z') \geq \lambda p' \cdot (z - z') \text{ for any } \lambda > 0. \tag{1}$$

[4]Note that a version of this result for supply functions with a finite image was first established in the working paper by Kushnir and Lokutsievskiy (2019).

For the sake of contradiction, assume that $p\cdot(z-z') < 0$. Then, we must also have $\lambda p'\cdot(z-z') < 0$ for any $\lambda > 0$. We arrive at a contradiction, as we can then find a sufficiently small λ such that (1) is violated. Therefore, $p\cdot(z-z') \geq 0$ or $p\cdot z \geq p\cdot z'$ for any $p, p' \in \mathbb{R}^N$, $z \in y(p)$ and $z' \in y(p')$. In particular, $p \cdot z$ must be constant over all $z \in y(p)$.

Set production set Y to be the closure of convex hull of $\mathrm{Im}(y)$. Since linear inequalities are preserved under linear combination and under the operation of taking closure, we obtain that for $z \in y(p)$ and $z' \in Y$, $p\cdot z \geq p \cdot z'$. Finally, as product $p \cdot z$ is constant over all $z \in y(p)$, we obtain $y(p) \subseteq \{z \in Y \,|\, p\cdot z = \sup_{z'\in Y} p\cdot z'\}$. □

Note that the law of supply is also a necessary condition for rationalizability (see, e.g., Mas-Colell, Whinston, and Green, 1995).[5] To see this, note that rationalizability implies that for any $p, p' \in P, z \in y(p), z' \in y(p')$, $p'\cdot z \leq p\cdot z$. Similarly, $p\cdot z' \leq p'\cdot z'$. Adding these two inequalities together yields $p'\cdot z + p\cdot z' \leq p\cdot z + p'\cdot z'$, which is equivalent to the statement of the law of supply.

Unfortunately, rationalizable supply correspondences might not satisfy the homogeneity of degree zero, as one could pick maximizers in set $\{z \in Y : p\cdot z = \sup_{z'\in Y} p\cdot z'\}$. Hence, it might be the case that $y(p) \neq y(\lambda p)$ for $\lambda > 0$. Such a situation is not possible for strongly rationalizable supply correspondences.

We now show our main result that the properties of being maximal monotone and homogeneous of degree zero fully characterize strongly rationalizable supply correspondences.

Theorem 2. *A correspondence $y : P \rightrightarrows \mathbb{R}^N$ is strongly rationalizable if and only if it is maximal monotone and homogeneous of degree zero.*

Proof. To prove sufficiency, consider $y : P \rightrightarrows \mathbb{R}^N$ that is maximal monotone and homogeneous of degree zero. As any maximal monotone correspondence satisfies the law of supply, Theorem 1 implies that y is rationalizable by some convex and closed set Y; that is, $y(p) \subseteq \{z \in Y \,|\, p\cdot z = \sup_{z'\in Y} p\cdot z'\}$. We consider production set Y constructed in the proof of Theorem 1 as being the closure of the convex hull of $\mathrm{Im}(y)$. It remains to establish that $y(p) = \{z \in Y \,|\, p\cdot z = \sup_{z'\in Y} p\cdot z'\}$ for any $p \in \mathbb{R}^N$.

Assume there exists (p^*, z^*) such that $p^*\cdot z^* = \sup_{z'\in Y} p\cdot z'$, $z^* \in Y$, and $z^* \notin y(p^*)$. Hence, for any $p' \in P$ and $z' \in y(p')$, we must have both $p^*\cdot(z^*-z') \geq 0$ and $p'\cdot(z'-z^*) \geq 0$. Therefore, $(z^*-z')\cdot(p^*-p') \geq 0$. Hence, we can construct a new correspondence y' that coincides with y for all $p \neq p^*$ and $y'(p^*) = y(p^*) \cup z^*$. Correspondence y' is monotone and its graph strictly includes the graph of y. This contradicts to y being maximal monotone.

[5] An alternative statement of Theorem 1 is that any homogeneous of degree one correspondence $y : P \rightrightarrows \mathbb{R}^N$ is *rationalizable* if and only if it satisfies the law of supply.

To prove necessity, we consider some strongly rationalizable supply correspondence, $y : P \rightrightarrows \mathbb{R}^N$. Hence, there exists a convex and closed set $Y \subset \mathbb{R}^N$ such that $y(p) = \{z \in Y \,|\, p \cdot z = \sup_{z' \in Y} p \cdot z'\}$ for any $p \in P$. Let us denote profit function

$$\pi(p) = \sup_{z' \in Y} p \cdot z' \text{ for any } p \in P.$$

Therefore, y is the subdifferential of profit function $y = \partial\pi$ (see Theorem 23.5 in Rockafellar, 1970a). Note that π is a proper lower semi-continuous convex function as the support function of set Y (see p. 206 in Phelps, 1997). Hence, y is maximal monotone as its subdifferential (see Theorem 2.15 in Phelps, 1997).[6]

Finally, for any scalar $\lambda > 0$, we have

$$y(p) = \{z \in Y : p \cdot z = \sup_{z' \in Y} p \cdot z'\} = \{z \in Y : \lambda p \cdot z = \sup_{z' \in Y} \lambda p \cdot z'\} = y(\lambda p),$$

which shows that any strongly rationalizable supply function y is homogeneous of degree zero. $\square$

Note that it is rather intuitive that the mathematical property of "maximality" is needed to characterize the property of strong rationalizability. Strong rationalizability aims to capture the idea that a given supply function contains all the maximization points of the production set. Similarly, the property of maximality ensures that one cannot properly extend a given monotone correspondence in the space of all monotone correspondences.

4 Discussion

In this section, we connect our results with mathematical literature on monotone operators and relate the results more closely to the recent contribution by Chambers and Echenique (2018).

The main contribution of this paper is to provide a simple characterization of rationalizable supply correspondences. When the classical characterizations in economics employ the properties of the Jacobian of supply functions, our characterization does not use any assumptions of differentiability. Instead, our characterization relies on the concept of *maximal monotonicity*, which a well-known and well-studied concept in mathematics. Monotone and maximal monotone operators play an important role in convex analysis, partial differential equations, optimization, and calculus of variations (see Phelps, 1997; Rockafellar and Wets, 2009; Simons, 2006; Zălinescu, 2002; Zeidler, 2013). The mathematical literature provides several characterizations of maximal monotone operators including the one using saddle functions (Krauss, 1985) and the one using Fitzpatrick

[6] An alternative proof that any strongly rationalizable support function y is maximal monotone can done using the Bishop-Phelps theorem (see p. 31 in Simons, 2006).

functions (Fitzpatrick, 1988).[7] The Fitzpatrick functions have become instrumental in providing simpler proofs of many deep results on maximal monotone operators (see, e.g., Borwein and Zhu, 2010; Burachik and Svaiter, 2003; Martinez-Legaz and Svaiter, 2005; Simons and Zălinescu, 2004) and cyclically monotone operators (see Bartz, Bauschke, Borwein, Reich, and Wang, 2007). For the modern treatment of the theory of monotone operators see Burachik and Iusem (2008) and Simons (2008).

An important class of maximal monotone operators are subdifferentials of convex functions. In fact, Rockafellar (1970b) proved a seminal result that a given operator is a subdifferential of a proper convex lower semicontinuous function if and only if it is maximal cyclically monotone. In one dimension, any monotone operator is also cyclically monotone. This is no longer the case in multi-dimensional spaces, where some skew linear operators (see, e.g., Simons, 2006) and some rotations (Bartz, Bauschke, Borwein, Reich, and Wang, 2007; Archer and Kleinberg, 2014) are monotone, but not cyclically monotone. In the finite-dimensional Euclidean spaces, however, any monotone function is cyclically monotone if it is defined on a convex domain and it has a finite range (Saks and Yu, 2005; Bikhchandani, Chatterji, Lavi, Mu'alem, Nisan, and Sen, 2006). This result has become important in economics literature as a way to characterize incentive compatible allocation rules in the fields of auctions and mechanism design. Some subsequent studies extend this result to non-convex domains (see, e.g., Kushnir and Lokutsievskiy, 2021; Mishra, Pramanik, and Roy, 2014) and to functions with infinite range (Müller, Perea, and Wolf, 2007; Carbajal and Müller, 2015, 2017). The main result of this paper (Theorem 2) contributes to this literature by showing that any correspondence that is maximal monotone and homogeneous of degree zero is also maximal cyclically monotone.

Also, we want to relate our characterization of suppply correspondences to the characterization of combinatorial demand correspondences by Chambers and Echenique (2018). Both characterizations have different properties: we characterize supply correspondences in terms of maximal monotonicity and homogeneity of degree zero, whereas Chambers and Echenique (2018) characterize combinatorial demand correspondences in terms of the law of demand and upper hemicontinuity. The homogeneity of the degree zero condition is absent in consumer choice theory, as the maximization objective is not linear in prices. The maximal monotonicity condition is a stronger form of the law of supply. Both the law of supply and the law of demand are monotonicity conditions, with the opposite signs reflecting that the supply function should increase in output prices and the demand function should decrease in good prices. At the same time, maximal monotonicity is stronger than uppper hemicontinuity, as any maximal monotone correspondence is upper hemicontinuous, but the reverse direction is generally not true (p. 203 in Phelps, 1997).

We cannot replace maximal monotonicity with a weaker condition of upper semicontinuity in

[7]See Penot (2004) for an alternative characterization.

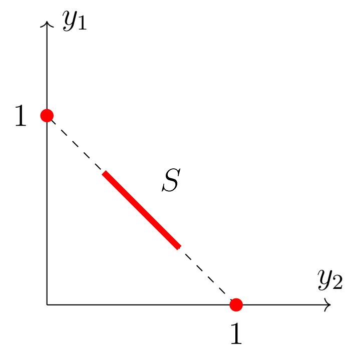

Figure 1. The graph of supply correspondence (2) that satisfies the law of supply, is homogeneous of degree zero, and is upper hemicontinuous, but that cannot be strongly rationalized with a convex or a closed set.

Theorem 2, as we can then no longer guarantee that the supply correspondence can be strongly rationalized with a convex closed set or even just a closed set (even if it satisfies the law of supply). To illustrate, let us consider supply correspondence $y : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$, defined as

$$y(p) = \begin{cases} (1,0) & \text{if } p_1 \geq p_2 \\ S & \text{if } p_1 = p_2 \\ (0,1) & \text{if } p_2 \geq p_1 \end{cases}, \tag{2}$$

where S is a subset of $Z = \{z \mid z = \alpha(0,1) + (1-\alpha)(1,0),\ \alpha \in [0,1]\}$ and we require $(0,1),(1,0) \in S$. Set S is depicted in red in Figure 1. It is straightforward to see that y satisfies the law of supply and is homogeneous of degree zero. Hence, y is rationalizable. For example, set Z rationalizes y. As $(0,1),(1,0) \in S$ supply correspondence is also upper hemicontinuous. If S is a strict subset of Z, then y cannot be strongly rationalized with a convex production set, because such a set has to coincide with Z. Moreover, if S is not closed, y cannot be strongly rationalized with a closed set, as such a set has to coincide with S. Overall, the above example illustrates that we cannot weaken the maximal monotonicity condition with the upper hemicontinuity condition in the settings when the image of supply correspondence is not finite.

Finally, we want to point out that the convexity and closedness requirements on a production set disciplines our strong rationalizability concept (Definition 2). If we drop these requirements on the production set then the proofs of Theorems 1 and 2 imply that any homogeneous of degree zero correspondence is strongly rationalizable by any set if and only if it satisfies the law of supply.

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